Time Reversal Method

from mathematical principles to waveform analysis applications

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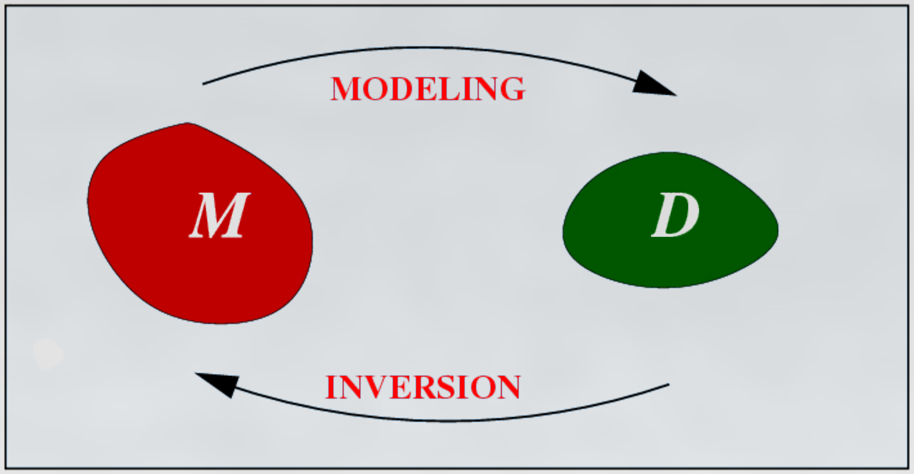
Plan of the talk

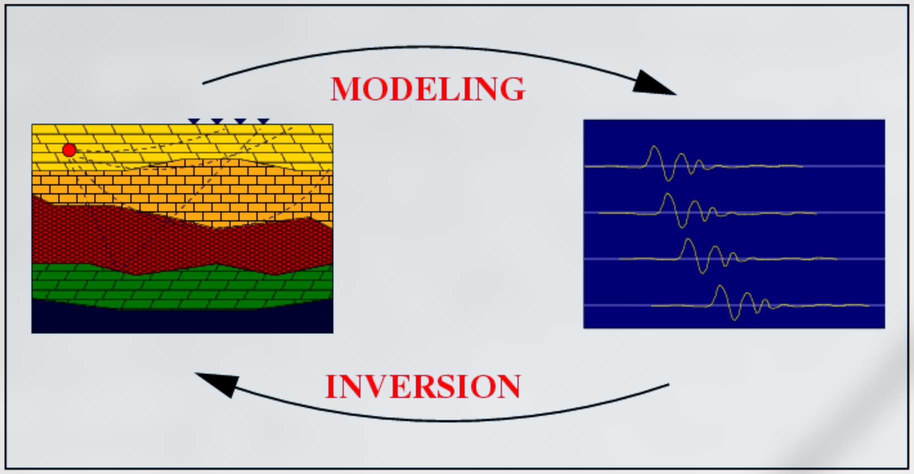
- When we cannot measure ....
- TRM - mathematical introduction
- TRM - numerical point of view
- Applications
  - Retrieving energy release pattern
  - Source localization
  - Tomography

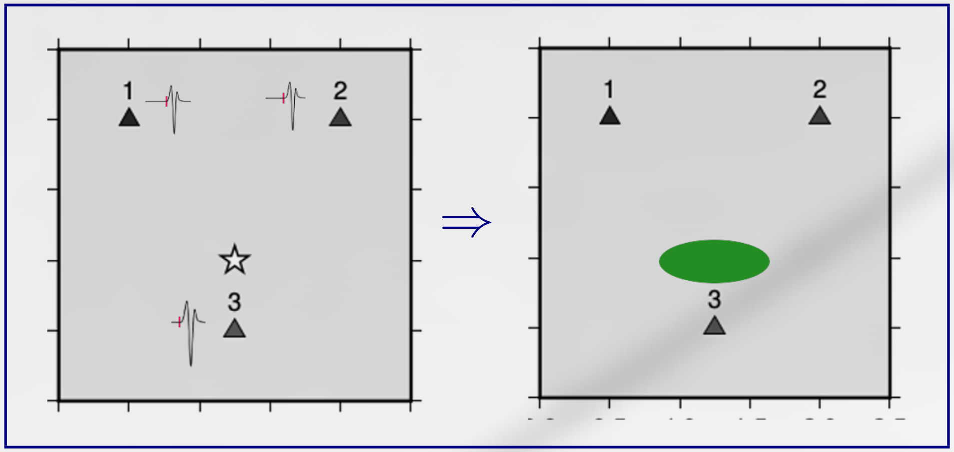
Instead of introduction

The method I will talk about has attracted my attention around 1995, while discussing with Albert Tarantola the spectacular acoustic experiment of M. Fink. Fink has proved experimentally that the mathematically predicted time reversal symmetry of acoustic equation is a reality. The point of my interest was (and still is) how the method can be used for inferring unknown properties of a media and sources (earthquakes) of waves.

Inverse theory





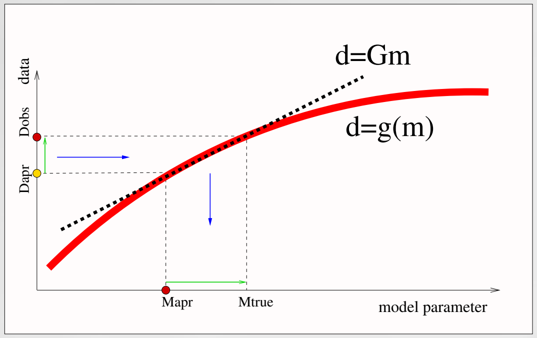


Back projection

$$\mathbf{m} = (m_1, m_2, \cdots m_M) \in \mathcal{M}$$

$$\mathbf{d} = (d_1, d_2, \cdots d_N) \in \mathcal{D}$$

$$\mathbf{d}(\mathbf{m}) = \mathbf{G} \cdot \mathbf{m}$$

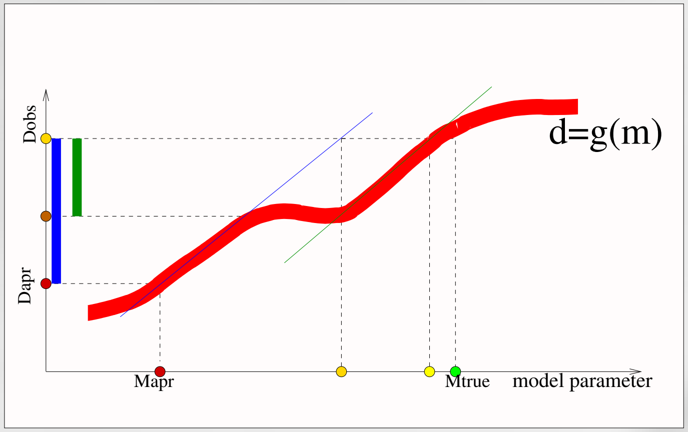


$$\mathbf{d} = \mathbf{d}^{obs} \quad \Rightarrow \mathbf{m} = ?$$

$$\mathbf{m}^{est} = (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \mathbf{d}^{obs}$$

Optimization inversion

$$\mathbf{d}(\mathbf{m}) = \mathbf{G}(\mathbf{m})$$



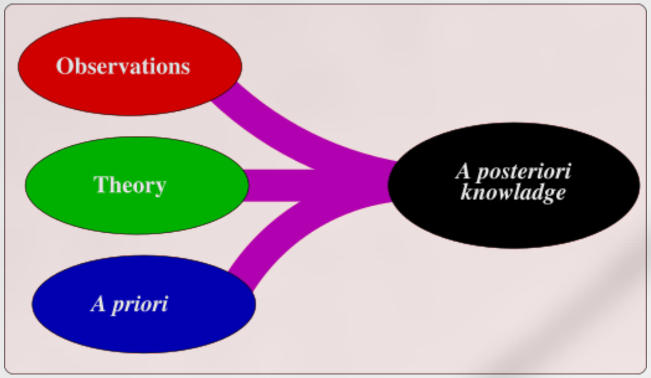
Search for $\mathbf{m}^{est} \in \mathcal{M}$:

$$||\mathbf{d}^{obs} - \mathbf{G}(\mathbf{m}^{est})|| = \min$$

Probabilistic (Bayesian)

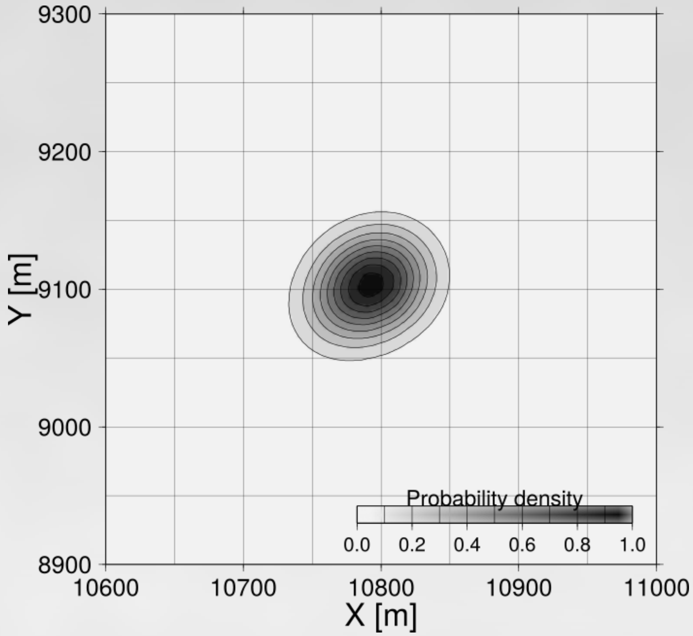
$$(\mathbf{m},\mathbf{d}) \in \mathcal{M} \times \mathcal{D} \quad \Rightarrow \quad \sigma(\mathbf{m},\mathbf{d})$$

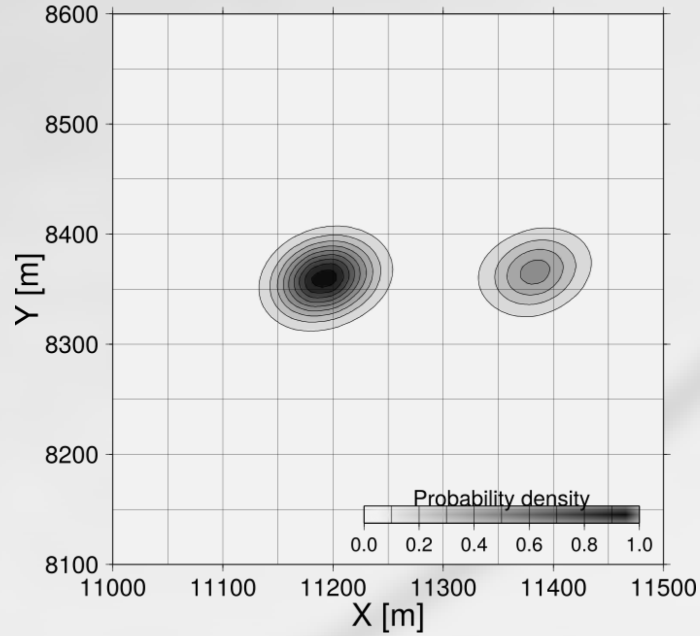
$$\sigma(\mathbf{m},\mathbf{d}) \;=\; \frac{p(\mathbf{m},\mathbf{d})\, q(\mathbf{m},\mathbf{d})\, f(\mathbf{m},\mathbf{d})}{\mu^2(\mathbf{m},\mathbf{d})}$$



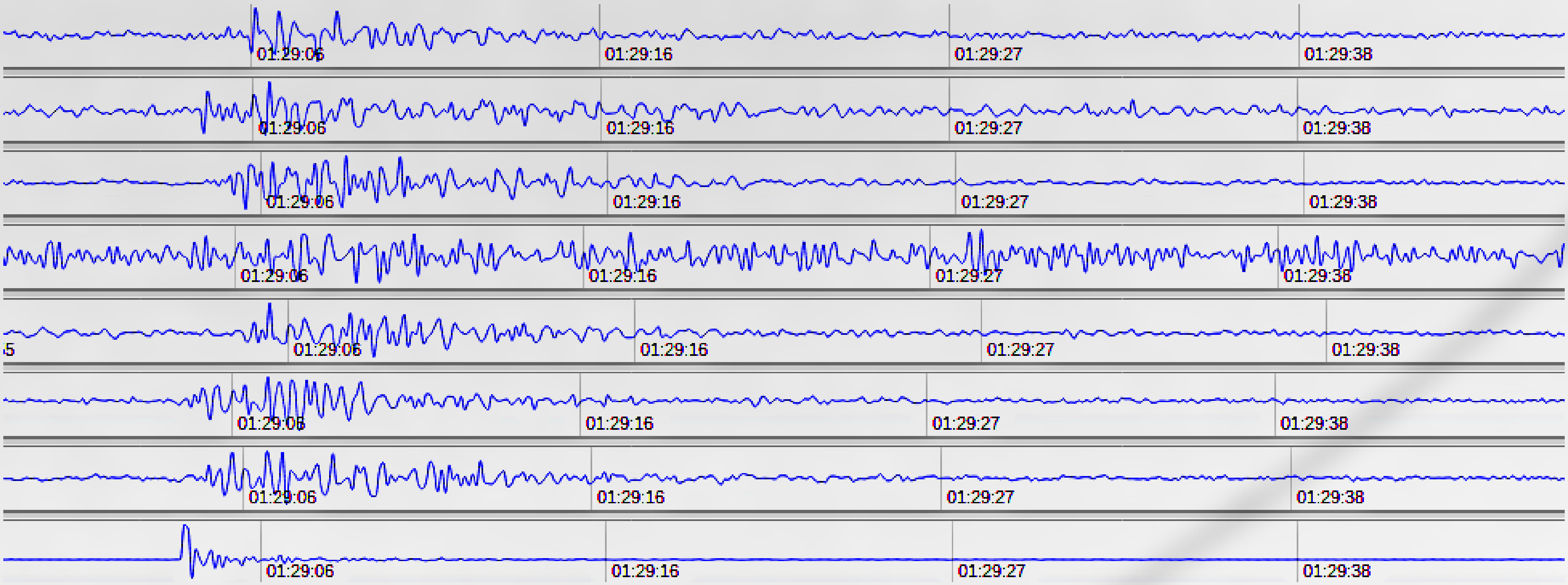
$$\sigma_m(\mathbf{m}) = f(\mathbf{m}) \int_D p(\mathbf{d},\mathbf{d}^{obs}) \frac{q(\mathbf{m},\mathbf{d})}{\mu(\mathbf{m},\mathbf{d})} \mathrm{d}\mathbf{d}$$

Probabilistic inversion - example





Acoustic/elastic waves -time series data



Rudna copper mine, 2013-11-13, Credit: IS-EPOS PLATFORM

Acoustic waves

$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)}_{\Box} u(x,t) = S(x,t)$$

non-uniqness

$$\Box P = S$$

$$\Box Q = 0$$

$$\Box (P+Q) = S$$

initial condition

$$u(x, t=0) = f(x);$$

$$\dot{u}(x, t=0) = g(x);$$

Radiation problem (Sommerfeld condition)

Assumption about $S(x,t)$

- finite size source (campact support): $[S|x \in V_o]$
- finite duration source: $[S|t \in [0,T]\,]$
- finite energy

$$\int\limits_0^T \mathrm{dt} \int\limits_{V_o} \mathrm{dx}\, |S(x,t)|^2 < \infty$$

"Zero" initial condition: $f(x) = g(x) = 0$

Green's function

$$\frac{1}{c^2}\frac{\partial^2 G(x,t)}{\partial t^2} - \Delta G(x,t) = \delta(x - x_o, t - t_o)$$

Solution:

$$u(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x' \, G(t,x;t',x') S(x',t')$$

$G(x,t;,x',t')$ "propagates" information from source (x') to receiver (x)

Green's function - symmetries

Energy conservation

(time translational invariance - Noether's theorem)

$$G(x,t;x',t') = G(x,x';t-t')$$

Homogeneous medium

(translational invariance)

$$G(x,t;x',t') = G(x-x';t-t')$$

Reciprocity

$$G(\color{red}{x,t}\color{black};x',t') = G(x',t';\color{red}{x,t}\color{black})$$

Green's function - initial condition

Casual: $u(x,t) = 0$ for $t < t'$

Anti-casual: $u(x,t) = 0$ for $t > t'$

Feynman ....

Retarded and advanced Green's functions

Retarded (casual) GF:

$$G^+(x - x', t - t') = \frac{1}{4\pi ||x - x'||} \delta \left(t - t' - \frac{||x - x'||}{c} \right) \Theta(t - t')$$

Advanced (anti-casual) GF:

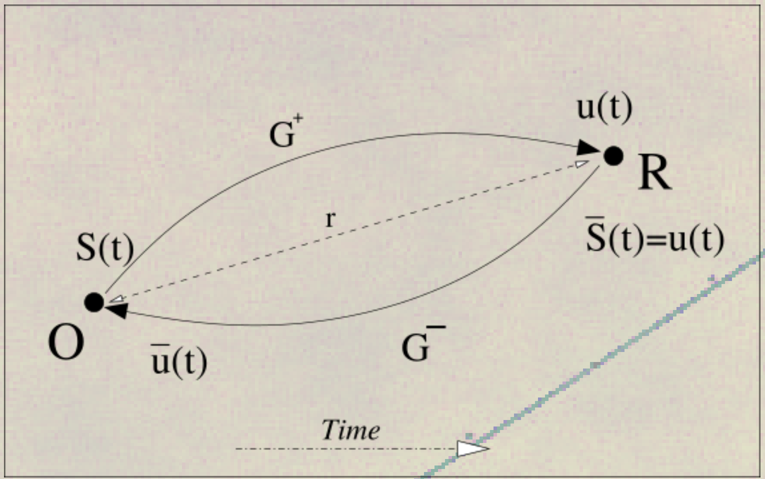
$$G^-(x - x', t - t') = \frac{1}{4\pi ||x - x'||} \delta \left(t - t' + \frac{||x - x'||}{c} \right) \Theta(t' - t)$$

$$G^+(x - x', t - t') = G^-(x - x', t' - t)$$

Time Reversal symmetry

$$G^{+}(x-x', t-t') = G^{-}(x-x', t'-t)$$

$$\Box u(x,t) = S(t) \implies \Box u(x,-t) = S(t)$$



Radiation from point source (O)

$$S(x,t) = S(t)\delta(x - x_o)$$

Solution at given receiver (R)

$$u(x_R,t) = \int_O^T G^+(x_R - x_o, t - t',)S(t')\mathrm{d}t'$$

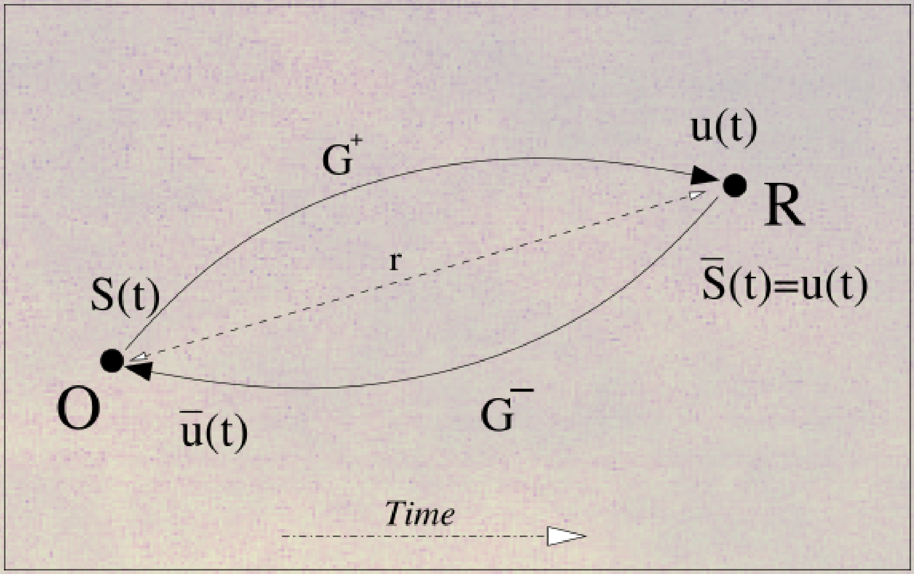
Anti-casual propagation from R to O

$$\bar{u}(x_o, t) = \int_0^T G^{-}(x_o - x_R, t - t',)\bar{S}(t')\mathrm{d}t'$$

From physical point of view it corresponds to recording at the point O incoming waves, for example, “refracted” at the point R

$$\bar{S}(t) = u_R(t)$$

Time Reversal Method (TRM)



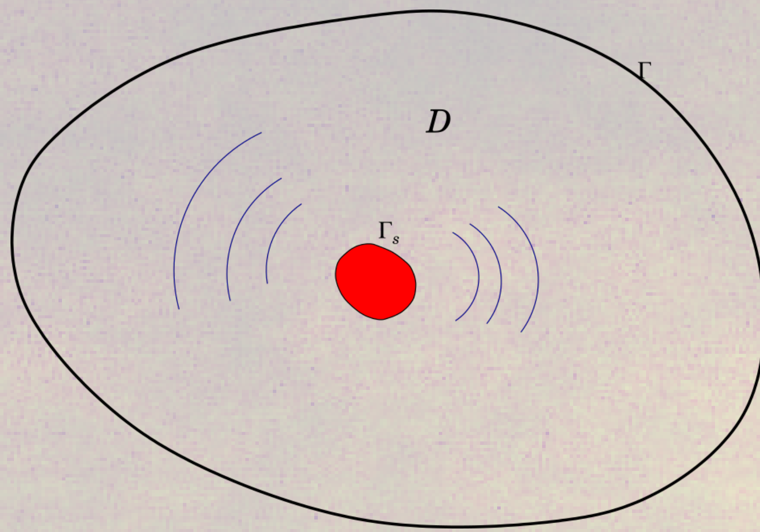
1. $u(r,t) = \int G^{+}(r, t-t')S(t')\mathrm{d}t'$
2. $\bar{S} = u(r,t)$
3. $\bar{u}(t) = \int G^{-}(r, t-t')\bar{S}(t')\mathrm{d}t'$

$\bar{\mathbf{u}}(\mathbf{r},\mathbf{t}) = \kappa\, \mathbf{S}(\mathbf{t})$

However: $G^{+}(\cdot, t-t') = G^{-}(\cdot, -(t-t'))$

$$S(t) = \int G^{+}(t-t', r)u(-t')\mathrm{d}t'$$

Time Reversal limitation - finite size source



$$u_D(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x' \; G^+(t,x;t',x')S(x',t')$$

$$u_D(x,t) = \int_{-\infty}^{\infty} dt' \int_{\partial\Gamma} ds' \Big(u(x',-t')\frac{\partial}{\partial n'}G^+(x,t;x',t') - G^+(x,t;x',t')\frac{\partial}{\partial n'}u(x',-t')\Big).$$

TRM as the inverse problem

$$\mathbf{m} = S(t) \qquad\qquad \mathbf{d} = u(t)$$

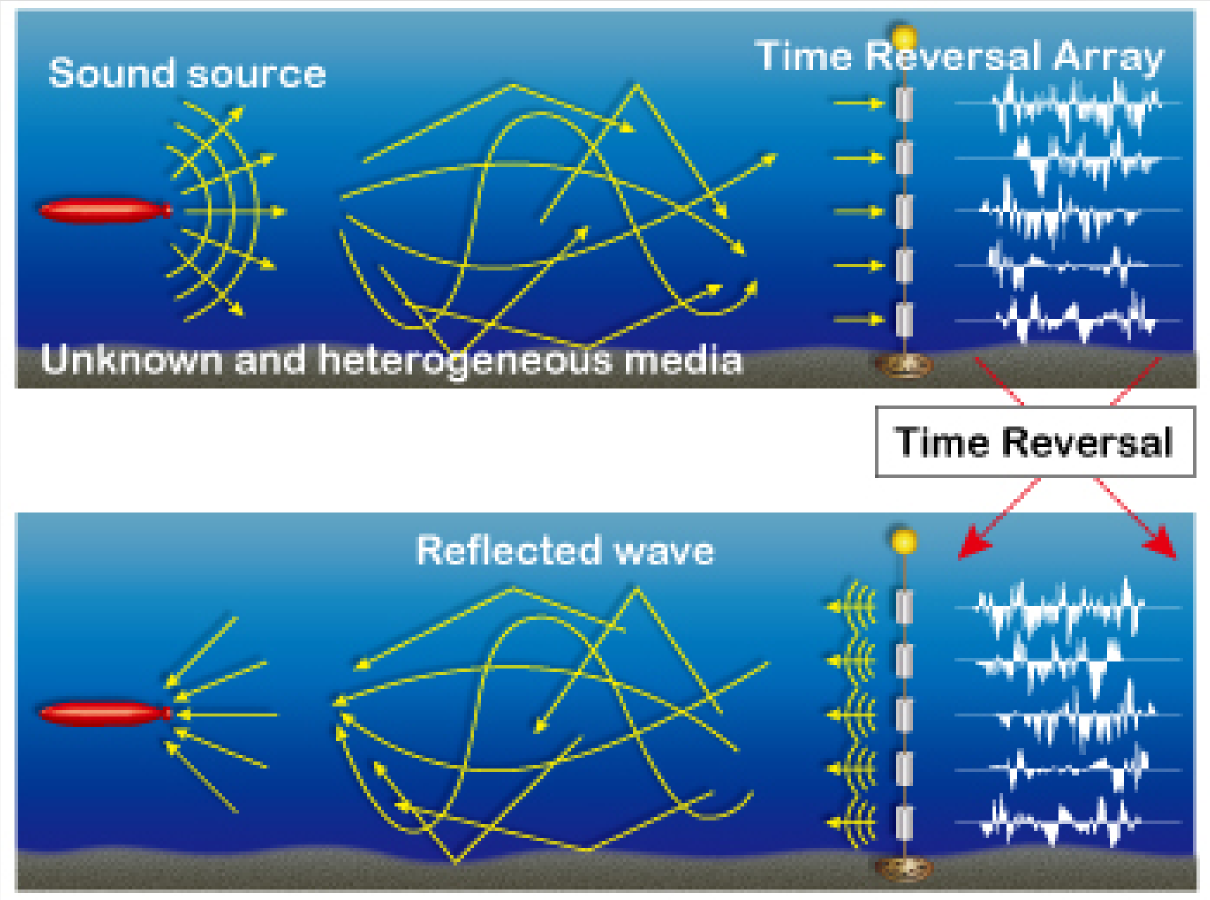
$$u(r,t) = \int G^{+}(r, t-t')\, S(t')\mathrm{d}t' \quad \Rightarrow \quad \mathbf{d} = \mathcal{G}\mathbf{m}$$

$$S(t) = \int G^{+}(r, t-t')\, u(-t')\mathrm{d}t' \quad \Rightarrow \quad \mathbf{m} = \mathcal{G}^{-1}\mathbf{d}$$

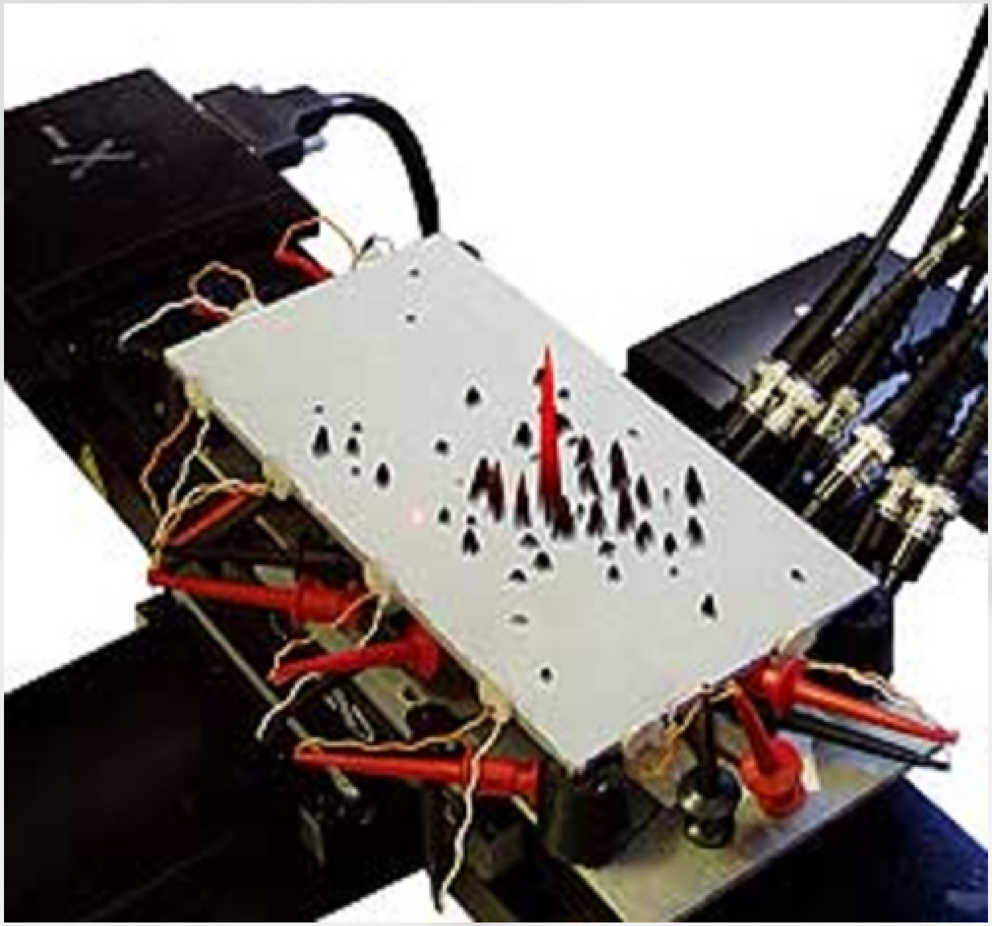
The method is computationally very efficient, but...

we need to know the medium: $G^{+}(\cdot,\cdot)$

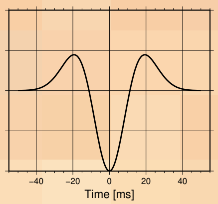
Physics of Time Reversal

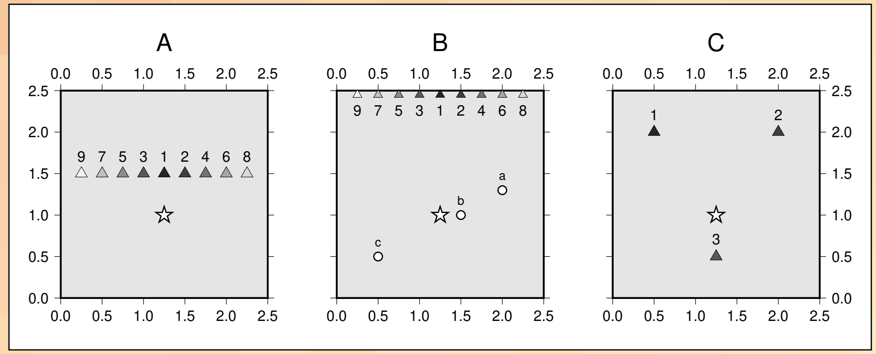


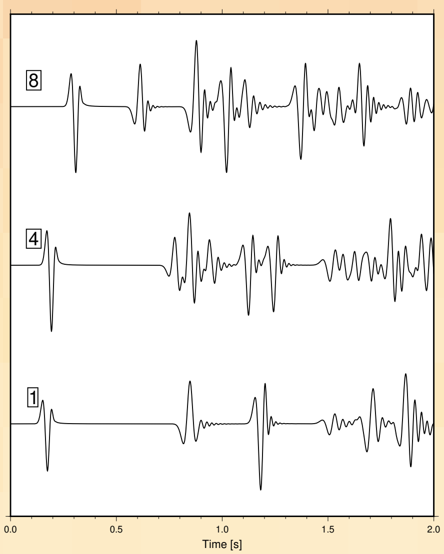
Experimental evidence (M. Fink)

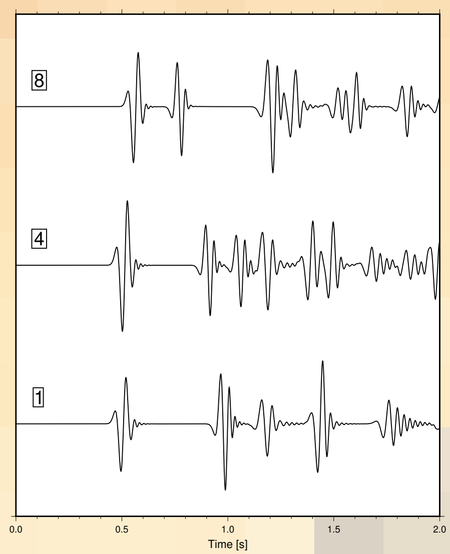


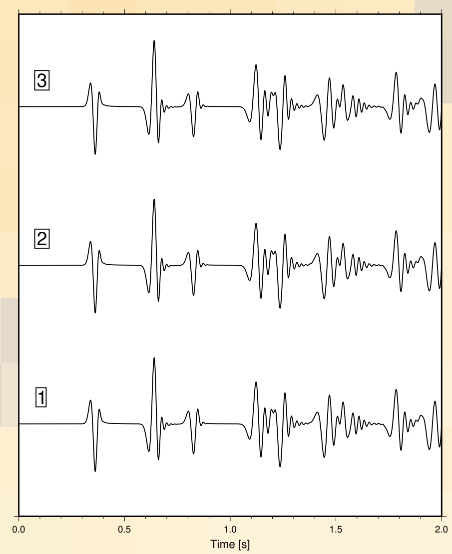
TRM and numerical simulations (I)



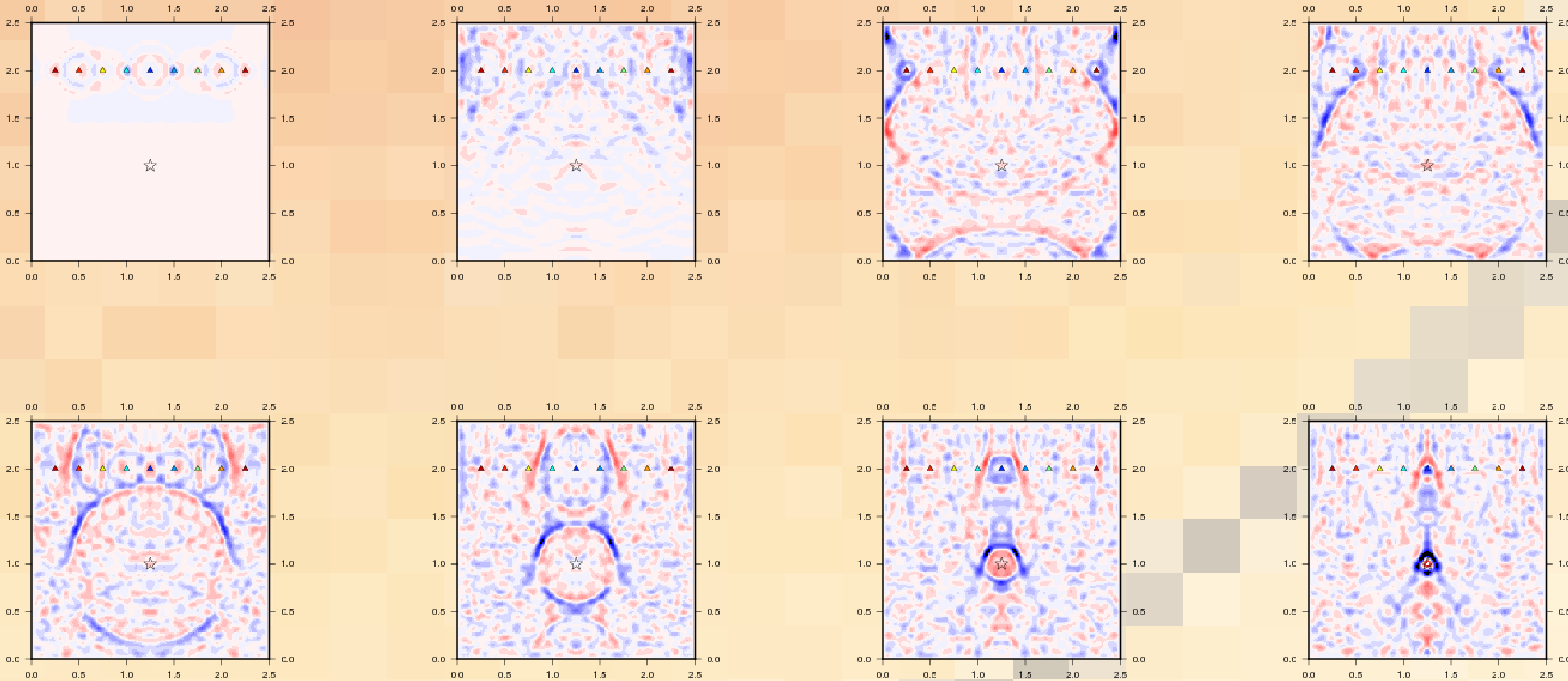




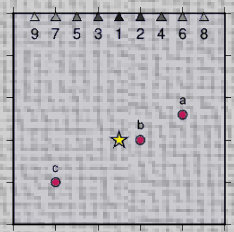


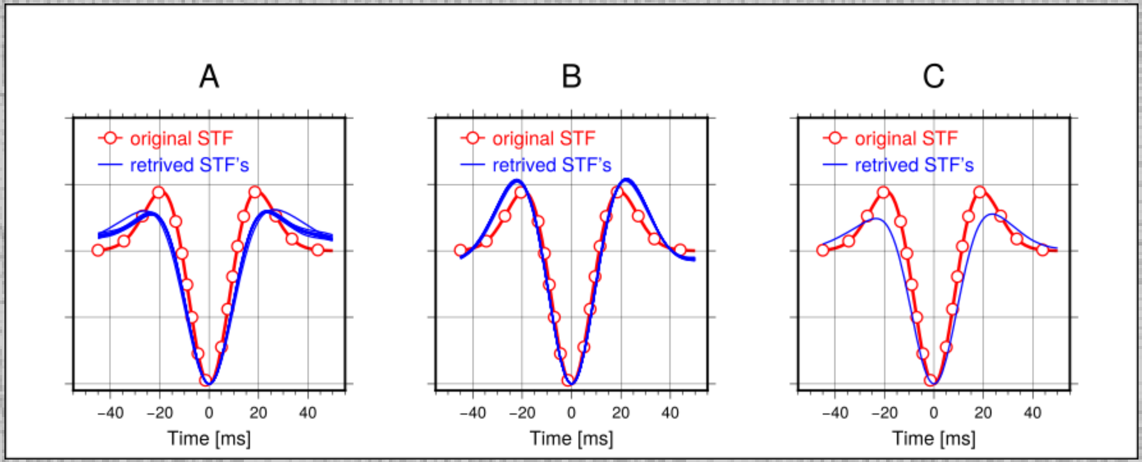


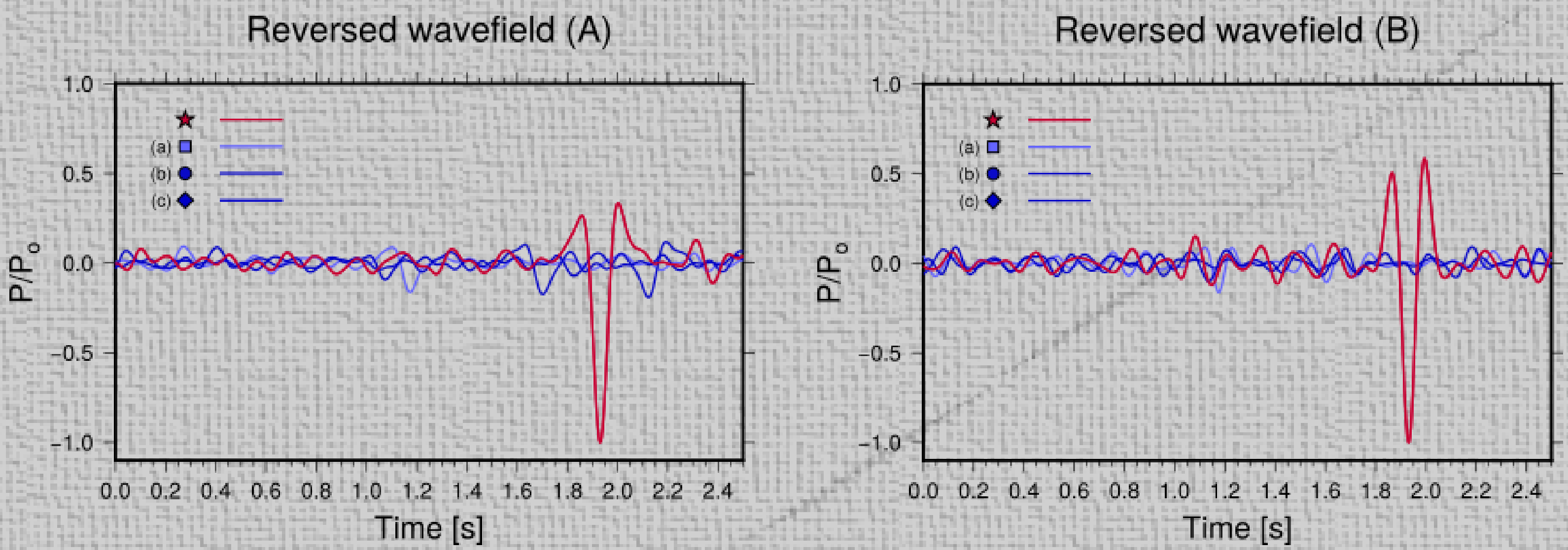
TRM and numerical simulations (II)



TRM: energy release pattern





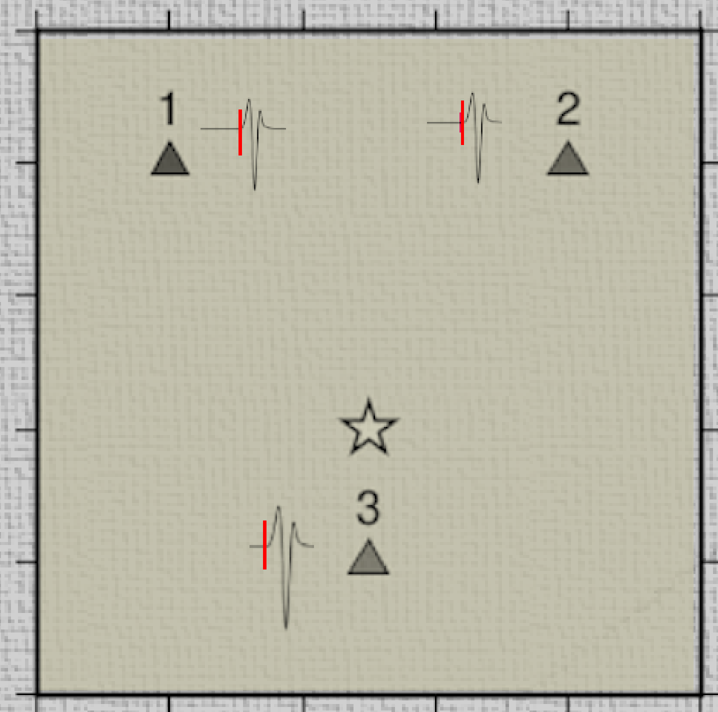


Source location - classical approach

$$\mathbf{m} = (x, y, z)$$

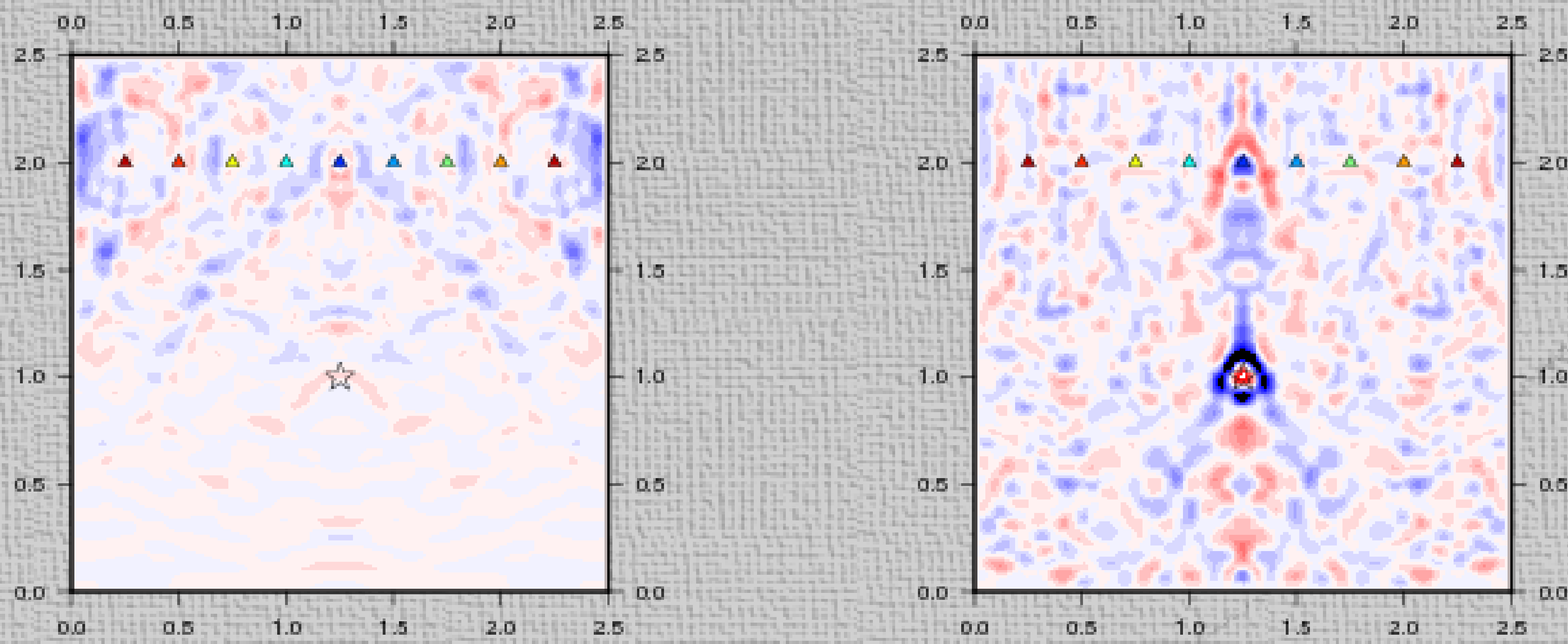
$$\mathbf{d} = (t_1^o, t_2^o, \ldots)$$

$$t_i(\mathbf{m}) = t_o + \int_{SR} \frac{dl}{v}$$



$$\left\{ (m_s,\ t_o) : \sum_i \left(t_i^o - t_i(\mathbf{m})\right)^2 = \min \right\}$$

TRM: source location



Solution: ***search for the point of maximum coherence of back-projected waves***

Iterative procedure (minimization $S(\mathbf{V})$)

$$\mathbf{v^0} = \mathbf{v^{apr}}$$

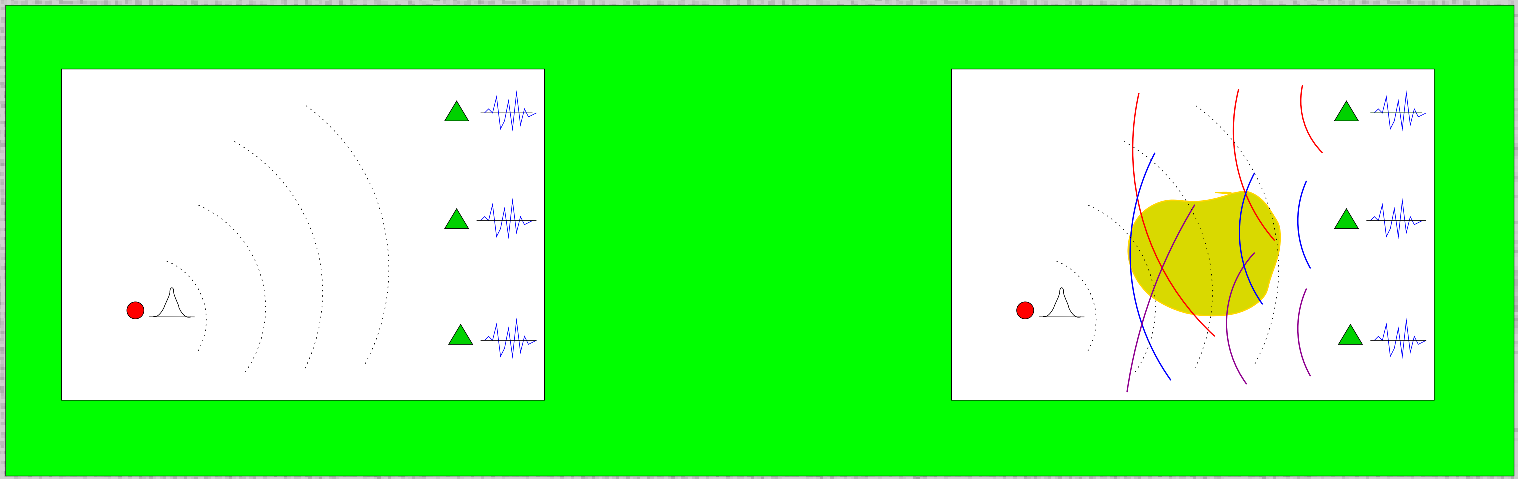
$$S(\mathbf{v}) \;=\; \int_t ||(d^{obs}(t) - d^{\mathbf{v}}(t))||$$

$$\nabla \mathbf{S} = \frac{\delta S}{\delta \mathbf{v}} \;\Rightarrow\; \delta \mathbf{v} = \hat{H}\, \nabla S$$

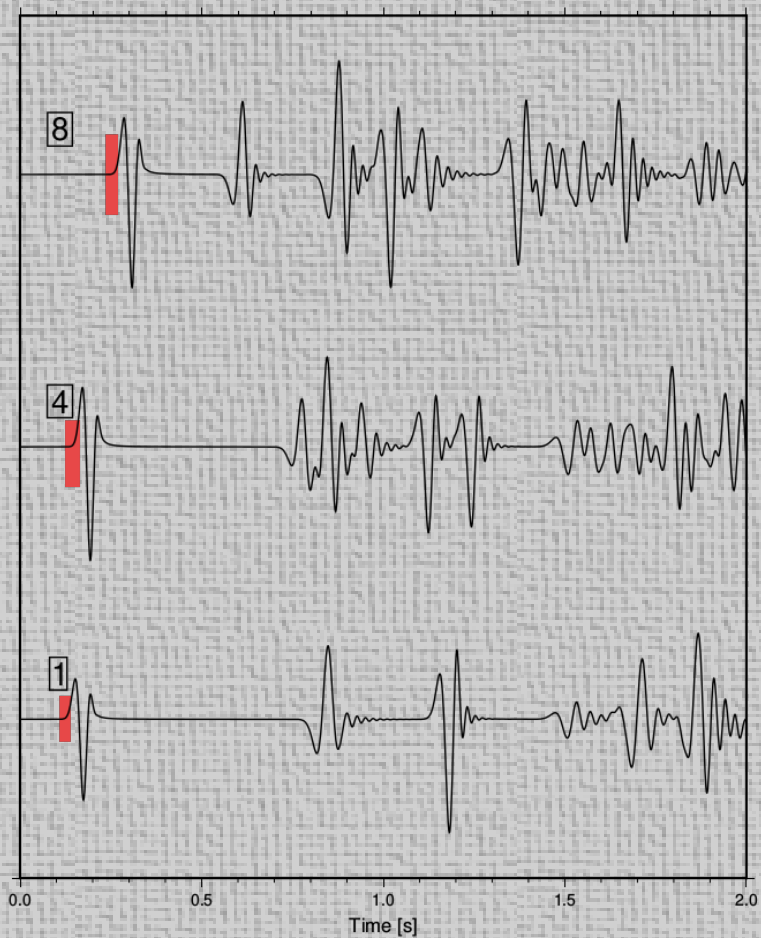
$$\mathbf{v^{i+1}} = \mathbf{v^i} + \alpha\, \delta \mathbf{v}$$

TRM: gradient calculation

$$\nabla \mathbf{S} = \int \Big\{ \underbrace{\left(G^{+}(t) * S\right)}_{\text{direct wave}} * \underbrace{\left(G^{-}(t) * (d^{obs} - d^{syn})\right)}_{\text{back propagated residua}} \Big\} \mathrm{d}t$$



Full waveform inversion



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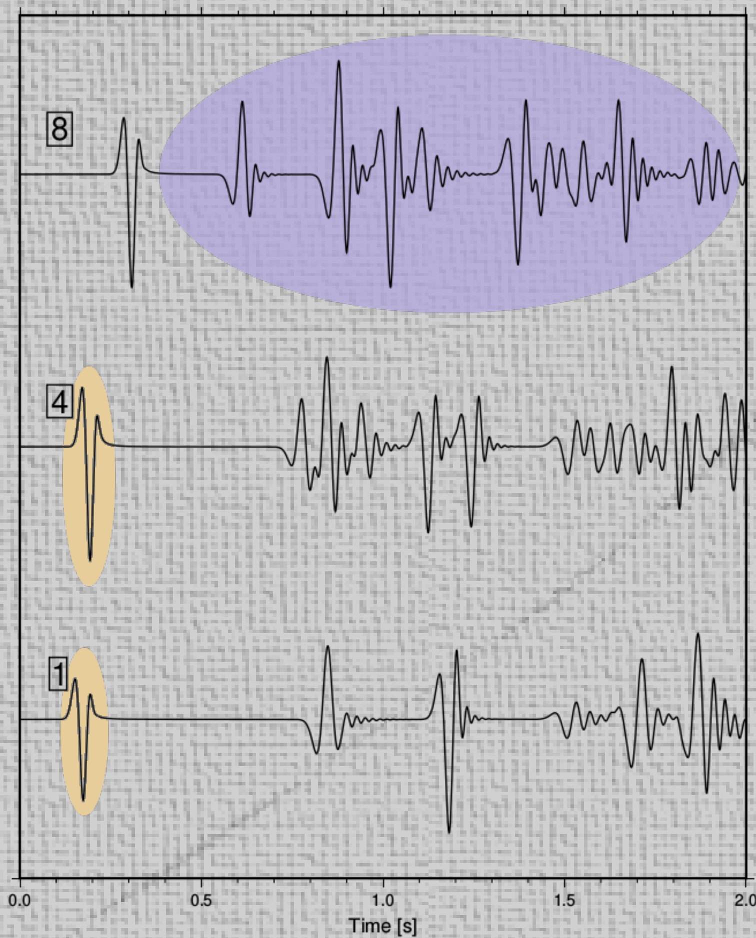

TRM - source of information

- arrival times
- secondary phases
- amplitudes of direct waves

More information → higher S/N ratio → better resolution

- only a few forward modelling required
- necessary search for maximum coherence point

Instead of conclusions

- TRM requires

  knowledge of G^+ (medium)

  or

  knowledge of source

- works in Euclidean (AW) and Minkovsky (EM) spaces

- can be used in curved spaces ???

- what if time is a dynamical parameter (g_{ij}) ???

Thank you for your attention

Prepared: January 17, 2019