Probabilistic Inverse Theory

Lecture 8

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Inversion - reanalysis of foundations

question

$\Downarrow$

observational data

$\Downarrow$

inference

$\Downarrow$

output ....

Probabilistic point of view

inverse problem

$$\Updownarrow$$

joining available information

Joining information according to Tarantola

Two distributions describing different pieces of information about the same object

1. $p(x)$
2. $q(x)$

$$\mathbf{p} \wedge \mathbf{q}(\mathbf{x}) = \frac{\mathbf{p}(\mathbf{x})\,\mathbf{q}(\mathbf{x})}{\mu(\mathbf{x})}$$

$\mu(x)$ - non-informative probability

A posteriori pdf

$$\bar{p} \in \Pi = M \times D$$

$$\sigma(\bar{p}) = \frac{p^{obs}(\bar{p})\, q^{th}(\bar{p})\, f^{apr}(\bar{p})}{\mu^2(\bar{p})}$$

Marginal *a posteriori* distribution

$$\sigma_j(p_j) = \int_{\Pi} \sigma(\bar{p}) \prod_{i \neq j} dp_i$$

Marginal*A posteriori* distribution

If

$$\bar{p} = (\mathbf{m}, \mathbf{d})$$

$$\sigma_m(m) = \int_D \sigma(m, d) dD$$

A posteriori pdf

$$\sigma_m(m) = f(m) \cdot L(m, d^{obs})$$

$$L(\mathbf{m}, \mathbf{d}^{obs}) = \int\limits_D p(\mathbf{d}, \mathbf{d}^{obs}) \frac{q(\mathbf{m}, \mathbf{d})}{\mu(\mathbf{m}, \mathbf{d})} \mathrm{d}\mathbf{d}$$

Probabilistic solution - features

The solution: *a posteriori* probability distribution

- ✦ always exists
- ✦ fully nonlinear
- ✦ include all uncertainties
- ✦ possible full error analysis
- ✦ physical well define meaning (and role) of *a priori* term
- ✦ requires methods of exploring $\sigma(\mathbf{m})$
- ✦ non-parametric inverse problems?

Exploring *a posteriori* probability

- searching for maximum of $\sigma(\mathbf{m})$
- calculate point estimators
- marginal distributions
- sampling $\sigma(\mathbf{m})$

Exploring *a posteriori* probability: m^{ml} solution

Basic characteristic of $\sigma(\mathbf{m})$:

location of the (global) maximum
- the most likelihood $\mathbf{m}^{ml}$ value

$$\mathbf{m}^{ml}: \quad \sigma(\mathbf{m}) = \max$$

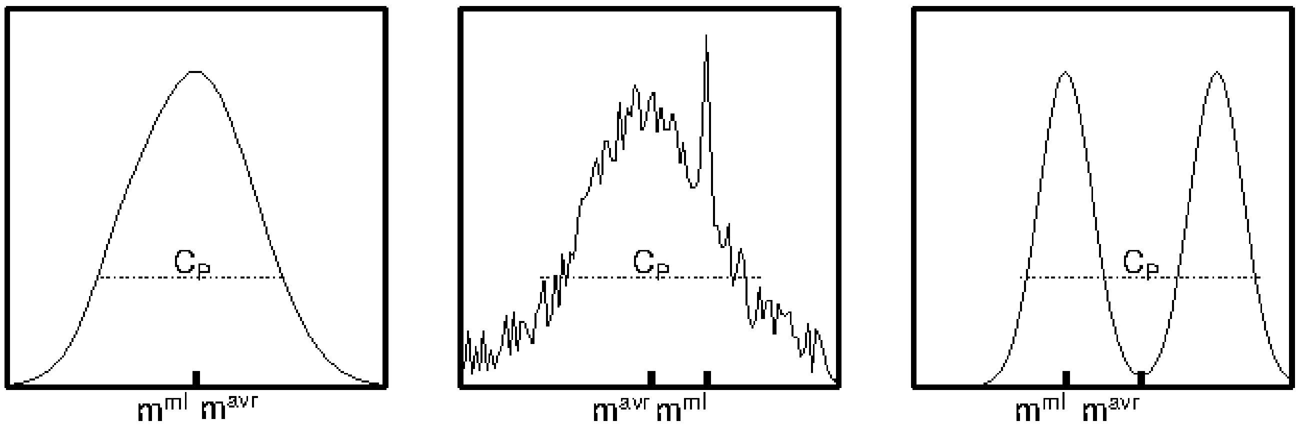
Problem reduced to optimization approach

Point estimators

Characterization of $\sigma(\mathbf{m})$ by its moments:

- average (expected) value: $\mathbf{m}^{avr} = \int_M \mathbf{m}\,\sigma(\mathbf{m})\mathrm{d}\mathbf{m}$
- covariance $C_{ij} = \int_M (m_i - m_i^{avr})(m_j - m_j^{avr})\,\sigma(\mathbf{m})\mathrm{d}\mathbf{m}$
- higher order moments

Point estimators (m^{ml}, m^{avr})



Marginal *a posteriori* distribution

- 1D marginals

$$\sigma_i(m_i) = \int\limits_{\mathbf{m} \neq m_i} \sigma(\mathbf{m})\, \mathrm{d}\mathbf{m}$$

- 2D marginals

$$\sigma_{ij}(m_i, m_j) = \int\limits_{\mathbf{m} \neq m_i, m_j} \sigma(\mathbf{m})\, \mathrm{d}\mathbf{m}$$

- higher dimension marginals

Require efficient methods of calculation multi-dimensional integrals

Probability distribution - do we know them?

$$L(\mathbf{m}, \mathbf{d}^{obs}) = \int_D p(\mathbf{d}, \mathbf{d}^{obs}) \frac{q(\mathbf{m}, \mathbf{d})}{\mu(\mathbf{m}, \mathbf{d})} \mathrm{d}\mathbf{d}$$

While $p()$ can often be estimated (e.g. by repeating measurements) the most problematic is estimating “theoretical” uncertainties $q()$

If pdf's are unknown ...

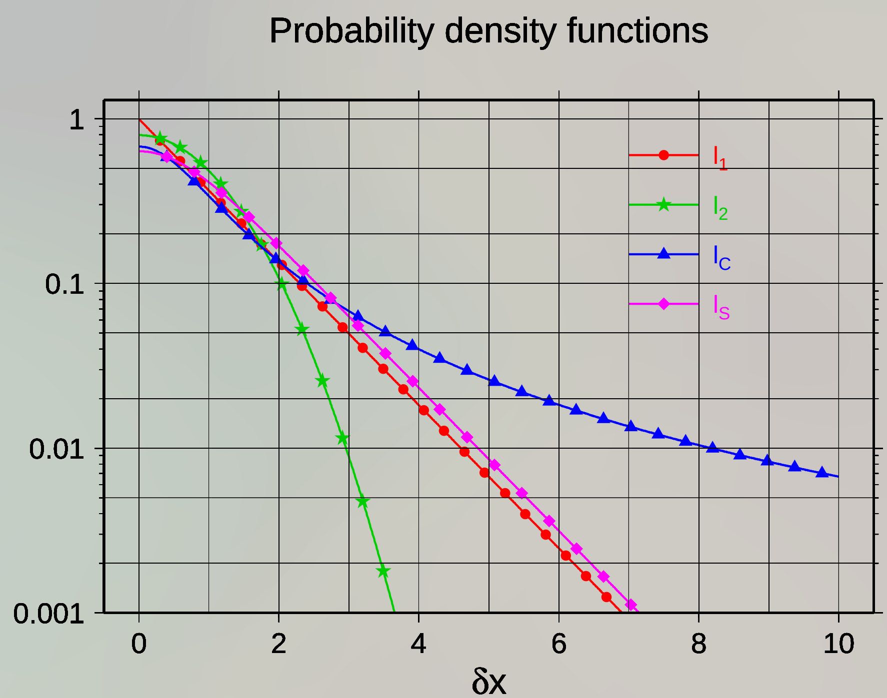
$$q(\mathbf{m},\mathbf{d}) \;= \delta(\mathbf{d}-\mathbf{G}(\mathbf{m})) \Rightarrow L(\mathbf{m}) \sim p(\mathbf{d}-\mathbf{G}(\mathbf{m}))$$

taking into account modelling errors:

$$L(\mathbf{m}) = k\exp(-||\mathbf{d}^{obs}-\mathbf{G}(\mathbf{m})||)$$

$$\sigma(\mathbf{m}) = f(\mathbf{m}) \times \exp(-||\mathbf{d}^{obs}-\mathbf{G}(\mathbf{m})||)$$

Probability distribution - generic forms



Still remaining task: how to chose the best norm ?

Discrete Inverse problems

Physical system:

$$p_1, p_2, \cdots p_K$$

parameters:

$$\mathbf{m} = (m_1, m_2, \cdots m_M)$$

predicted (measureable) quantities:

$$\mathbf{d} = (d_1, d_2, \cdots d_N)$$

Forward modelling: $\mathbf{d}^{th} = f(\mathbf{m}, \mathbf{m}^{fix})$

Inversion : $\mathbf{m} = ... \pm ...$

Continuous Inverse problems

What if thought quantity is a continuous function?

$$(m_1, m_2, \cdots m_M) \Rightarrow m(t)$$

What if data is a continuous function?

$$(d_1, d_2, \cdots d_N) \Rightarrow d(t)$$

Discretization (naive)

Introduce grid (1D case for illustration):

$$t_i \; = \; t_o + i * dt$$

Calculate values of $m(t)$ at grid nodes

$$m_i \; = \; m(t_i)$$

continuous $\rightarrow$ discrete

$$m(t) \rightarrow (m_1, m_2 \cdots)$$

Naive discretization - problems

However, this way we loose the very important "continuous" property of the function $m(t)$ Discrete approach treats all m_i as independent (or weakly correlated) variables. In case of continuous problems all $m(t)$ are strongly correlated. Actually, in most cases all the value $m(t)$ for any t is determined by $m(0)$ and $m'(0)$ if $m(t)$ is solution of partial differential equation (e.g. seismograms)

To retrieve this property we need to put additional constraints to the inverse problem. It usually leads to serious complications and sever diminishing computational efficiency of the method.

In addition the non-uniqness and dependences of the solution on used discretization method becomes an important issue.

Discretization (spectral)

Choose a complete set of “kernel” functions $\psi_i(t)$ so that

$$m(t) \;=\; \sum_{i=1}^{N} a_i \psi_i(t)$$

continuous $\rightarrow$ discrete

$$m(t) \rightarrow (a_1, a_2 \cdots)$$

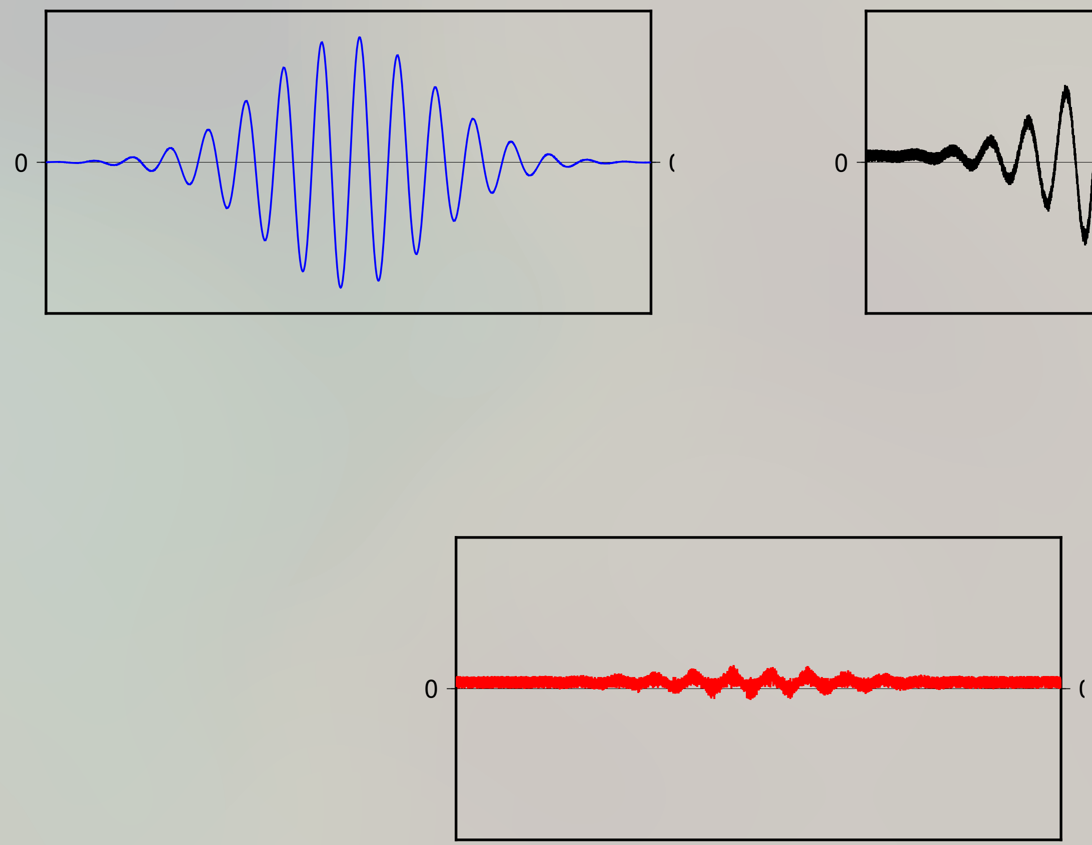
Continuous “data”

$$||\mathbf{d}^{obs} - \mathbf{d}^{th}(\mathbf{m})|| \; = \; \min$$

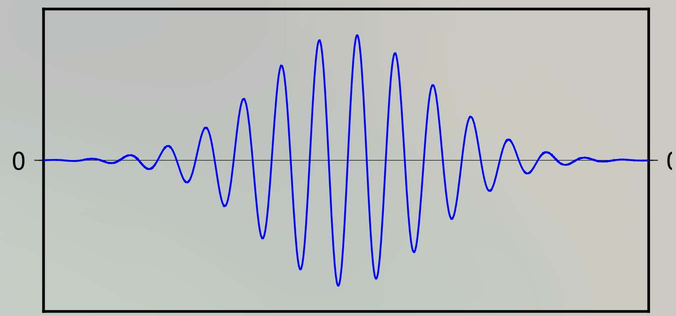
$$||\mathbf{d}^{obs} - \mathbf{d}^{th}|| \; = \begin{cases} \sum\limits_{i=1}^{N} (d_i^{obs} - d_i^{th})^2 & \text{discrete} \\ \\ \int\limits_0^T \left(d^{obs}(t) - d^{th}(t)\right)^2 \mathrm{d}t & \text{continuous} \end{cases}$$

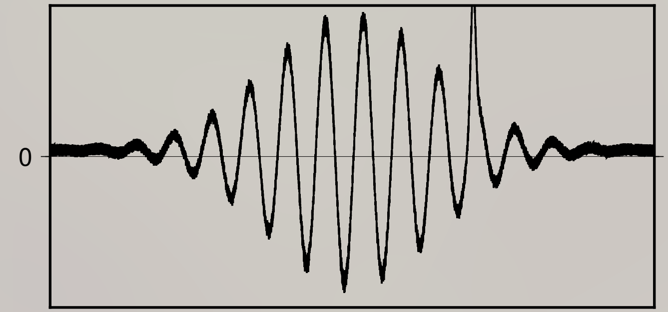
The method seems to be OK but ...

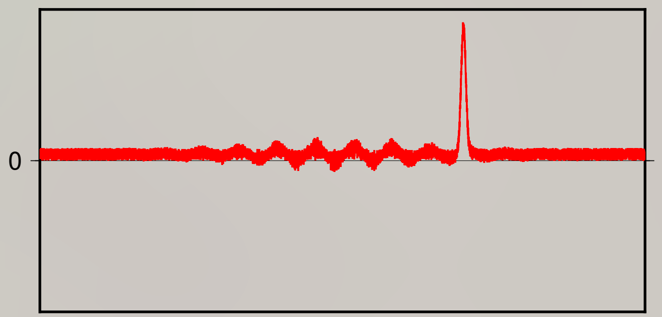
Continuous “data”



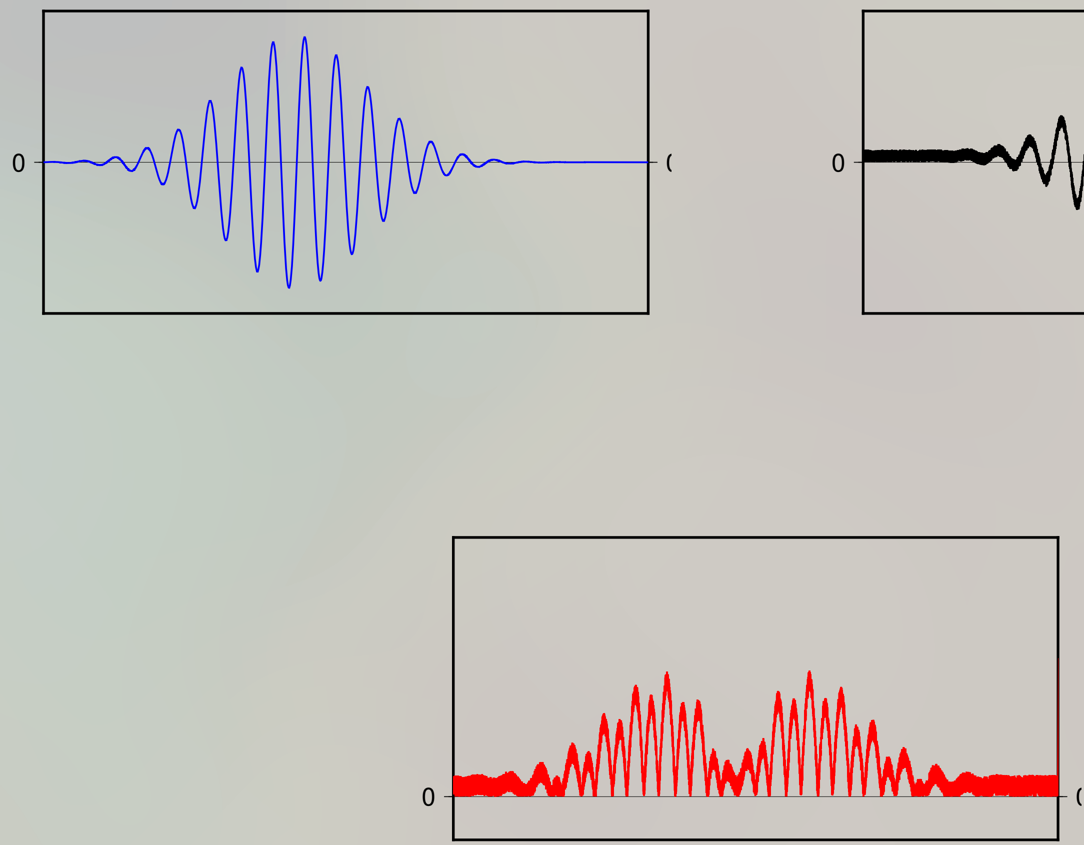
Continuous “data” -I







Continuous "data" - II



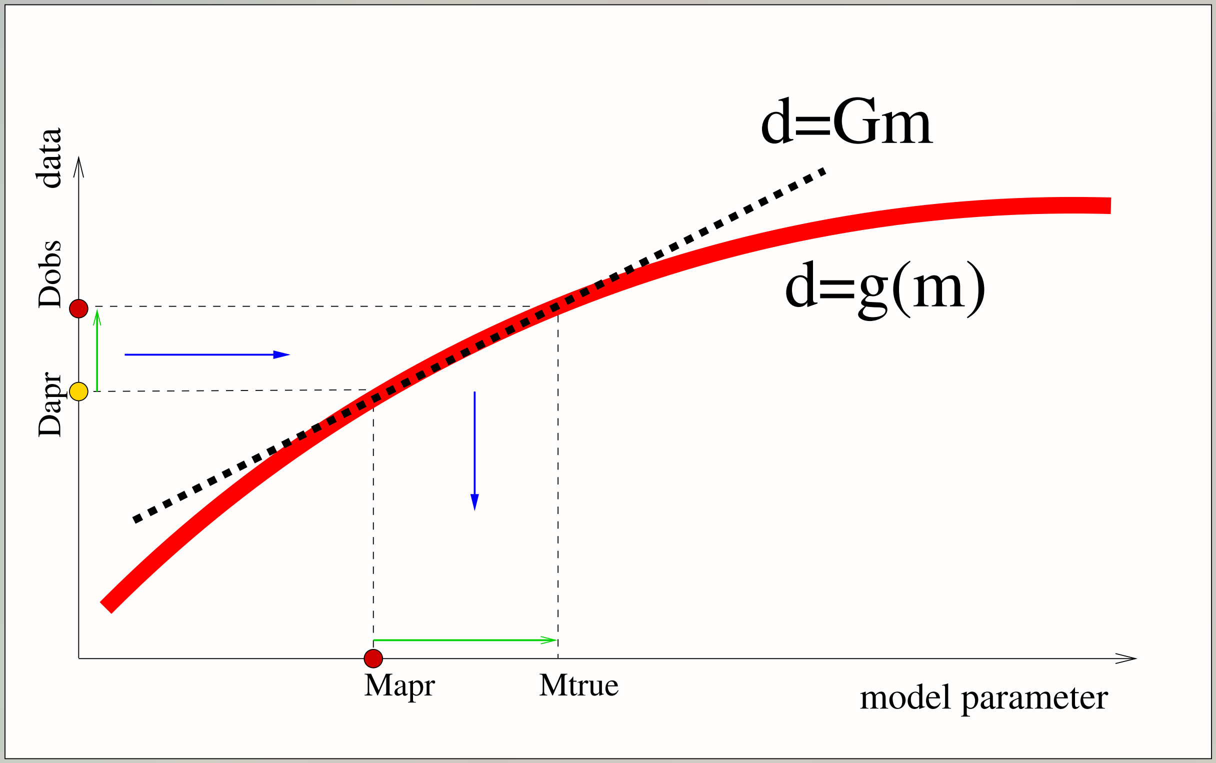
Continuous “data” - (spectral measure)

$$||\mathbf{d}^{obs}(x) - \mathbf{d}^{th}(x)||^2 = \langle \mathbf{d}^{obs} - \mathbf{d}^{th}, \mathbf{d}^{obs} - \mathbf{d}^{th} \rangle$$

$$< h, g > = \text{Re} \int_0^{+\infty} \frac{\bar{h}(f) * g(f)}{S_n(f)} \, \mathrm{d}f$$

$S_n(f)$ - observational noise power spectrum

Continuous linear inverse problems - back projection



Example of continuous back projection

Time Reversal Method

$$\rho \ddot{u}_i \;=\; \partial_j \tau_{ij} + S_i$$

Inverse task:

$$u_i^{obs}(t) \rightarrow S(t)$$

Time reversal - Green's function

$$\frac{1}{c^2}\frac{\partial^2 G(x,t)}{\partial t^2} - \Delta G(x,t) = \delta(x - x_o, t - t_o)$$

Solution:

$$u(x,t) = \int_0^T \mathrm{d}t' \int_{V_o} \mathrm{d}x'\, G(t,x;t',x')S(x',t')$$

$G(x,t;,x',t')$ "propagates" information from source (x') to receiver (x)

Green's function cd.

Assuming time translation invariance

(energy conservation - Noether's theorem)

$$G(x,t;x',t') = G(x,x';t-t')$$

Homogeneous medium

translational invariance

$$G(x,t;x',t') = G(x-x';t-t')$$

Retarded and advanced Green's functions

Retarded (casual) GF:

$$G^+(x - x', t - t') = \frac{1}{4\pi||x - x'||}\delta\left(t - t' - \frac{||x - x'||}{c}\right)\Theta(t - t')$$

Advanced (anti-casual) GF:

$$G^-(x - x', t - t') = \frac{1}{4\pi||x - x'||}\delta\left(t - t' + \frac{||x - x'||}{c}\right)\Theta(t' - t)$$

$$G^+(x - x', t - t') = G^-(x - x', t' - t)$$

Time Reversal - step I

Radiation from point source (O)

$$S(x,t) = S(t)\delta(x - x_o)$$

Solution at given receiver (R)

$$u_R(x,t) = \int_{OR} G^+(x - x_o, t - t',)S(t')\mathrm{d}t'$$

Time Reversal - step II

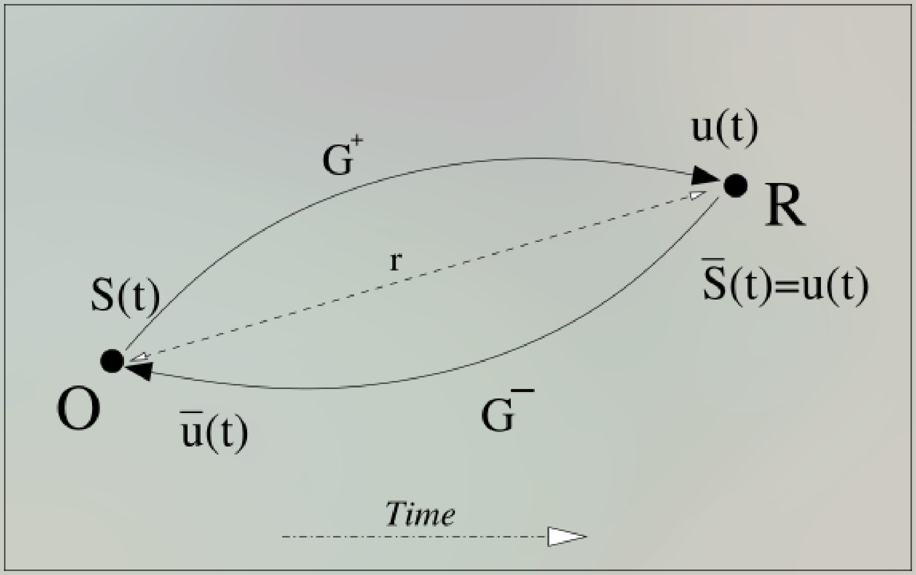
Anti-casual propagation from R to O

$$\bar{u}(x,t) = \int_{RO} G^{-}(x - x_R, t - t',)\bar{S}(t')\mathrm{d}t'$$

From physical point of view it corresponds to recording at the point O incoming waves, for example, “refracted” at the point R

$$\bar{S}(t) = u_R(t)$$

Time Reversal Method (TRM) - basic principle



1. $u(r,t) = \int G^{+}(r, t-t')S(t')\mathrm{d}t'$
2. $\bar{S} = u(r,t)$
3. $\bar{u}(t) = \int G^{-}(r, t-t')\bar{S}(t')\mathrm{d}t'$

$\mathbf{\bar{u}(r,t) = k(r)\ S(t)}$

However: $G^{+}(\cdot, t-t') = G^{-}(\cdot, -(t-t'))$

$$S(t) = \int G^{+}(t-t', r)u(-t')\mathrm{d}t'$$

TRM as the inverse problem

$$\mathbf{m} = S(t) \qquad \mathbf{d} = u(t)$$

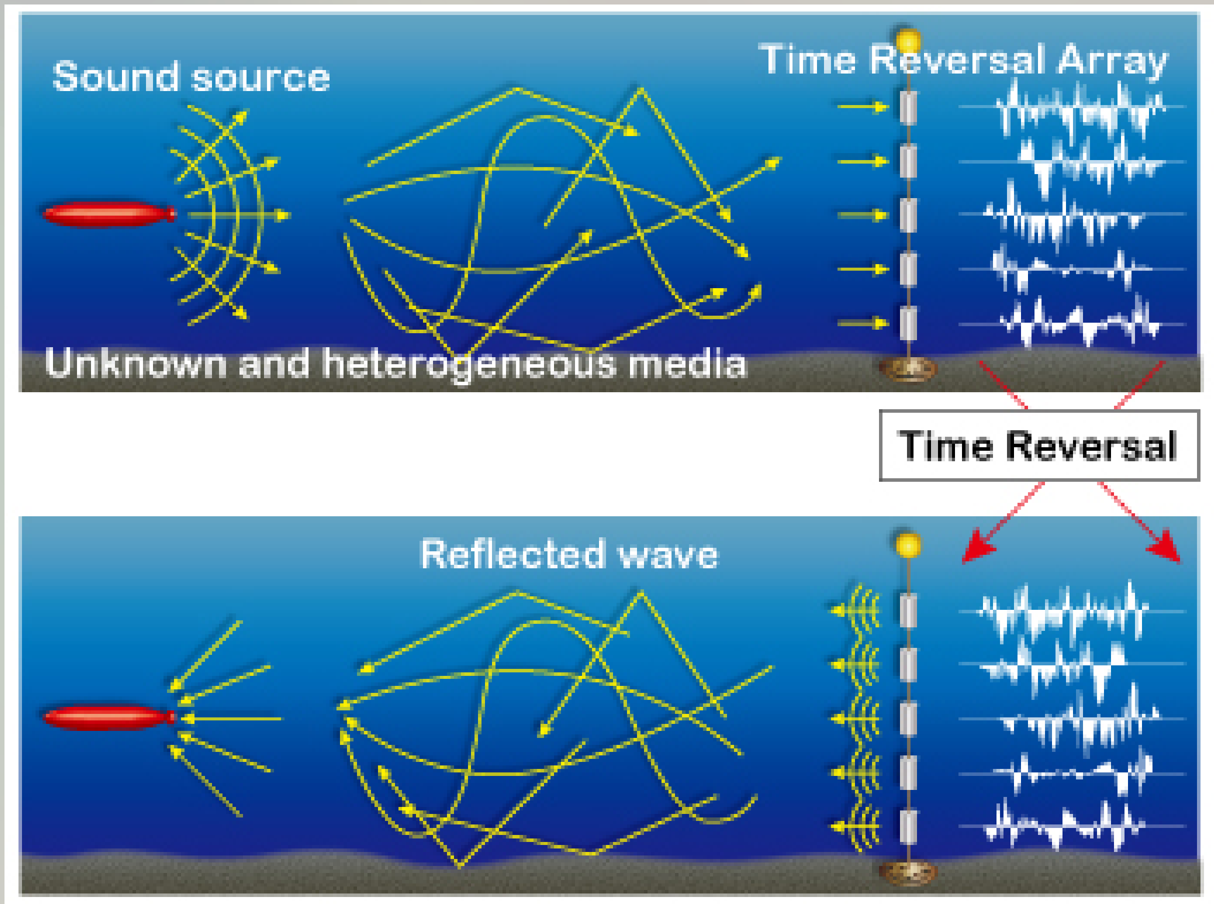
$$u(r,t) = \int G^{+}(r, t-t')\, S(t') \mathrm{d}t' \quad \Rightarrow \quad \mathbf{d} = \mathcal{G}\mathbf{m}$$

$$S(t) = \int G^{+}(r, t-t')\, u(-t') \mathrm{d}t' \quad \Rightarrow \quad \mathbf{m} = \mathcal{G}^{-1}\mathbf{d}$$

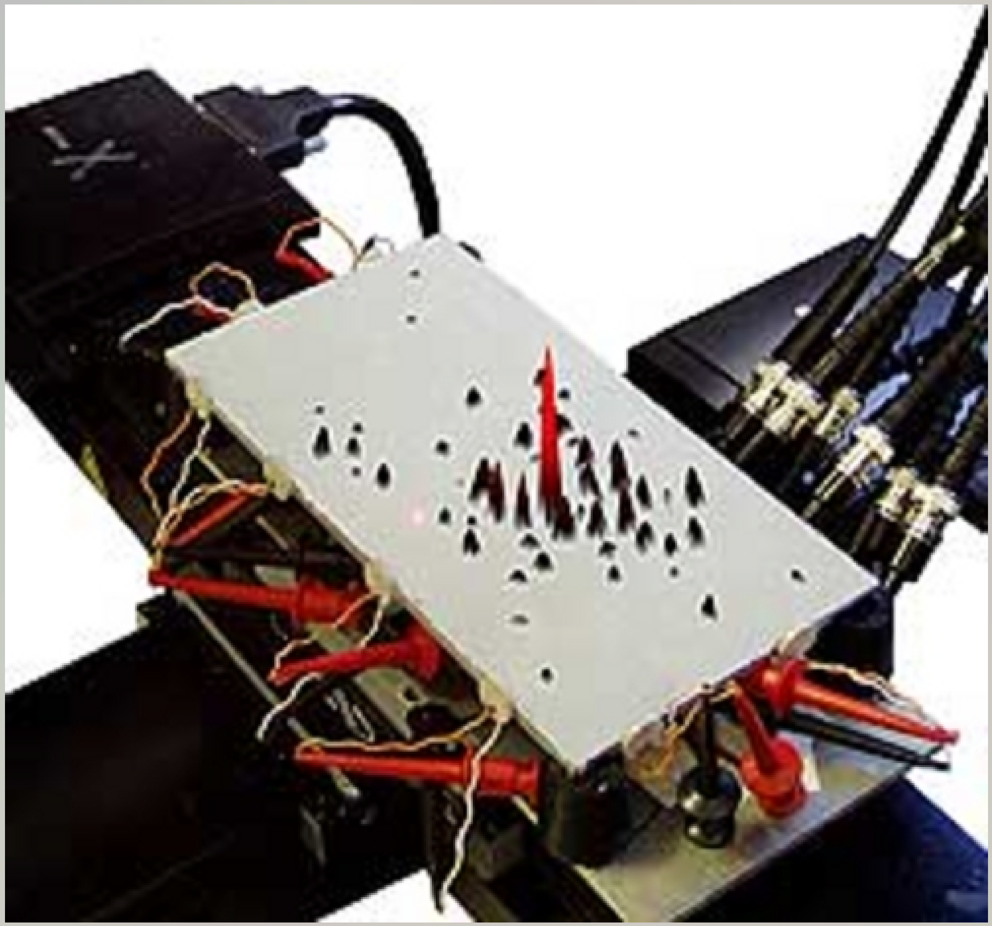
The method is computationally very efficient, but ...

we need to know the medium: $G^{+}(\cdot,\cdot)$

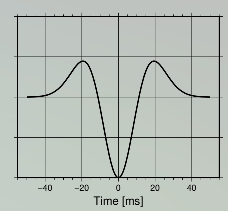
Physics of Time Reversal

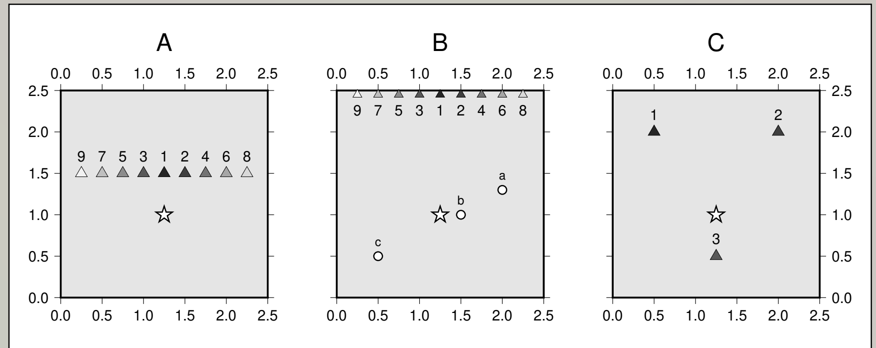


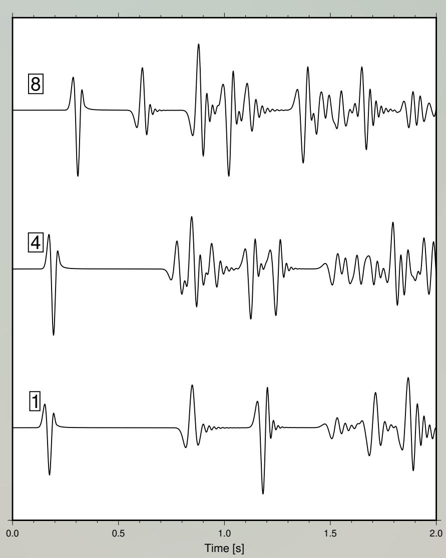
Experimental evidence (M. Fink)

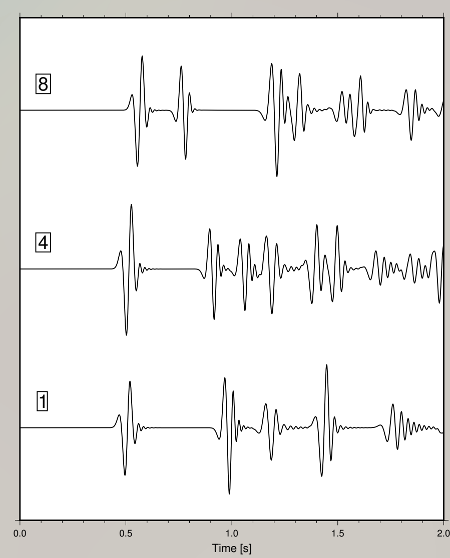


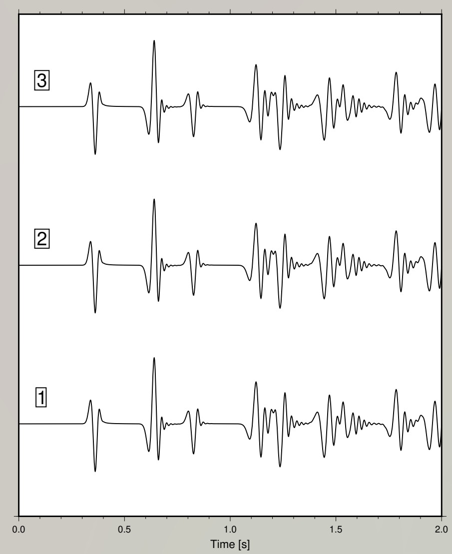
TRM and numerical simulations (K. Waskiewicz)











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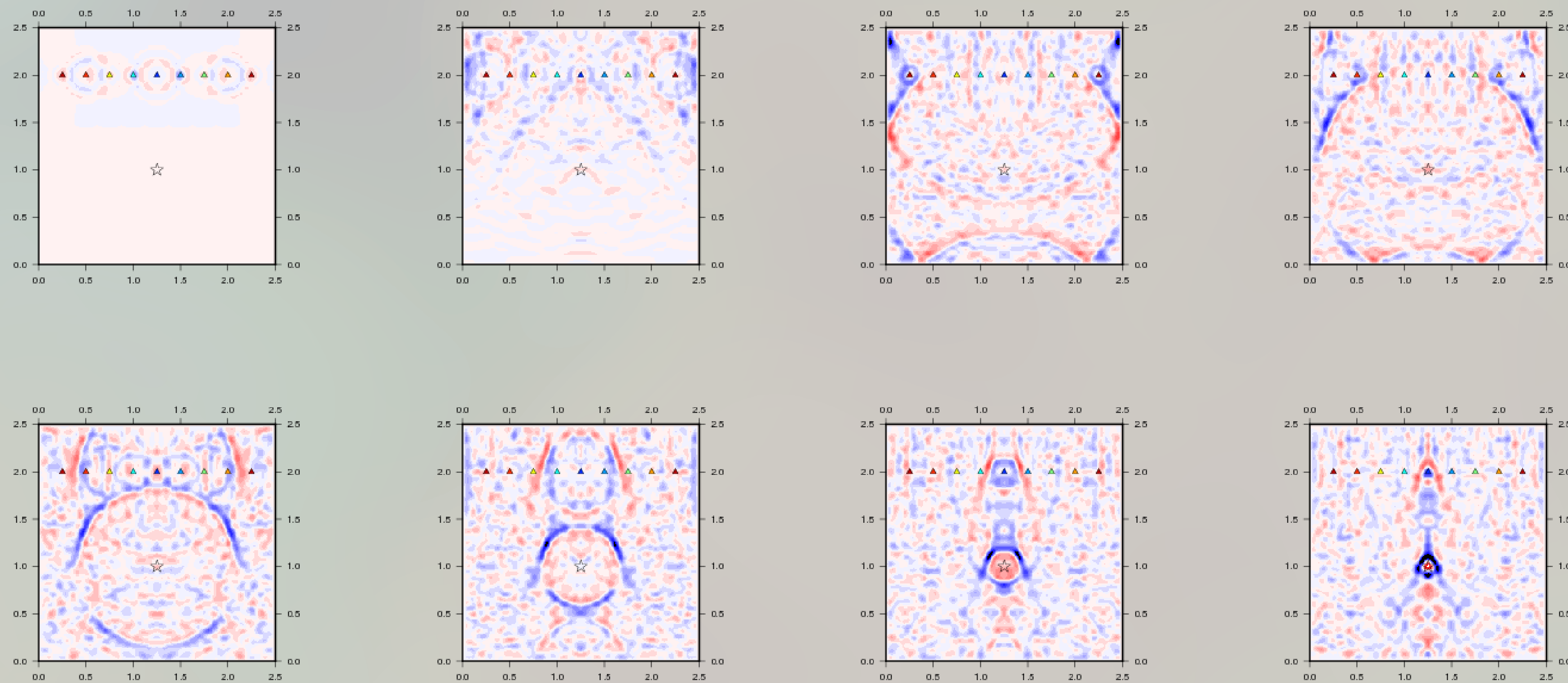

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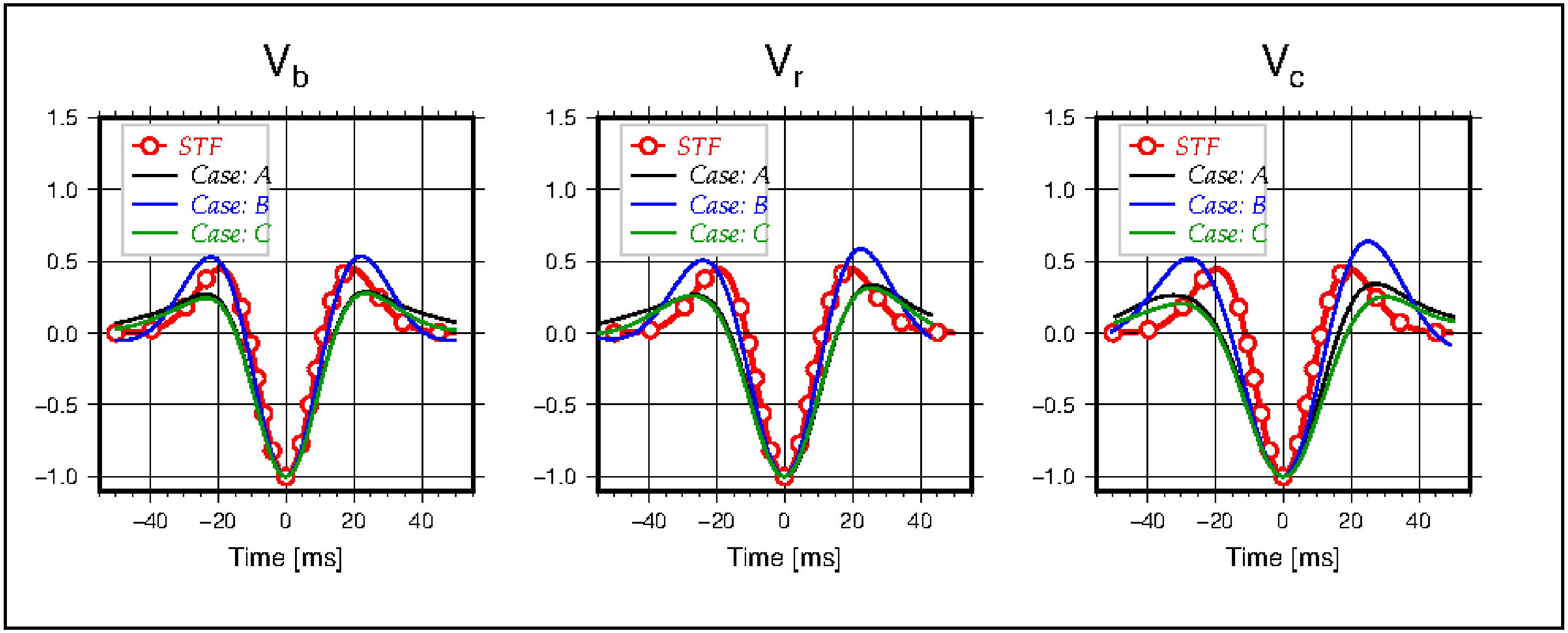

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