Source time function and time reversal technique

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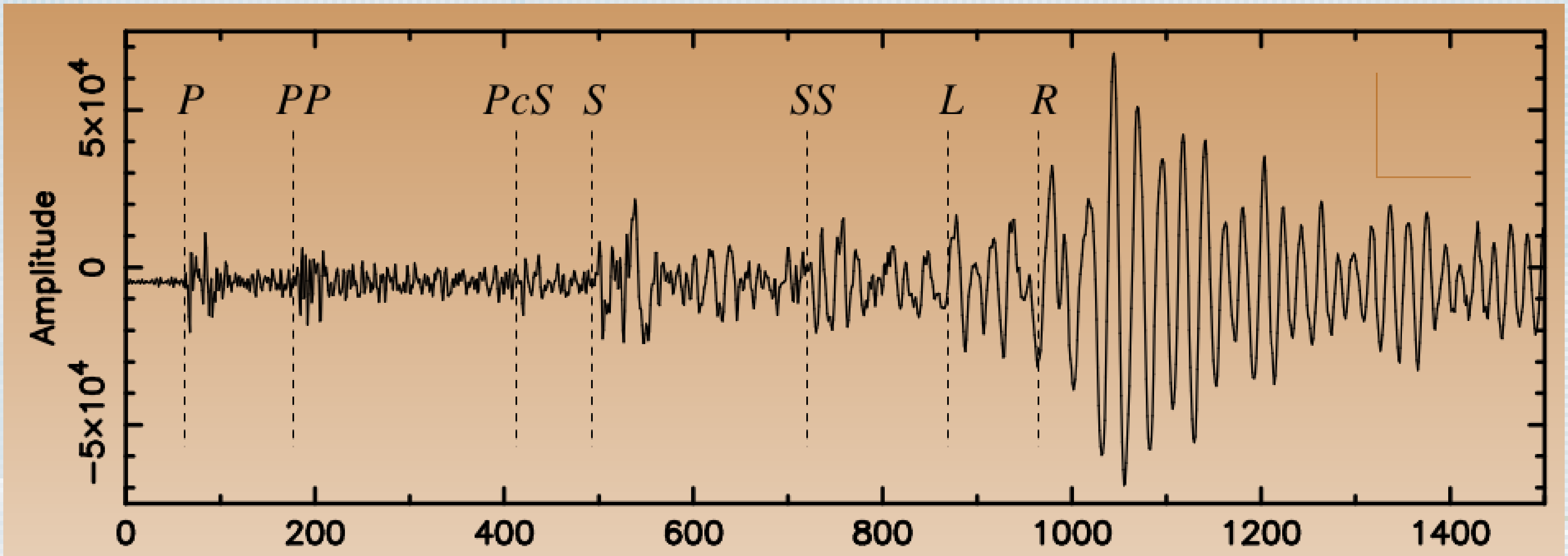
Seismological inference

Seismology deals with two broad classes o problems:

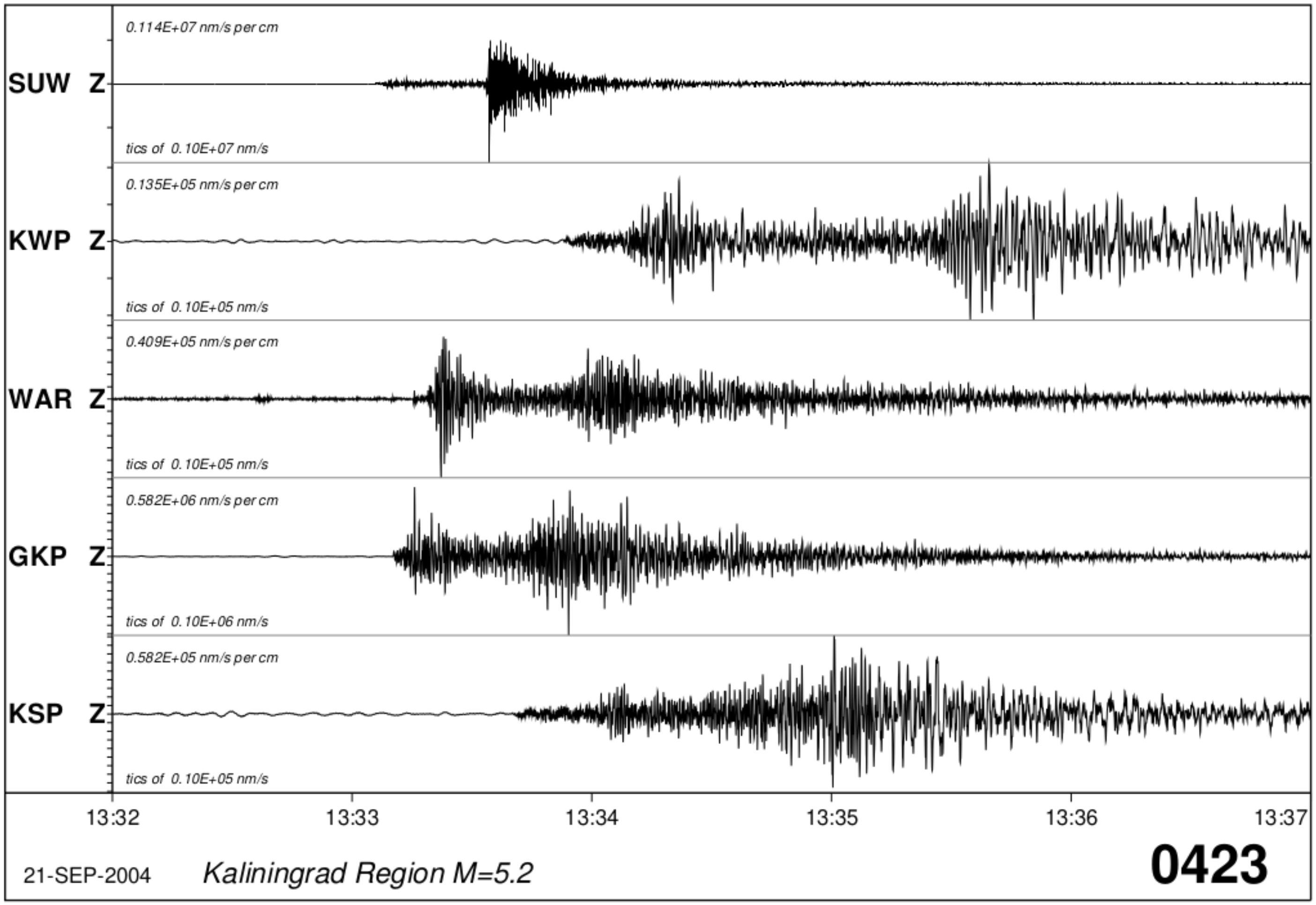
- Structure and composition of the Earth in various scales
  global, regional ($\sim$ 2000 km), local ($\sim$ 100 km), etc.
- Earthquakes: their origins, physics, effects and (possible) prediction

Inference is always based on records made on surface (space)
There is no possibility of methodological direct measurements
(especially seismic sources).

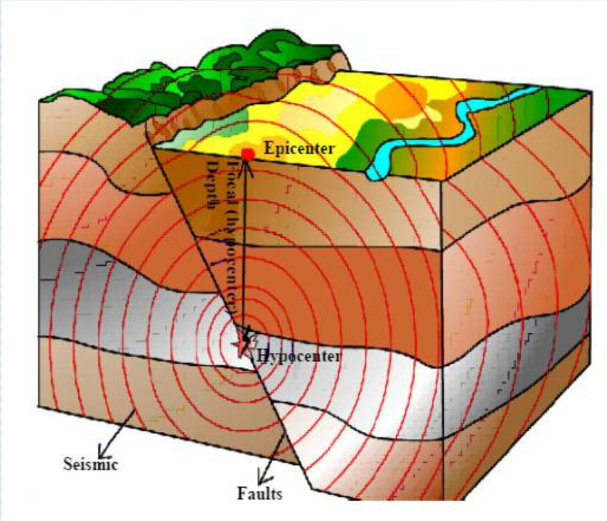
Seismic waves - primary source of information



Seismic waves - primary source of information

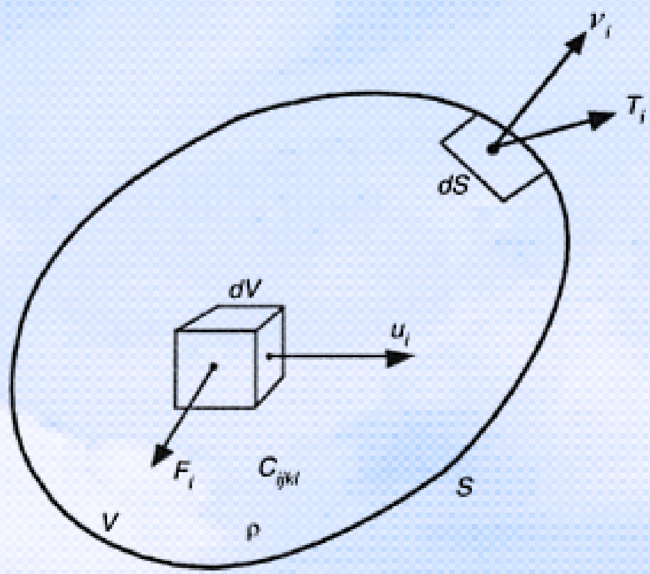


Seismic waves - information contents



$$\mathbf{u}(\mathbf{r},t) = \int_{R',T'} G(\mathbf{r},\mathbf{r}',t-t')\; S(\mathbf{r}',t')\, \mathrm{d}\mathbf{r}'\mathrm{d}t'$$

Mathematics of elastic waves



$$T_i = \tau_{ij} n_j$$

$$\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

$$\frac{\mathrm{d}}{dt} \int_V \rho v_i(x,t) \mathrm{d}V = \int_S T_i(x,t) \mathrm{d}S + \int_V F_i(x,t) \mathrm{d}V$$

Mathematics of elastic waves

Using Stock's theorem

$$\int_V \rho \frac{\partial^2 u_i}{\partial t^2} \mathrm{d}V = \int_V \frac{\partial \tau_{ij}}{\partial x_j} \mathrm{d}V + \int_V F_i(x,t) \mathrm{d}V$$

$$\rho \ddot{u}_i = \partial_j \tau_{ij} + F_i$$

Homogeneous isotropic elastic medium

Hook's law: $\tau_{ij} = c^{ijkl}\,\epsilon_{kl}$

Isotropic materials: $c^{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl}$

Homogeneous medium: $\lambda, \mu = \text{const.}$

$$\rho\ddot{\mathbf{u}} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{F}$$

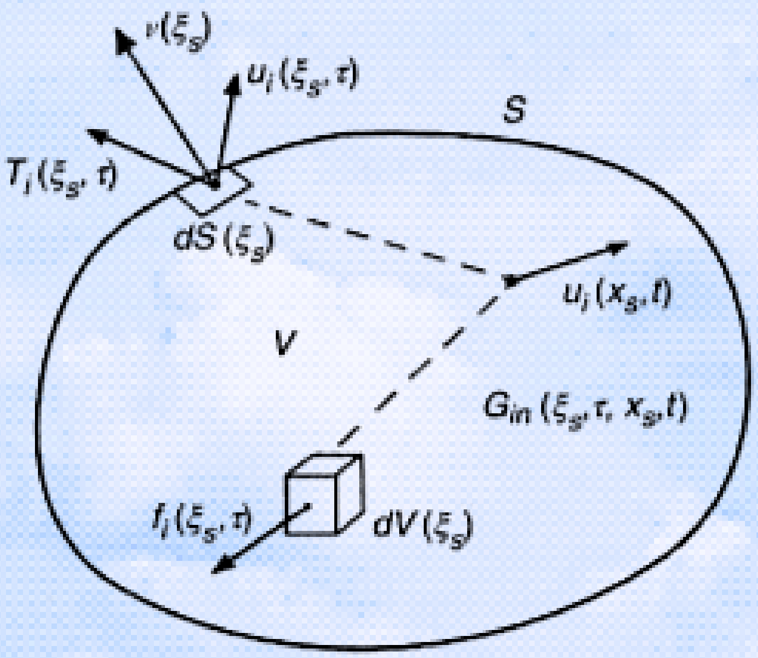
Representation theorem

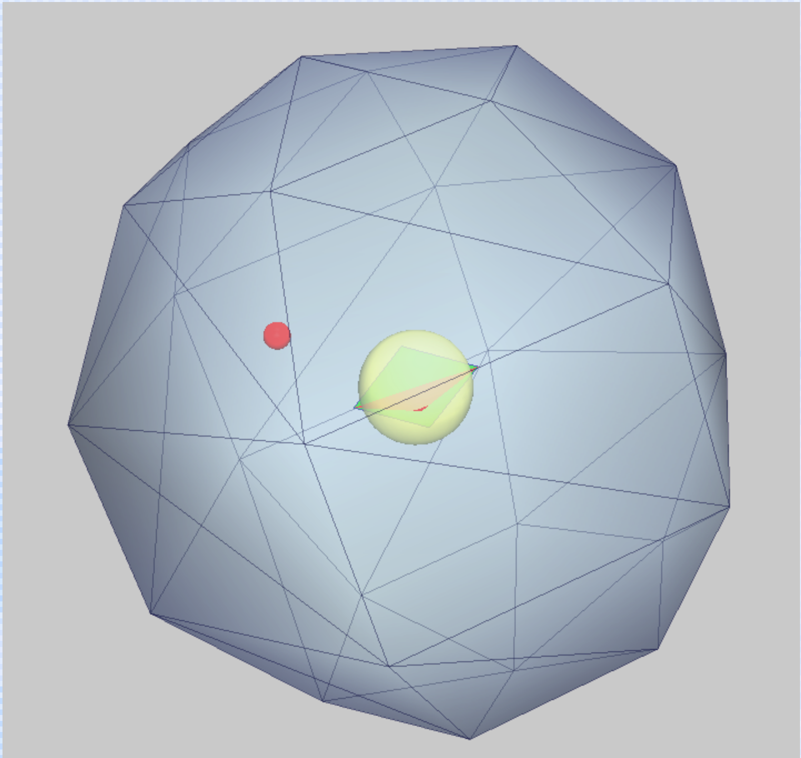
Desired explicit solution for given source term (F_i)

Green's function: $\left(\rho\delta_{ik}\partial_t^2 + \partial_j\tau_{ij}\right) G_{ik} = e_k\delta(\mathbf{r},\mathbf{t})$

$$
\begin{aligned}
u_n(x_s,t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \int_V f_i(\xi,t') G_{ni}(\xi,t';x_s,t)\mathrm{d}\xi \\
&+ \int_{-\infty}^{\infty} \mathrm{d}t' \int_S G_{ni}(\xi,t';x_s,t) T_i(\xi,t')\mathrm{d}\xi \\
&- \int_{-\infty}^{\infty} \mathrm{d}t' \int_S C_{ijkl}\, u_i(\xi,t') G_{nk,l}(\xi,t';x_s,t) n_j(\xi)\mathrm{d}\xi
\end{aligned}
$$

Interpretation - forward problem





Given $f_i, T_{|\Sigma}, u_{|\Sigma}$ one can uniquely determine $u(\mathbf{x})$

Interpretation - inverse problem

Given $u(\mathbf{x}, t)$ can source be determined uniquely ?

NO!

Three **equivalent** class of source models:

1. f_i - body force (intuitive)
2. $u_{|\Sigma}$ - kinematic (dislocation source)
3. $T_{|\Sigma}$ - dynamic (stress drop, friction, etc.)

Equivalence: Body force model

Point source approximation: $L << \lambda$, $L << R$

$$u_n(x_r, t) = \int_{-\infty}^{\infty} f_i(x_s, t') G_{ni}(x_s, t'; x_r, t) \mathrm{d}t'$$

For DC source system (4 elementary sources)

$$u_n(x_r, t) = (e_k e_j + e_j e_k) \int_{-\infty}^{\infty} M(t') \frac{\partial G_{nk}(x_s - x_r, t - t')}{\partial x_j} \mathrm{d}t'$$

Equivalence: Kinematic source model

$$u_n(x_s,t) = \int_{-\infty}^{\infty} \mathrm{d}t' \int_S C_{ijkl}\, \Delta u_i(\xi,t') G_{nk,l}(\xi,t';x_s,t) n_j(\xi) \mathrm{d}\xi$$

Equivalence: Kinematic source model

Assumptions:

- point source, homogeneous medium, non-rotating rupture
- $\mathbf{e}^s$ - slip unit vector ($\Delta\mathbf{u} = \mathbf{e}^s \Delta u$)
- $\mathbf{e}^n$ - vector normal to rupture plane
- $\mathbf{e}^n \cdot \mathbf{e}^s = 0$ - pure shearing

$$u_n(x_s, t) = \int_{-\infty}^{\infty} S\Delta u \left(\lambda e_l^s e_l^n \delta_{ij} + \mu(e_i^s e_j^n + e_j^s e_i^n)\right) G_{ni,j}(x_s, t'; x_r, t)\mathrm{d}t'$$

Equivalence:

Kinematic model:

$$u_n(x_s,t) = \int_{-\infty}^{\infty} (e^s_k e^n_j + e^s_j e^n_k) \underbrace{\mu S \Delta u(t')}_{M_o(t')} G_{nk,j}(x_s,t';x_r,t)\mathrm{d}t'$$

Body force model:

$$u_n(x_r,t) = (e_k e_j + e_j e_k) \int_{-\infty}^{\infty} M(t') G_{nk,j}(x_s,t';x_r,t)\mathrm{d}t'$$

Double couple $\equiv$ shear dislocation source

Far field solution - homogeneous medium

Moment tensor

$$M_{ij}(t) \;=\; (e^s_k e^n_j + e^s_j e^n_k)\mu S \Delta u(t) = (e^s_k e^n_j + e^s_j e^n_k) M_o(t)$$

Source time function

$$M_{ij}(t) \;=\; m_{ij}\, M_o S(t)$$

Solution

$$u_i(x_s, t) = \frac{M_o}{4\pi\rho v_p^3 r}\gamma_i\gamma_j\gamma_k m_{jk}\dot{S}(t - r/v_p)$$

Moment tensor inversion

$$u_n(x_r,t) = \int_{-\infty}^{\infty} M_{ij}(t')\, G_{ni,j}(x_s,t';x_r,t)\mathrm{d}t'$$

$$M_{ij}(t) = (e^s_k e^n_j + e^s_j e^n_k) M_o S(t)$$

Moment tensor inversion should comprise

- Nm=1 Seismic scalar moment $M_o = \mu S \Delta u_o$
- Nm=6 *"Fault plane solution"* $e^s_k e^n_j + e^s_j e^n_k$
- Nm=(∞) Source time function $S(t) = \Delta\dot{u}/u_o$

Inversion seismograms for STF

- Standard approach (waveform inversion)

$$||u_{syn}(x_r, x_s, S(t)) - u_{obs}(x_r, x_s)|| + [\text{a priori}] \;=\; \min$$

- Empirical Green Function (Hartzell)

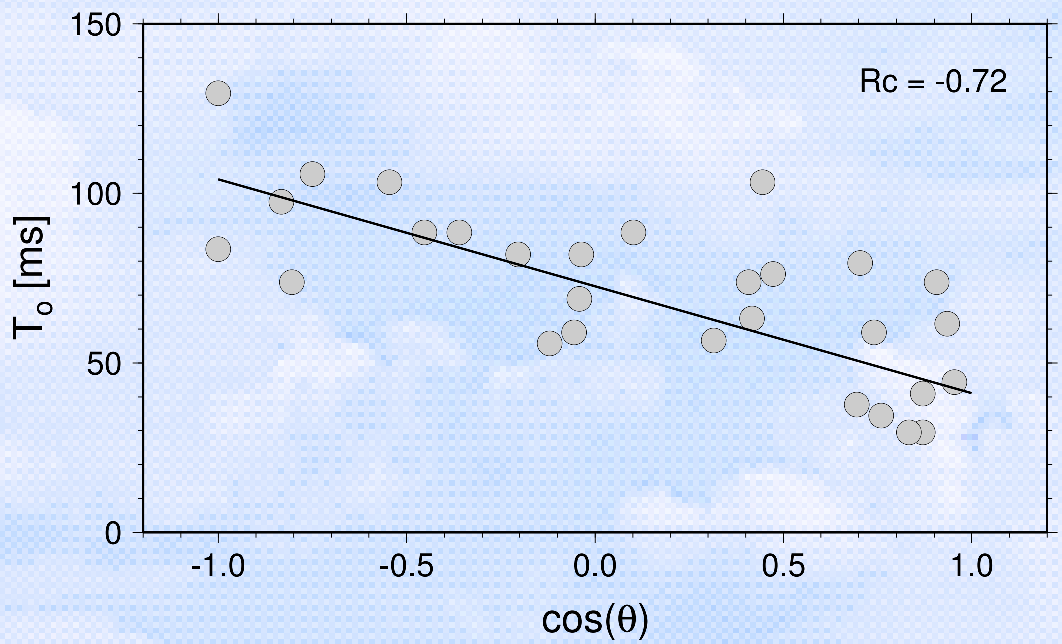
$$S(t) \;=\; \frac{u_{main}(t)}{u_{small}(t)} \qquad (S_{small}(t) = \delta(t)$$

- Back projection (time reversal)

Physics inferred from STF

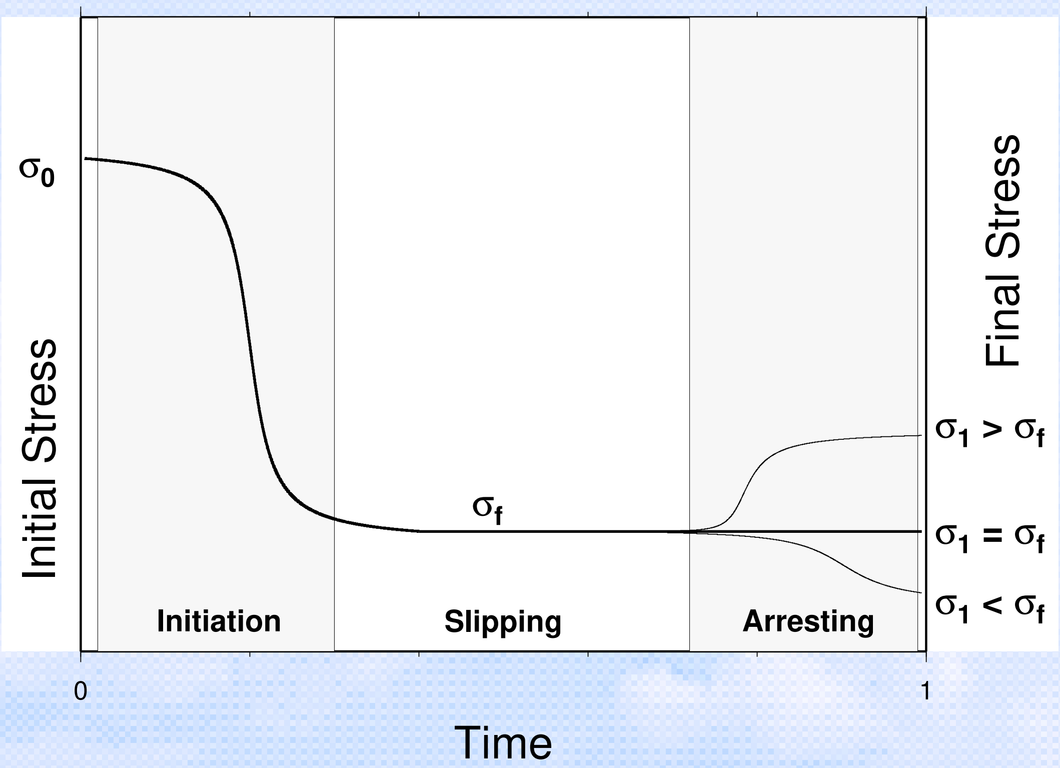
- rapture velocity (Ben-Menahem)

$$T_i \ = \ T_o - \delta T \cos(\theta)$$



Physics inferred from STF

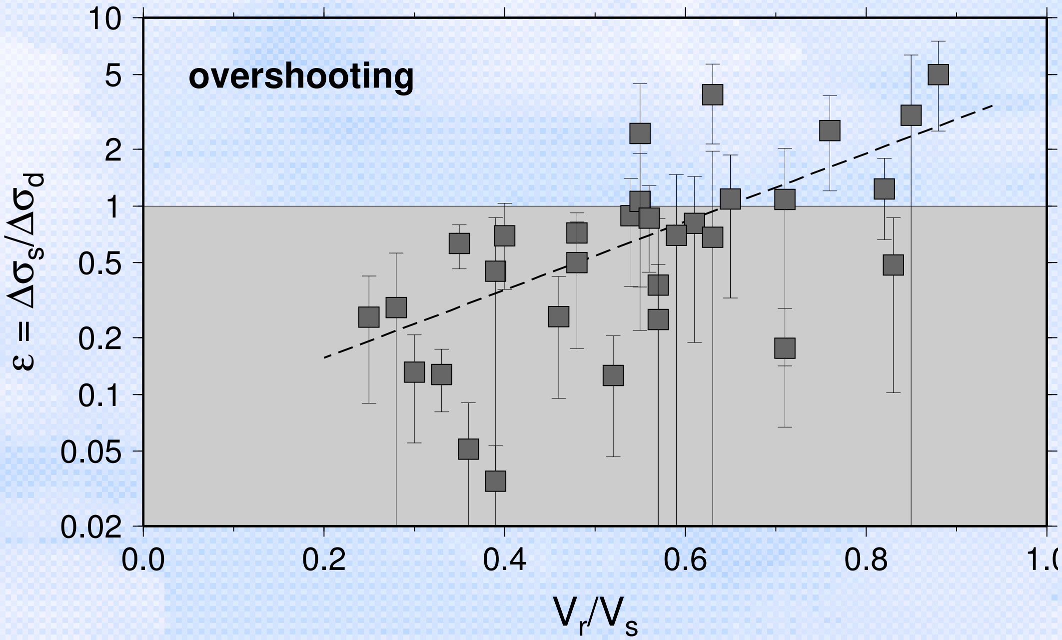
✦ arresting mechanisms



Physics inferred from STF

✦ dynamic stress drop

$$\Delta\sigma_d = \frac{M_o}{4\pi v_r^3}\xi\frac{\delta S}{\bar{S}}$$



Time reversal - Green's function

$$\frac{1}{c^2}\frac{\partial^2 G(x,t)}{\partial t^2} - \Delta G(x,t) = \delta(x - x_o, t - t_o)$$

Solution:

$$u(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x'\, G(t,x;t',x') S(x',t')$$

$G(x,t;,x',t')$ "propagates" information from source (x') to receiver (x)

Green's function cd.

Assuming time translation invariance

(energy conservation - Noether's theorem)

$$G(x,t;x',t') = G(x,x';t-t')$$

Homogeneous medium

translational invariance

$$G(x,t;x',t') = G(x-x';t-t')$$

Retarded and advanced Green's functions

Retarded (casual) GF:

$$G^{+}(x-x',t-t')=\frac{1}{4\pi||x-x'||}\delta\left(t-t'-\frac{||x-x'||}{c}\right)\Theta(t-t')$$

Advanced (anti-casual) GF:

$$G^{-}(x-x',t-t')=\frac{1}{4\pi||x-x'||}\delta\left(t-t'+\frac{||x-x'||}{c}\right)\Theta(t'-tt)$$

$$G^{+}(x-x',t-t')=G^{-}(x-x',t'-t)$$

Time Reversal - step I

Radiation from point source (O)

$$S(x,t) = S(t)\delta(x - x_o)$$

Solution at given receiver (R)

$$u_R(x,t) = \int_{OR} G^+(x - x_o, t - t',)S(t')\mathrm{d}t'$$

Time Reversal - step II

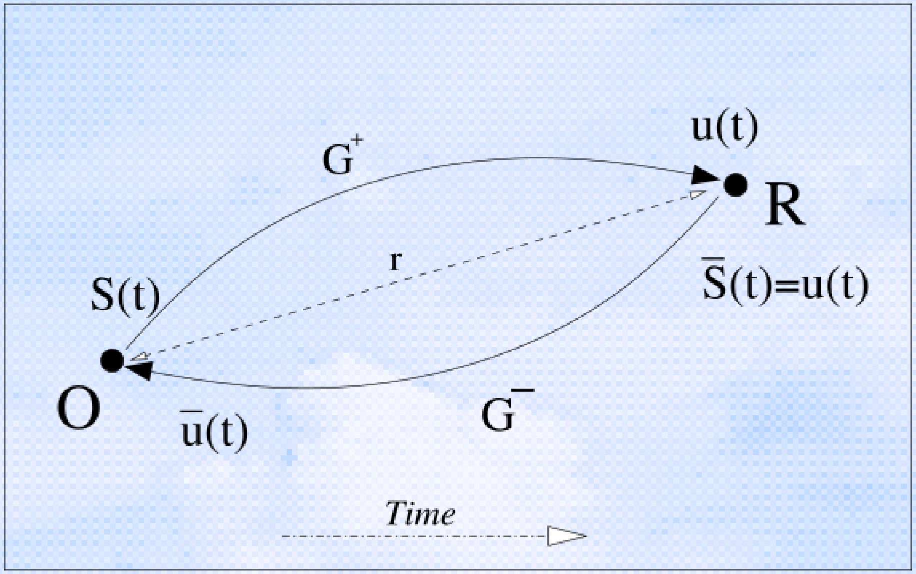
Anti-casual propagation from R to O

$$\bar{u}(x,t) = \int_{RO} G^{-}(x - x_R, t - t',)\bar{S}(t')\mathrm{d}t'$$

From physical point of view it corresponds to recording at the point O incoming waves, for example, "refracted" at the point R

$$\bar{S}(t) = u_R(t)$$

Time Reversal Method (TRM) - basic principle



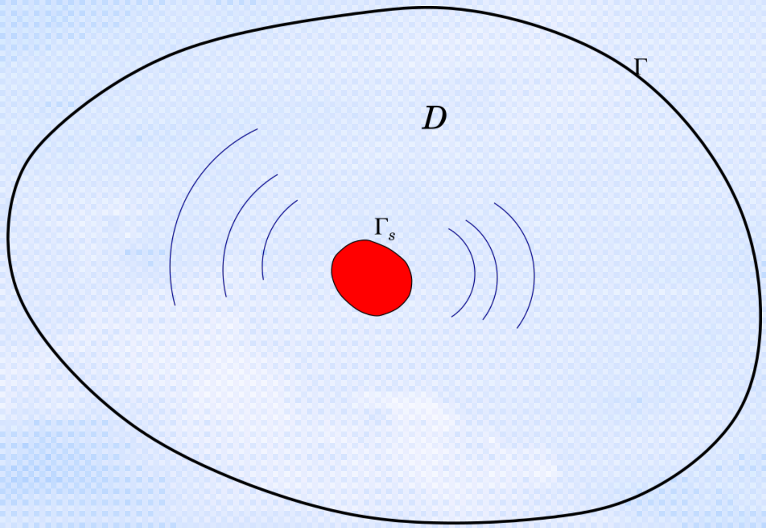
1. $u(r,t) = \int G^{+}(r, t-t')S(t')\mathrm{d}t'$
2. $\bar{S} = u(r,t)$
3. $\bar{u}(t) = \int G^{-}(r, t-t')\bar{S}(t')\mathrm{d}t'$

$\mathbf{\bar{u}(r,t) = k(r)\ S(t)}$

However: $G^{+}(\cdot, t-t') = G^{-}(\cdot, -(t-t'))$

$$S(t) = \int G^{+}(t-t', r)u(-t')\mathrm{d}t'$$

Time Reversal limitation - finite size source



$$u_D(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x' \; G^+(t,x;t',x')S(x',t')$$

$$u_D(x,t) = \int_{-\infty}^{\infty} dt' \int_{\partial\Gamma} ds' \Big(u(x',-t') \frac{\partial}{\partial n'} G^+(x,t;x',t') - G^+(x,t;x',t') \frac{\partial}{\partial n'} u(x',-t') \Big).$$

TRM as the inverse problem

$$\mathbf{m} = S(t) \qquad\qquad \mathbf{d} = u(t)$$

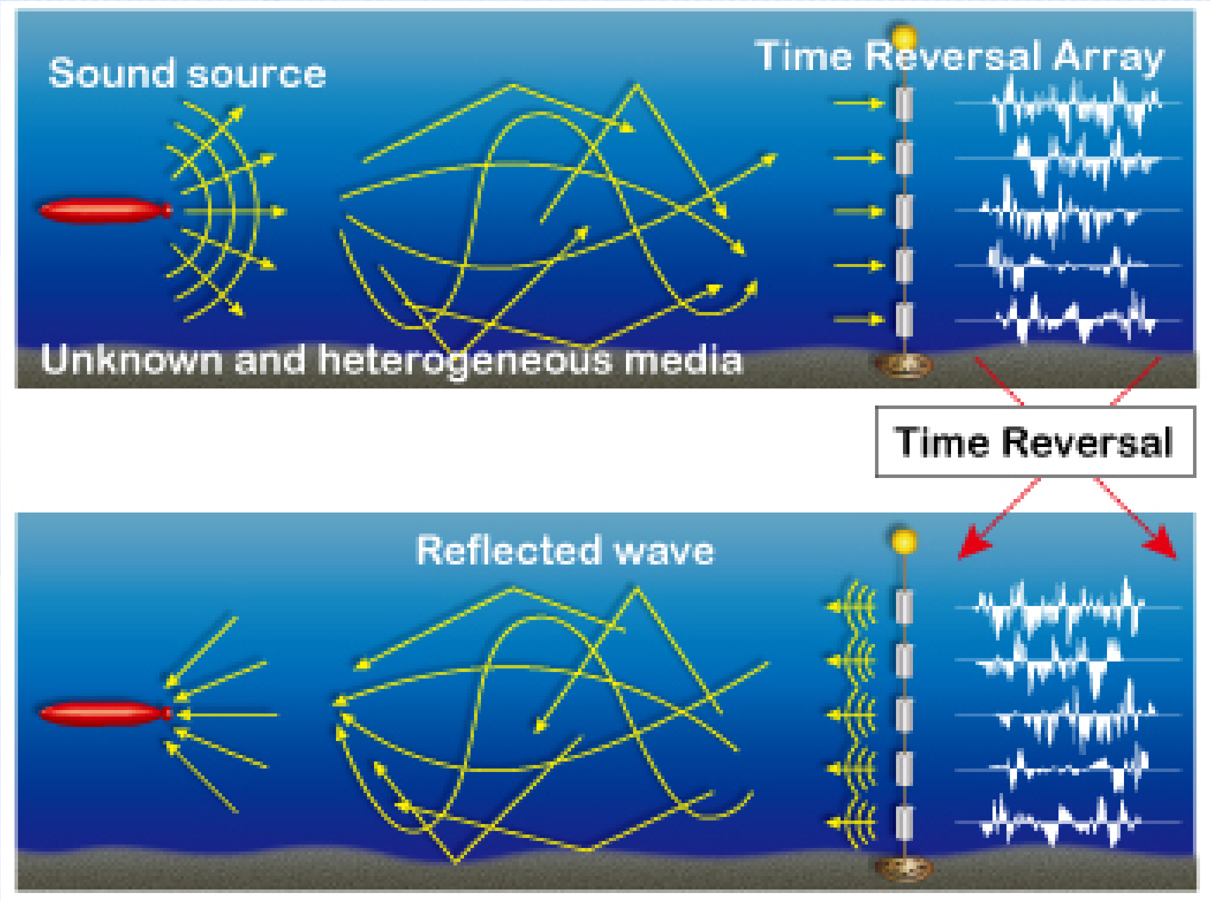
$$u(r,t) = \int G^+(r, t-t')\, S(t')\mathrm{d}t' \quad \Rightarrow \quad \mathbf{d} = \mathcal{G}\mathbf{m}$$

$$S(t) = \int G^+(r, t-t')\, u(-t')\mathrm{d}t' \quad \Rightarrow \quad \mathbf{m} = \mathcal{G}^{-1}\mathbf{d}$$

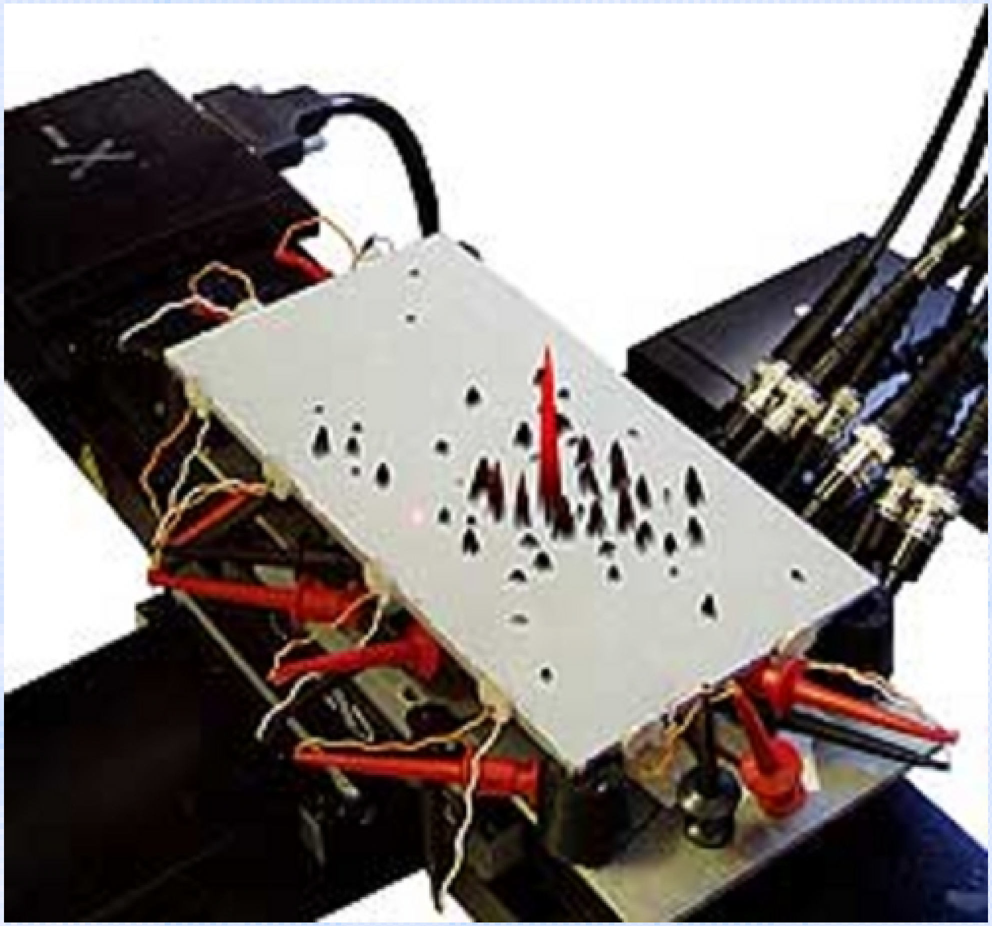
The method is computationally very efficient, but...

we need to know the medium: $G^+(\cdot, \cdot)$

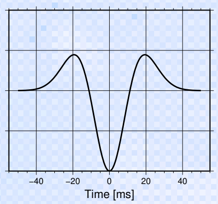
Physics of Time Reversal

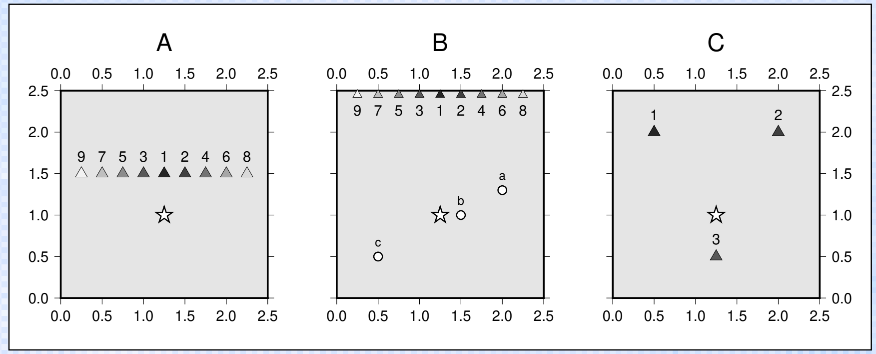


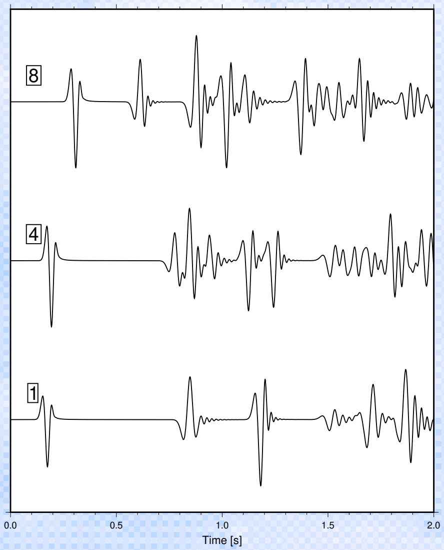
Experimental evidence (M. Fink)

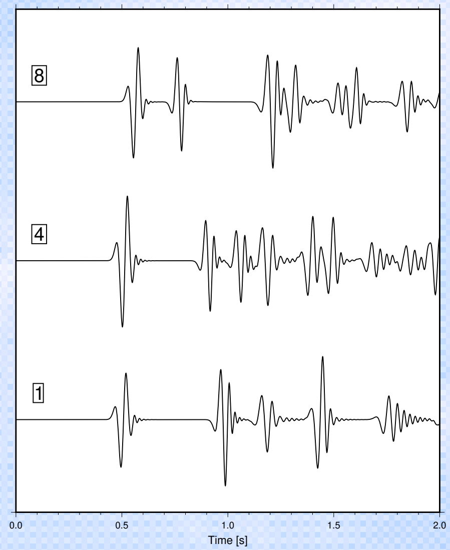


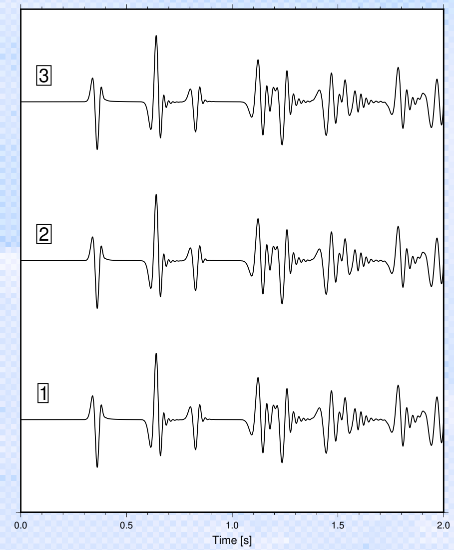
TRM and numerical simulations (K. Waskiewicz)











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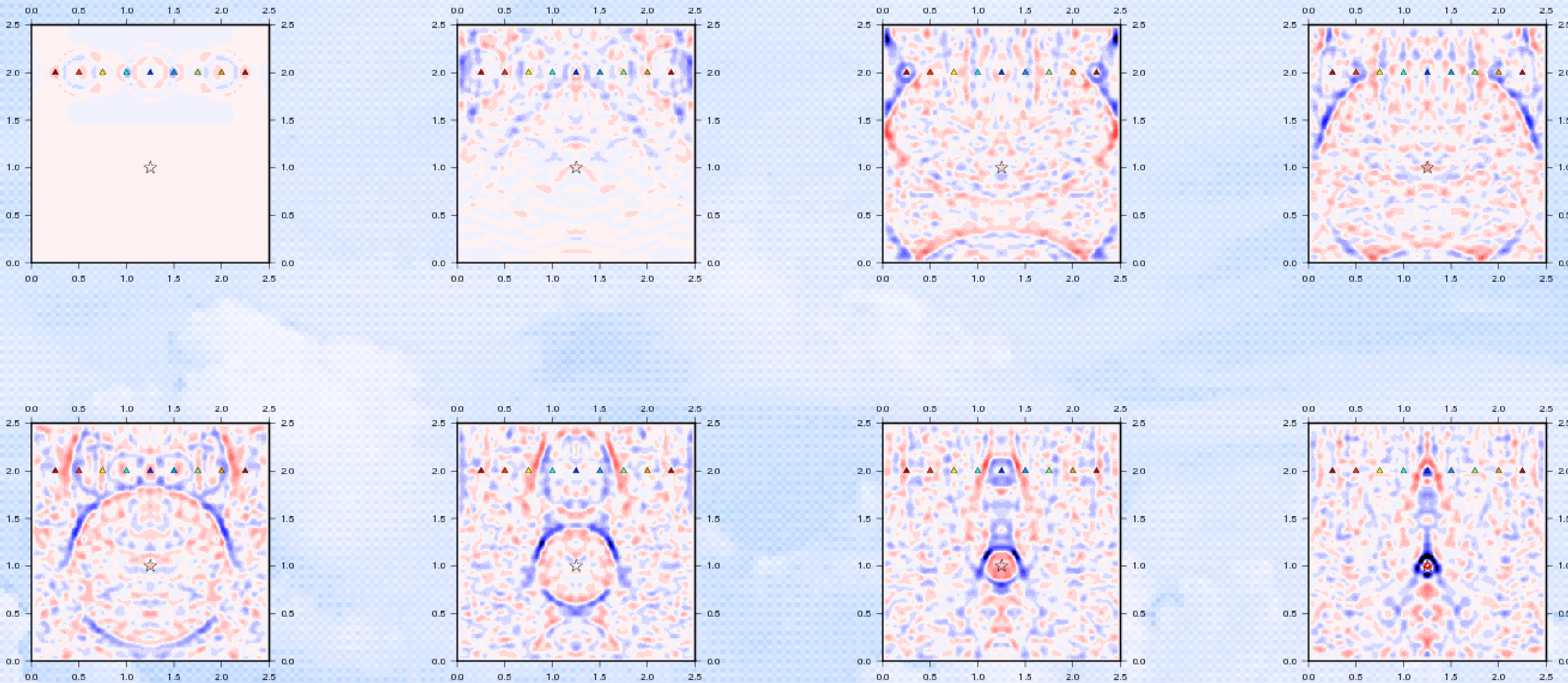

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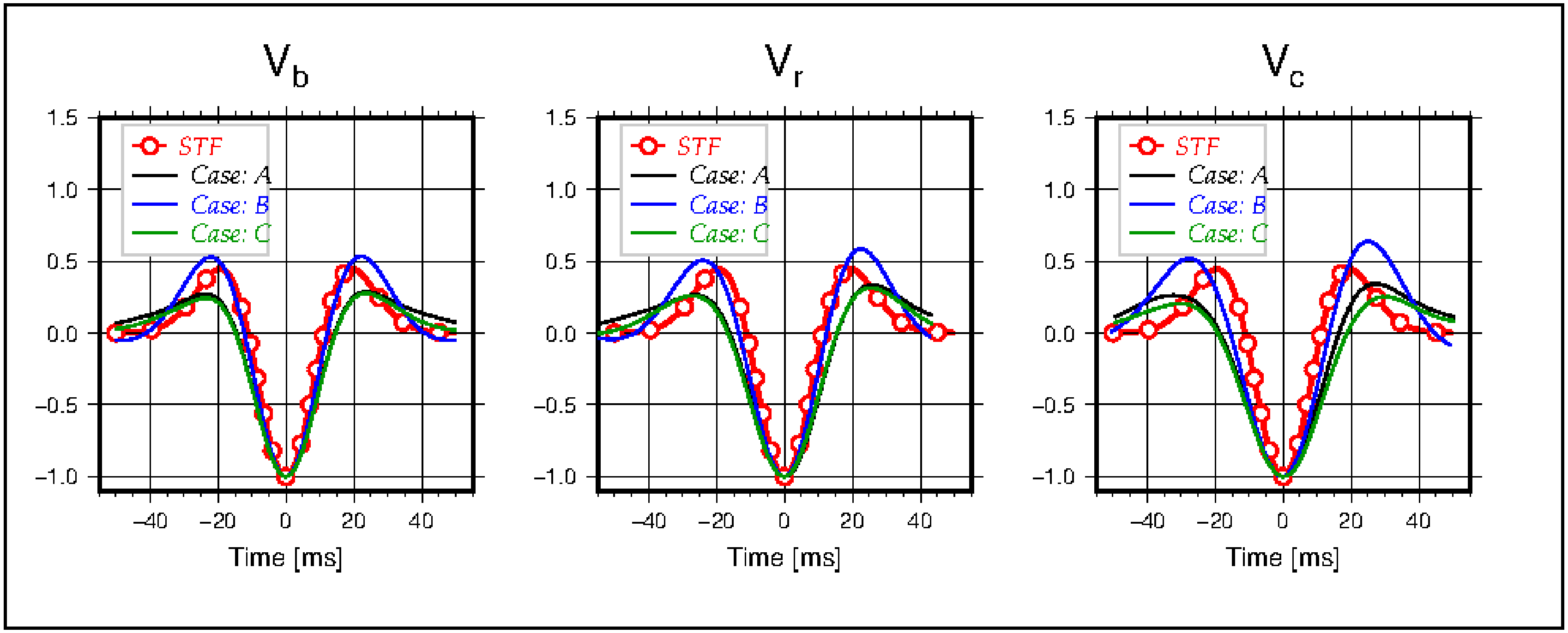

Thank you