Time Reversal

an efficient way of data analysis

W. Dębski
Inst. Geophys. PAS

Plan of the talk

- Basic physics (observation)
- Mathematical introduction
- Numerical point of view
- Applications (at least geophysical)
  - Source localization
  - Retrieving energy release pattern
  - Tomography

Motivation

The method I will talk about has attracted my attention around 1995, while discussing with Albert Tarantola the spectacular acoustic experiment of M. Fink. He has proved experimentally that the mathematically predicted time reversal symmetry of acoustic equation is a reality. The point of my interest was (and still is) how the method can be used for inferring unknown properties of a media (Earth, rock masses) and sources (earthquakes) of waves.

Acoustic waves

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) P(x,t) = S(x,t)$$

non-uniqness

$$\Box P = S$$

$$\Box Q = 0$$

$$\Box (P + Q) = S$$

initial condition

$$P(x, t = 0) = u(x);$$

$$\dot{P}(x, t = 0) = v(x);$$

Radiation problem (Sommerfeld condition)

Assumption about $S(x, t)$

- finite size source (compact support): $[S|x \in V_o]$
- finite duration source: $[S|t \in [0, T]\,]$
- finite energy

$$\int_0^T \mathrm{dt} \int_{V_o} \mathrm{dx}\, |S(x,t)|^2 < \infty$$

"Zero" initial condition: $u(x) = v(x) = 0$

Green's function

$$\frac{1}{c^2}\frac{\partial^2 G(x,t)}{\partial t^2} - \Delta G(x,t) = \delta(x - x_o, t - t_o)$$

Solution:

$$u(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x'\, G(t,x;t',x')S(x',t')$$

$G(x,t;,x',t')$ "propagates" information from source (x') to receiver (x)

Assuming time translation invariance

(energy conservation - Noether's theorem)

$$G(x,t;x',t') = G(x,x';t-t')$$

Homogeneous medium

translational invariance

$$G(x,t;x',t') = G(x-x';t-t')$$

Retarded and advanced Green's functions

Retarded (casual) GF:

$$G^{+}(x-x',t-t') = \frac{1}{4\pi||x-x'||}\delta\left(t-t'-\frac{||x-x'||}{c}\right)\Theta(t-t')$$

Advanced (anti-casual) GF:

$$G^{-}(x-x',t-t') = \frac{1}{4\pi||x-x'||}\delta\left(t-t'+\frac{||x-x'||}{c}\right)\Theta(t'-tt)$$

$$G^{+}(x-x',t-t') = G^{-}(x-x',t'-t)$$

Radiation from point source (O)

$$S(x,t) = S(t)\delta(x - x_o)$$

Solution at given receiver (R)

$$u_R(x,t) = \int\limits_{OR} G^+(x - x_o, t - t',)S(t')\mathrm{d}t'$$

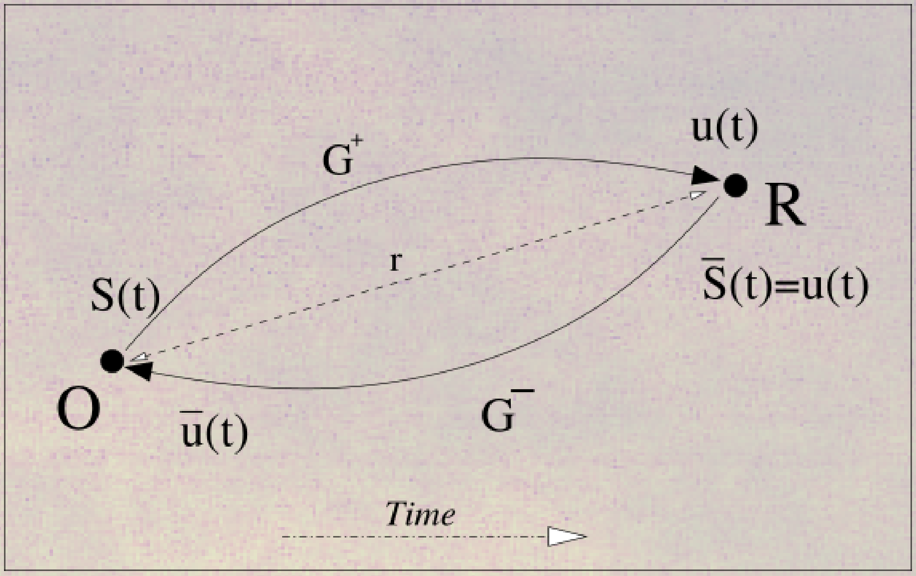
Anti-casual propagation from R to O

$$\bar{u}(x,t) = \int\limits_{RO} G^{-}(x - x_R, t - t',)\bar{S}(t')\mathrm{d}t'$$

From physical point of view it corresponds to recording at the point O incoming waves, for example, “refracted” at the point R

$$\bar{S}(t) = u_R(t)$$

Time Reversal Method (TRM) - basic principle



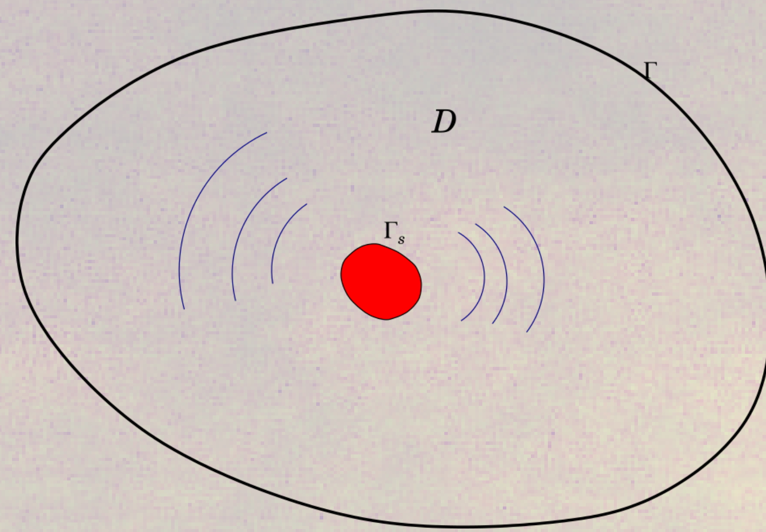
1. $u(r,t) = \int G^+(r, t-t')S(t')\mathrm{d}t'$
2. $\bar{S} = u(r,t)$
3. $\bar{u}(t) = \int G^-(r, t-t')\bar{S}(t')\mathrm{d}t'$

$\bar{\mathbf{u}}(\mathbf{r},\mathbf{t}) = \mathbf{k}(\mathrm{r})\ \mathbf{S}(\mathbf{t})$

However: $G^+(\cdot, t-t') = G^-(\cdot, -(t-t'))$

$$\boxed{S(t) = \int G^+(t-t', r)u(-t')\mathrm{d}t'}$$

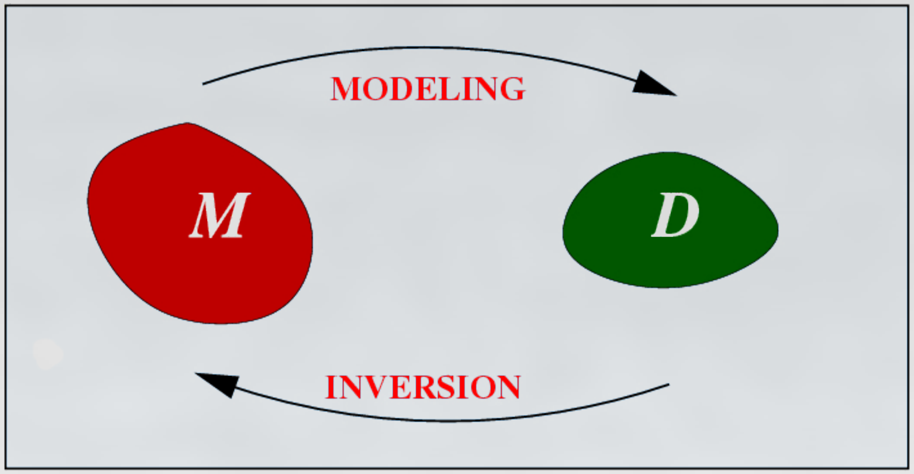
Time Reversal limitation - finite size source

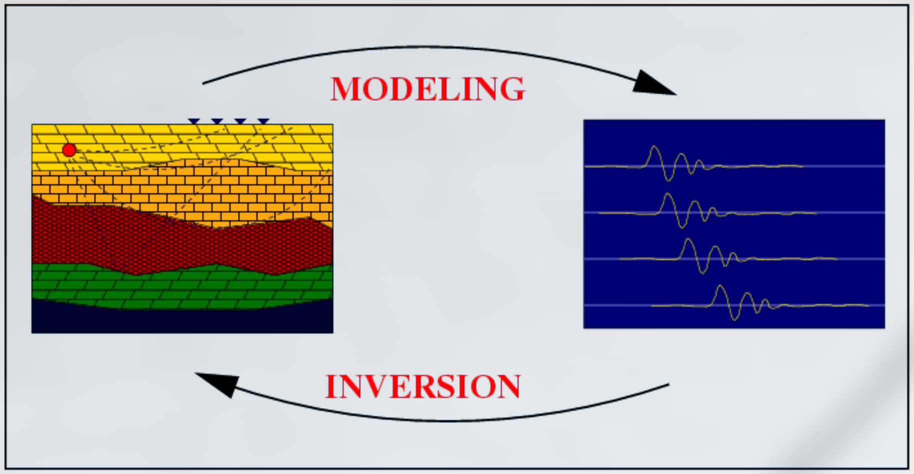


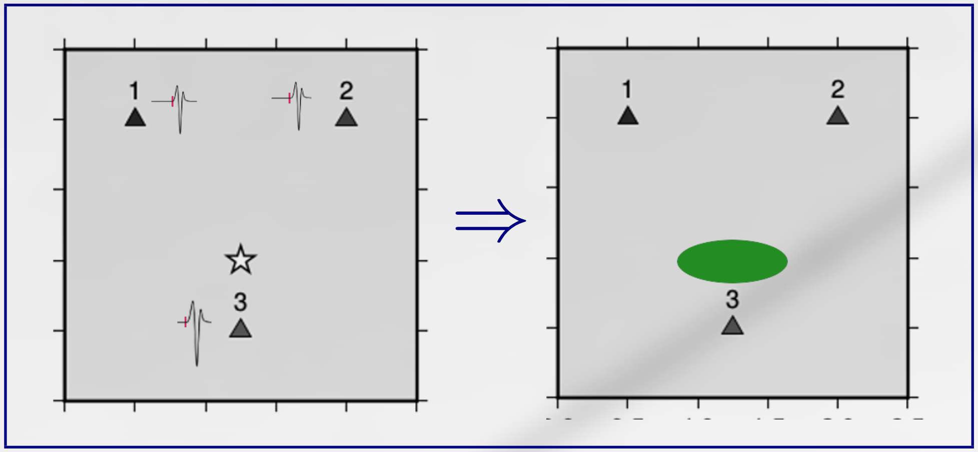
$$u_D(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x'\; G^+(t,x;t',x') S(x',t')$$

$$u_D(x,t) = \int_{-\infty}^{\infty} dt' \int_{\partial\Gamma} ds' \Big(u(x',-t') \frac{\partial}{\partial n'} G^+(x,t;x',t') - G^+(x,t;x',t') \frac{\partial}{\partial n'} u(x',-t') \Big).$$

Inverse theory





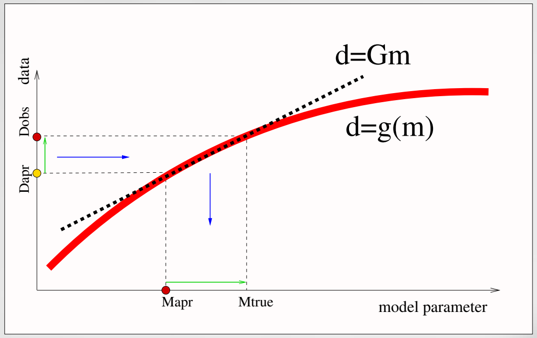


Back projection

$$\mathbf{m} = (m_1, m_2, \cdots m_M) \in \mathcal{M}$$

$$\mathbf{d} = (d_1, d_2, \cdots d_N) \in \mathcal{D}$$

$$\mathbf{d}(\mathbf{m}) = \mathbf{G} \cdot \mathbf{m}$$

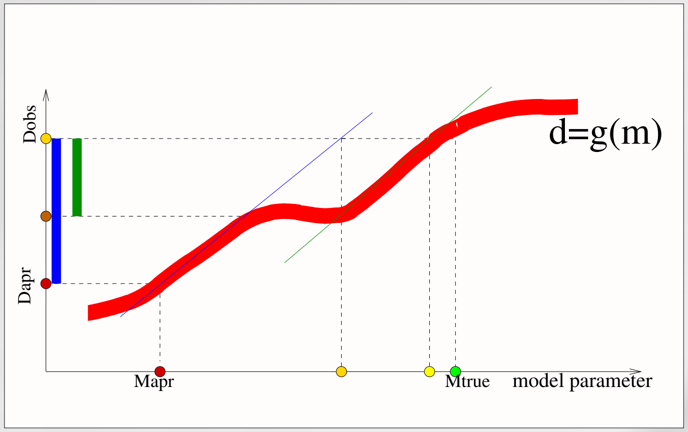


$$\mathbf{d} = \mathbf{d}^{obs} \qquad \Rightarrow \mathbf{m} = ?$$

$$\mathbf{m}^{est} = (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \mathbf{d}^{obs}$$

Optimization inversion

$$\mathbf{d}(\mathbf{m}) = \mathbf{G}(\mathbf{m})$$



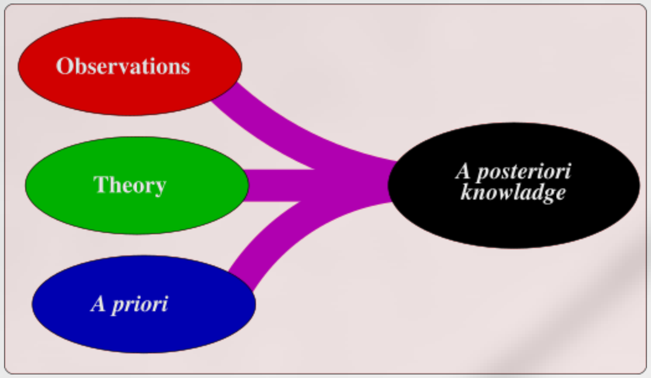
Search for $\mathbf{m}^{est} \quad \in \quad \mathcal{M}$:

$$||\mathbf{d}^{obs} - \mathbf{G}(\mathbf{m}^{est})|| = \min$$

Probabilistic (Bayesian)

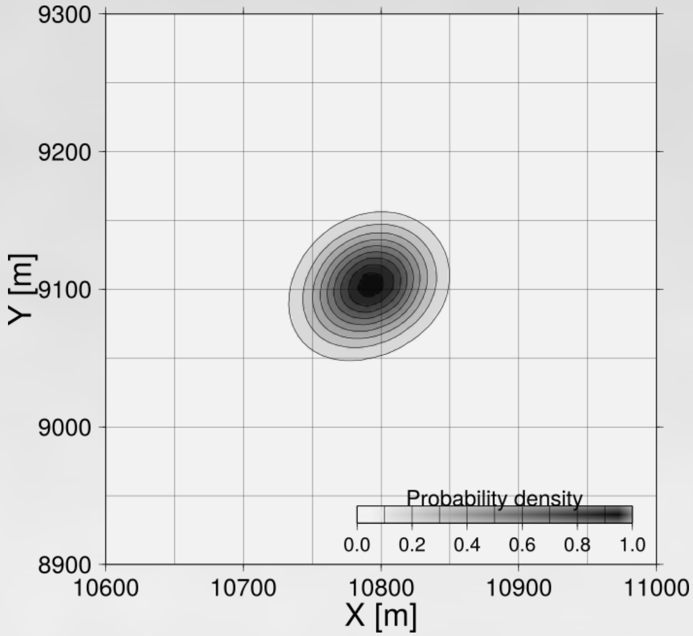
$$(\mathbf{m},\mathbf{d}) \in \mathcal{M} \times \mathcal{D} \quad \Rightarrow \quad \sigma(\mathbf{m},\mathbf{d})$$

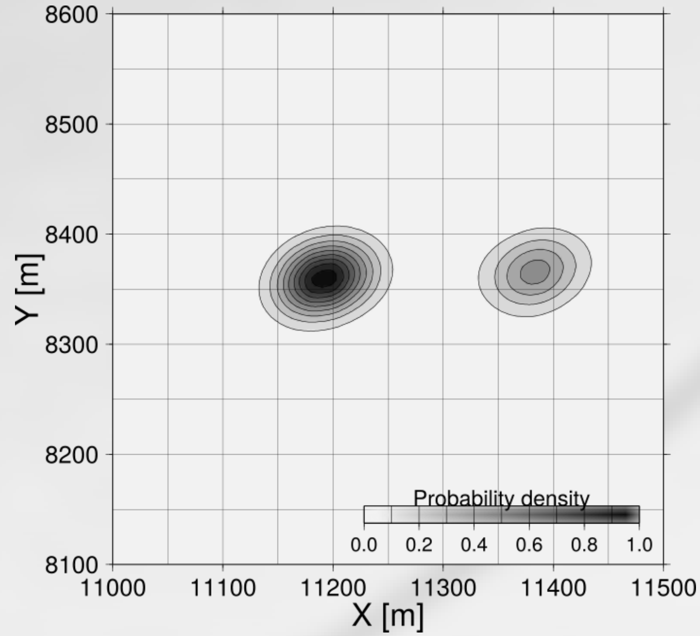
$$\sigma(\mathbf{m},\mathbf{d}) \; = \; \frac{p(\mathbf{m},\mathbf{d})\, q(\mathbf{m},\mathbf{d})\, f(\mathbf{m},\mathbf{d})}{\mu^2(\mathbf{m},\mathbf{d})}$$



$$\sigma_m(\mathbf{m}) = f(\mathbf{m}) \int_D p(\mathbf{d},\mathbf{d}^{obs}) \frac{q(\mathbf{m},\mathbf{d})}{\mu(\mathbf{m},\mathbf{d})} \mathrm{d}\mathbf{d}$$

Probabilistic inversion - example





TRM as the inverse problem

$$\mathbf{m} = S(t) \qquad\qquad \mathbf{d} = u(t)$$

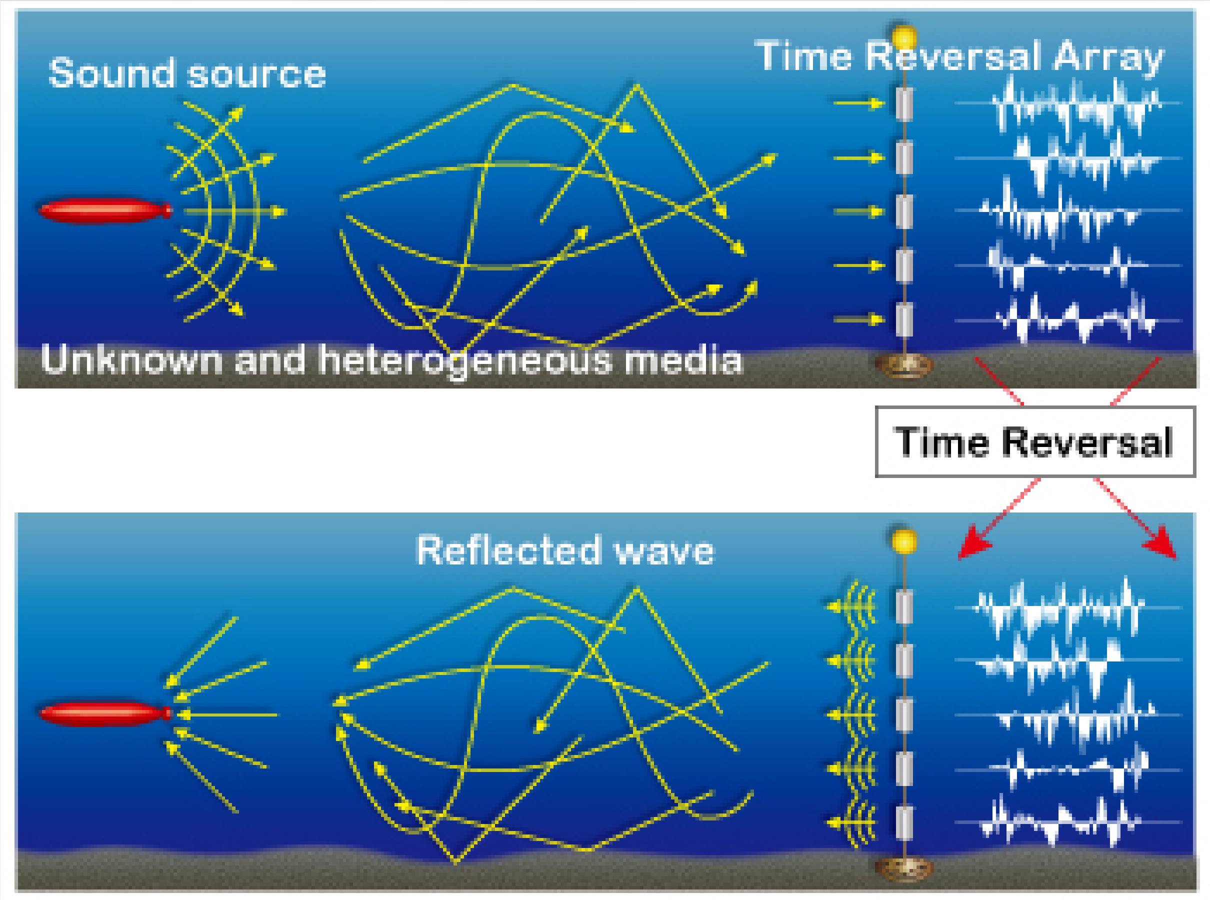
$$u(r,t) = \int G^{+}(r, t - t')\, S(t')\mathrm{d}t' \quad \Rightarrow \quad \mathbf{d} = \mathcal{G}\mathbf{m}$$

$$S(t) = \int G^{+}(r, t - t')\, u(-t')\mathrm{d}t' \quad \Rightarrow \quad \mathbf{m} = \mathcal{G}^{-1}\mathbf{d}$$

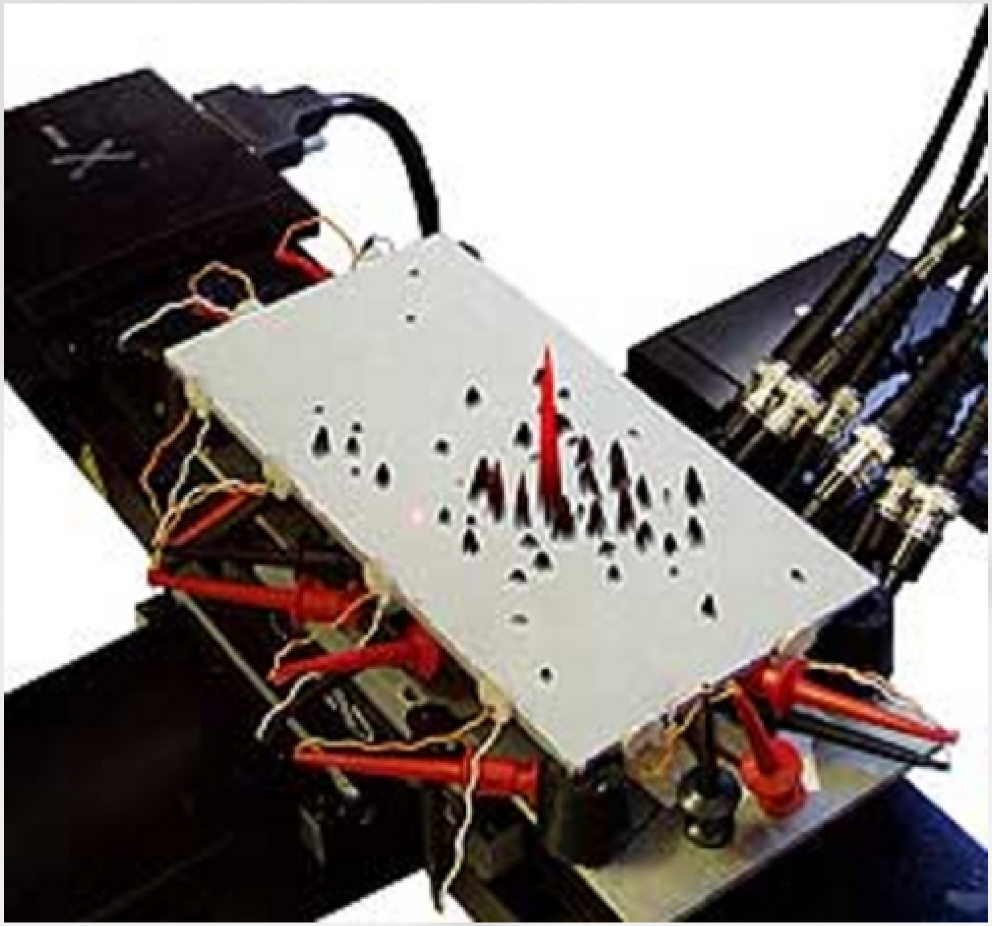
The method is computationally very efficient, but...

we need to know the medium: $G^{+}(\cdot, \cdot)$

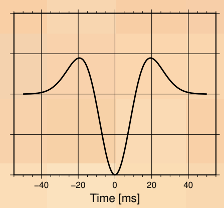
Physics of Time Reversal

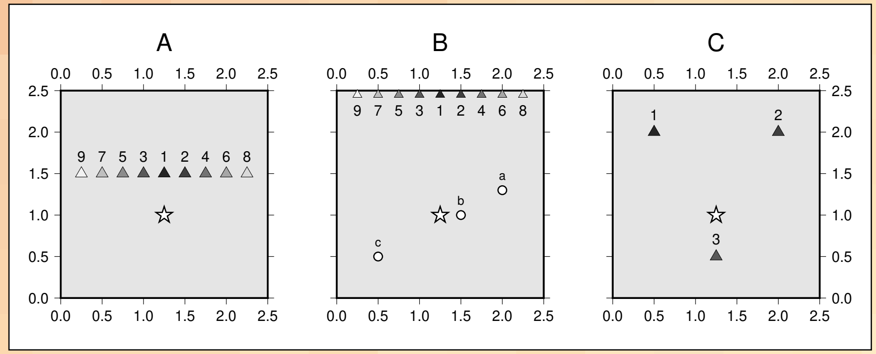


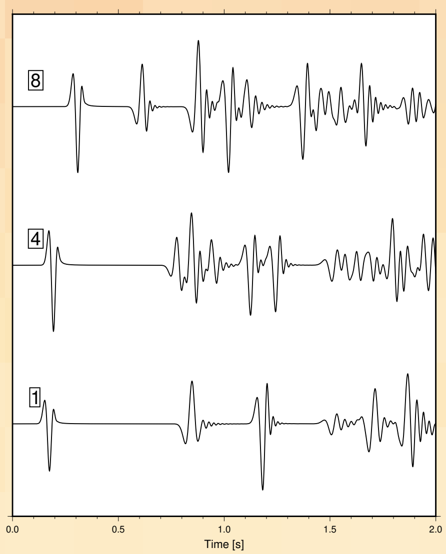
Experimental evidence (M. Fink)

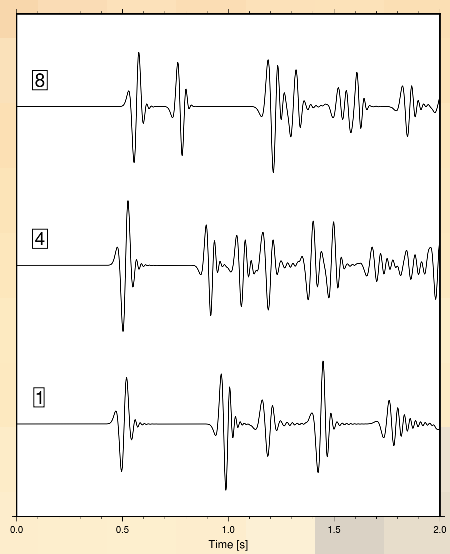


TRM and numerical simulations (I)

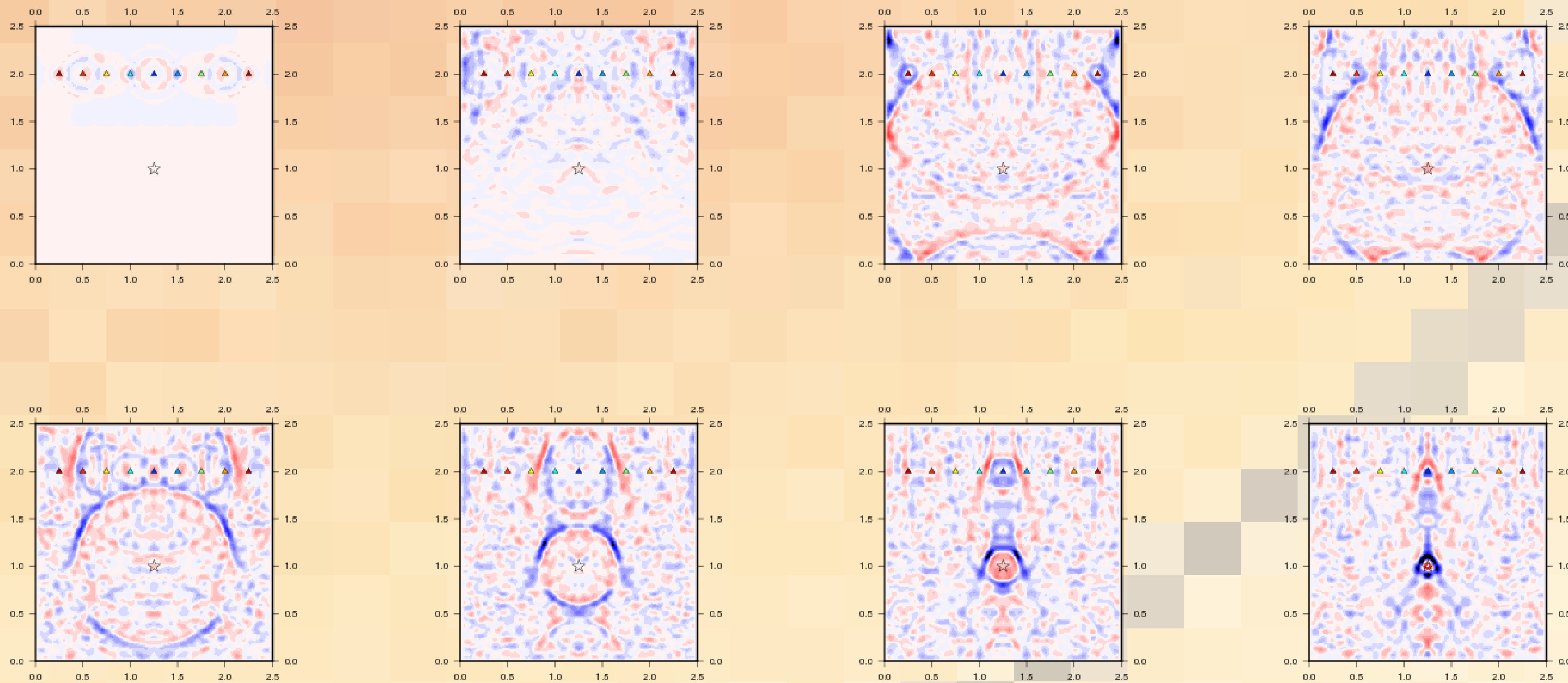




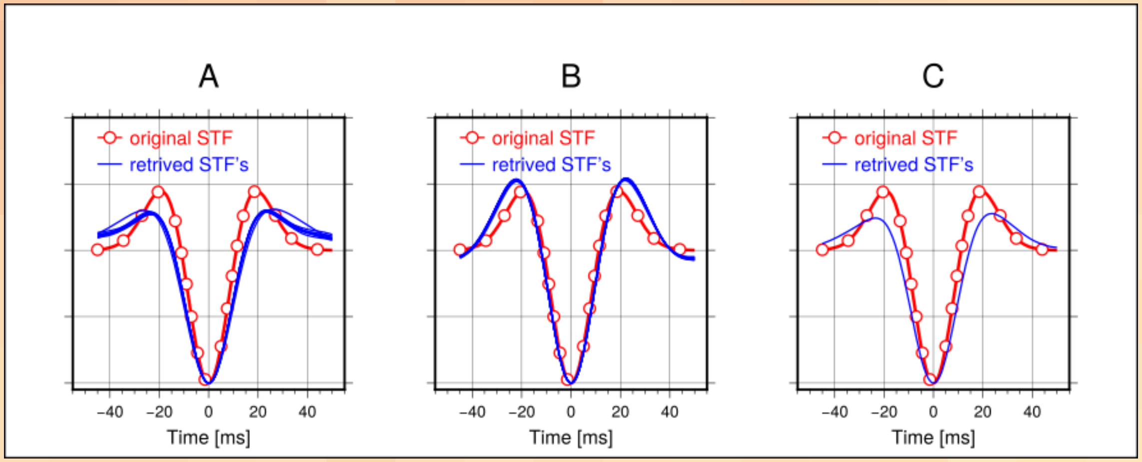


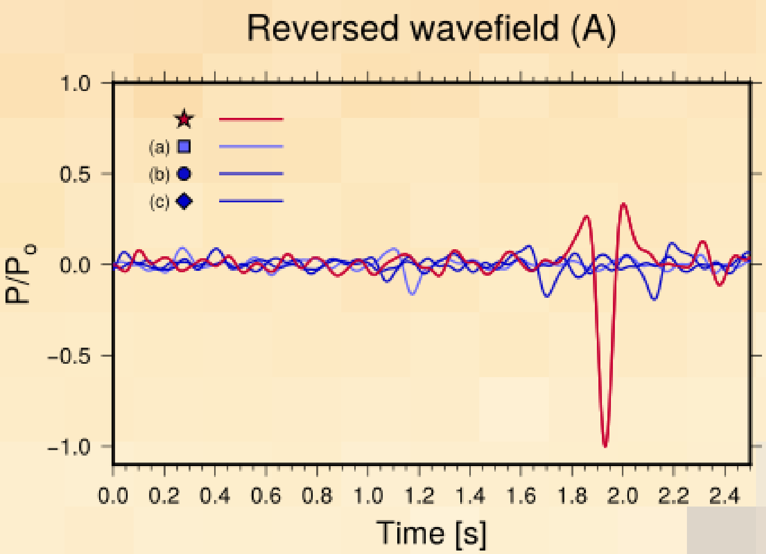


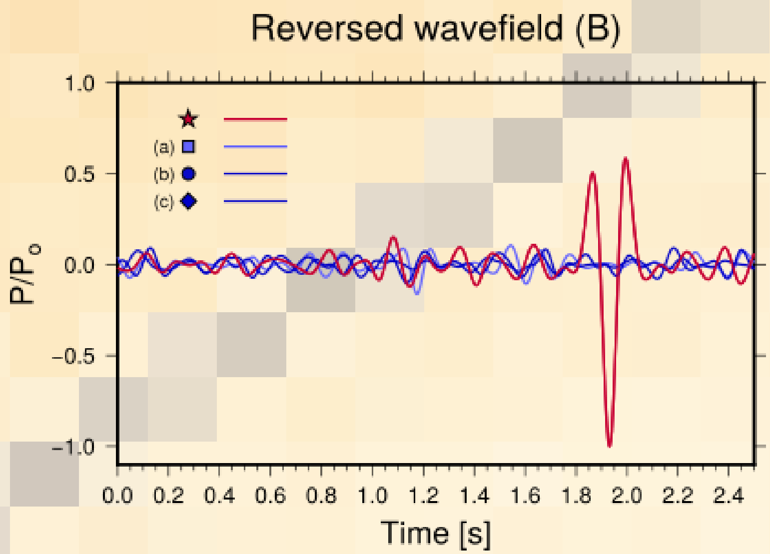
TRM and numerical simulations (II)



TRM and numerical simulations (III)





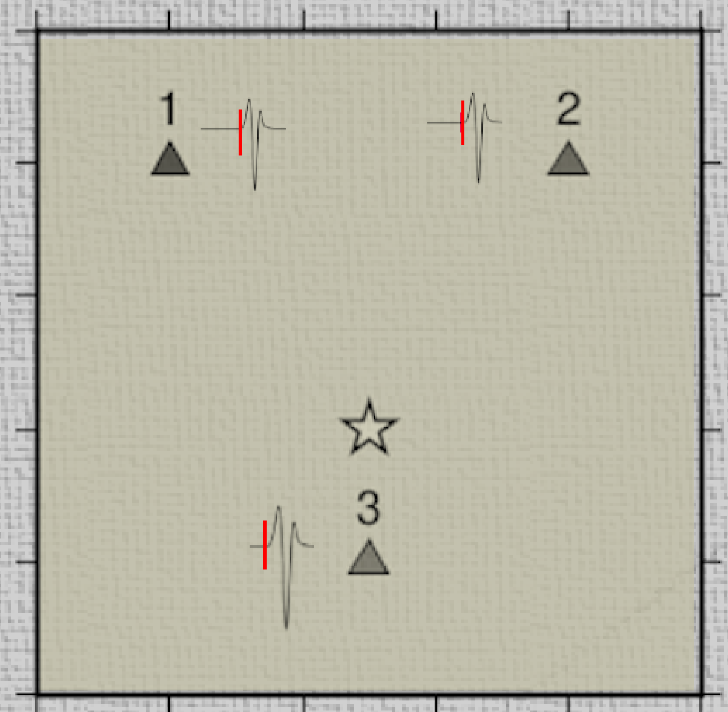


Source location - classical approach

$$\mathbf{m} = (x, y, z)$$

$$\mathbf{d} = (t_1^o, t_2^o, \ldots)$$

$$t_i(\mathbf{m}) = t_o + \int\limits_{SR} \frac{dl}{v}$$



$$\left\{ (m_s,\ t_o) : \sum_i \left(t_i^o - t_i(\mathbf{m})\right)^2 = \min \right\}$$

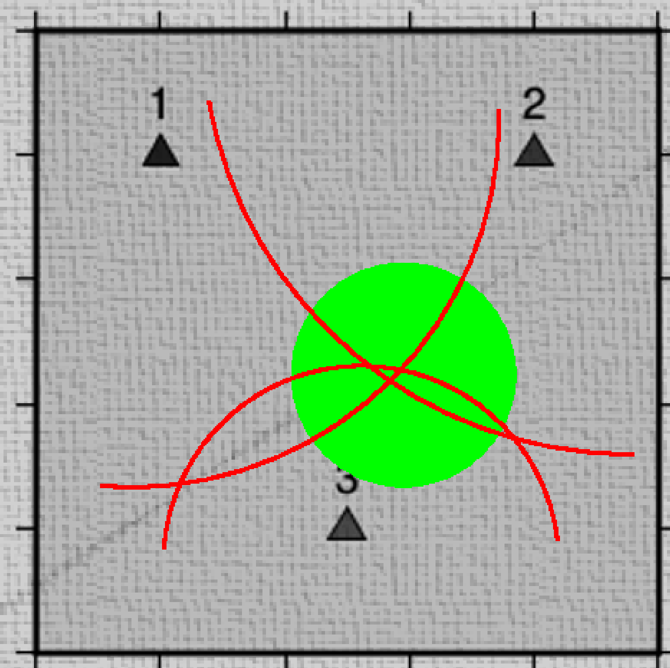
Source location - triangulation

Search for $\mathbf{m}_s = (x, y, z)$

$$t_o \;=\; t_i^o - \Delta t_i = const.$$

$$\Delta t_i = R_i / v$$

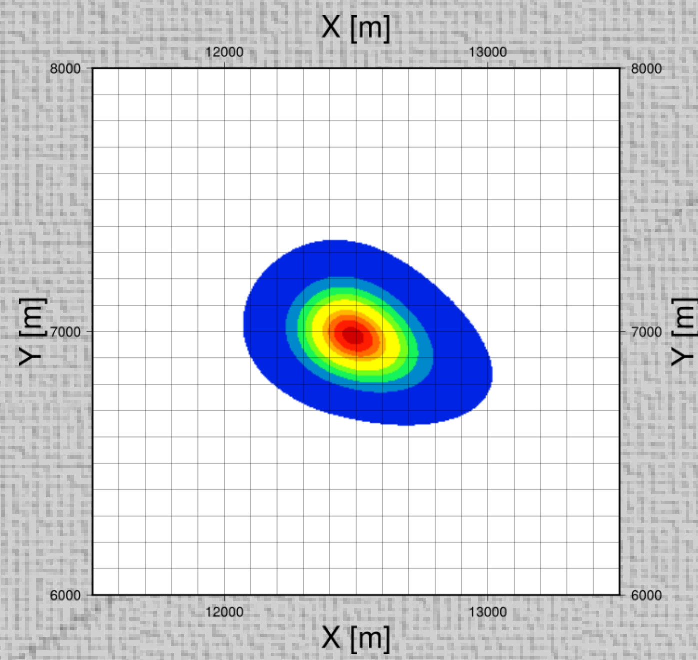
$$\Downarrow$$



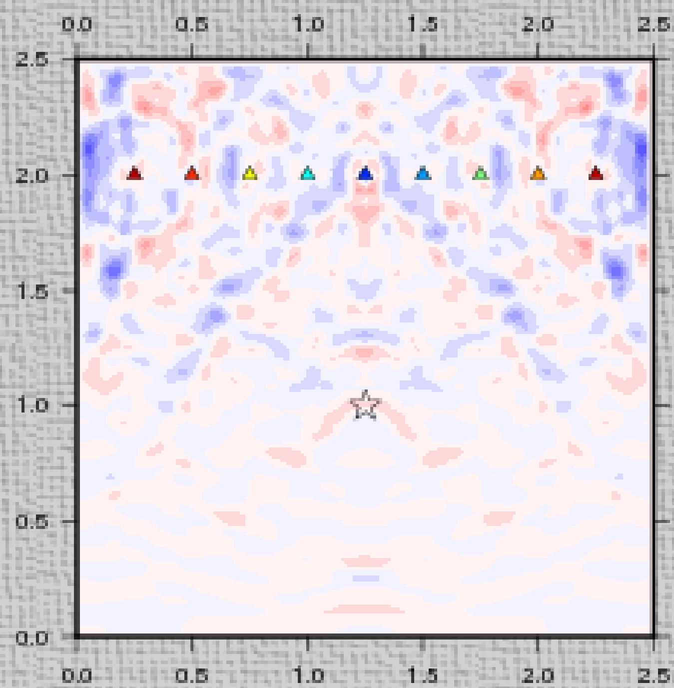
This is TRM-like approach but uses limited information only t_i

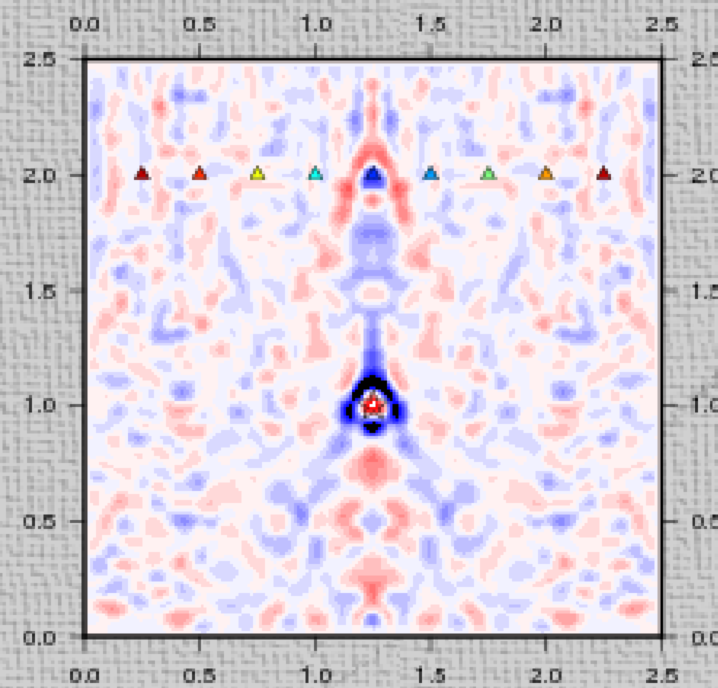
Construct *a posteriori* pdf

$$\sigma(\mathbf{m}) = f(\mathbf{m}) \exp\left(-||t_i^o - \Delta t_i(\mathbf{m})||\right)$$



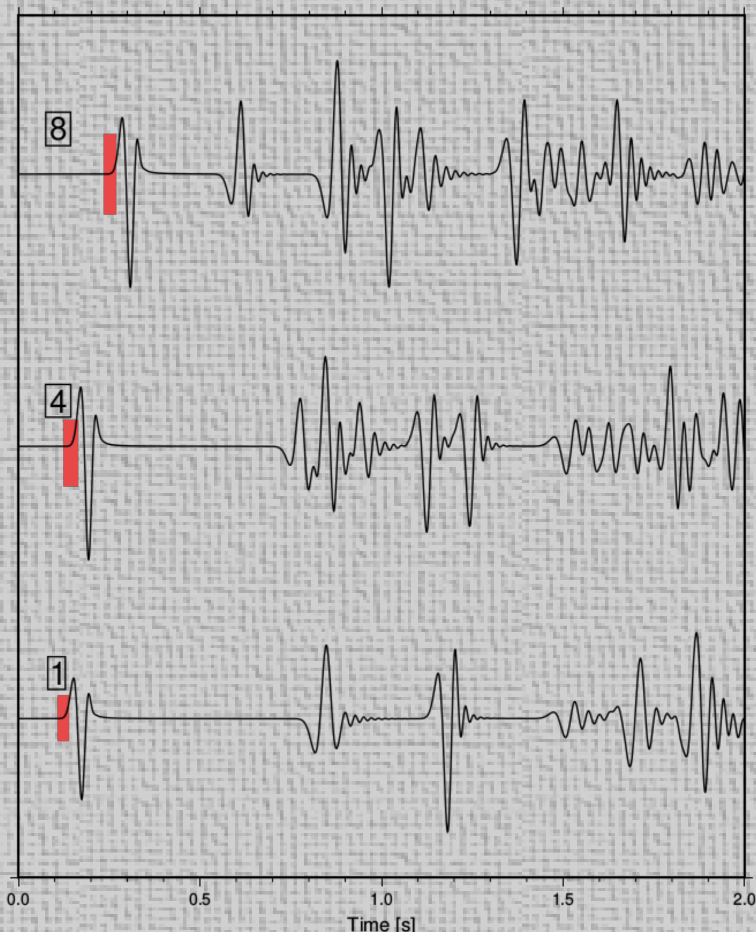
TRM location





Solution: ***search for the point of maximum coherence of back-projected waves***

TRM based full waveform inversion



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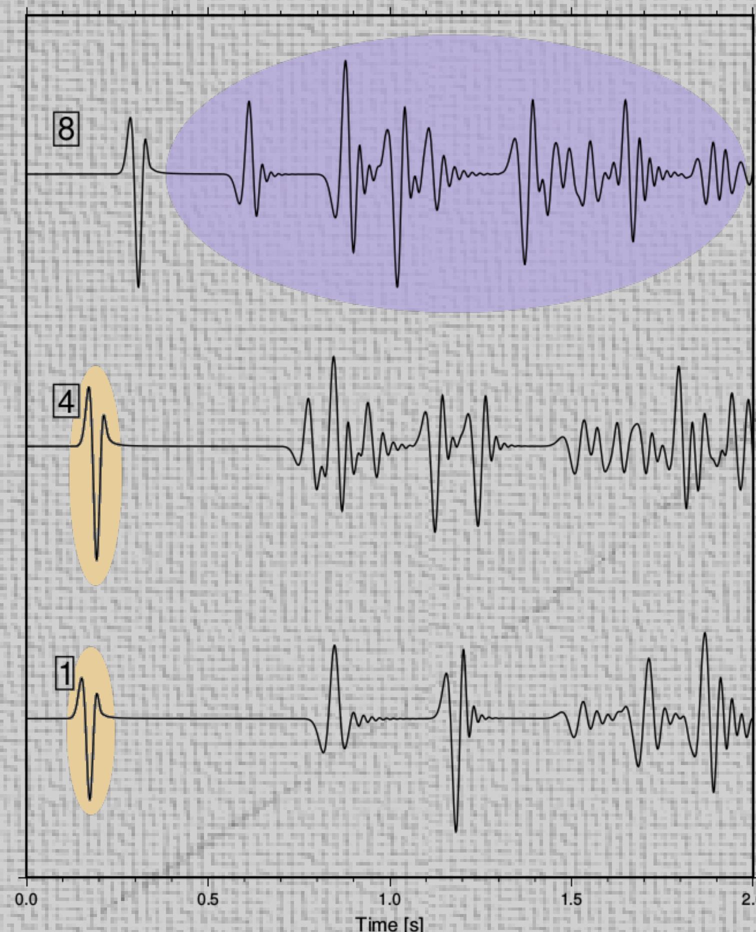

TRM - source of information

- arrival times
- secondary phases
- amplitudes of direct waves

More information → higher S/N ratio → better resolution

- only a few forward modelings required
- necessary search for maximum coherence point

TRM based location - features

- use all available information - higher spatial resolution
- less sensitive to noise (time picking accuracy)
- numerically very efficient - even when used with Bayesian inversion framework requires only a few forward modelings

Instead of conclusions

- requires knowledge of G^+ (medium)
- works for acoustic and EM waves
- can be used in curved spaces ???
- what about TRM if time is a dynamical parameter ???

Thank you for your attention

Prepared: January 13, 2021