Probabilistic Inverse Theory

Lecture 7

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Inversion - probabilistic point of view

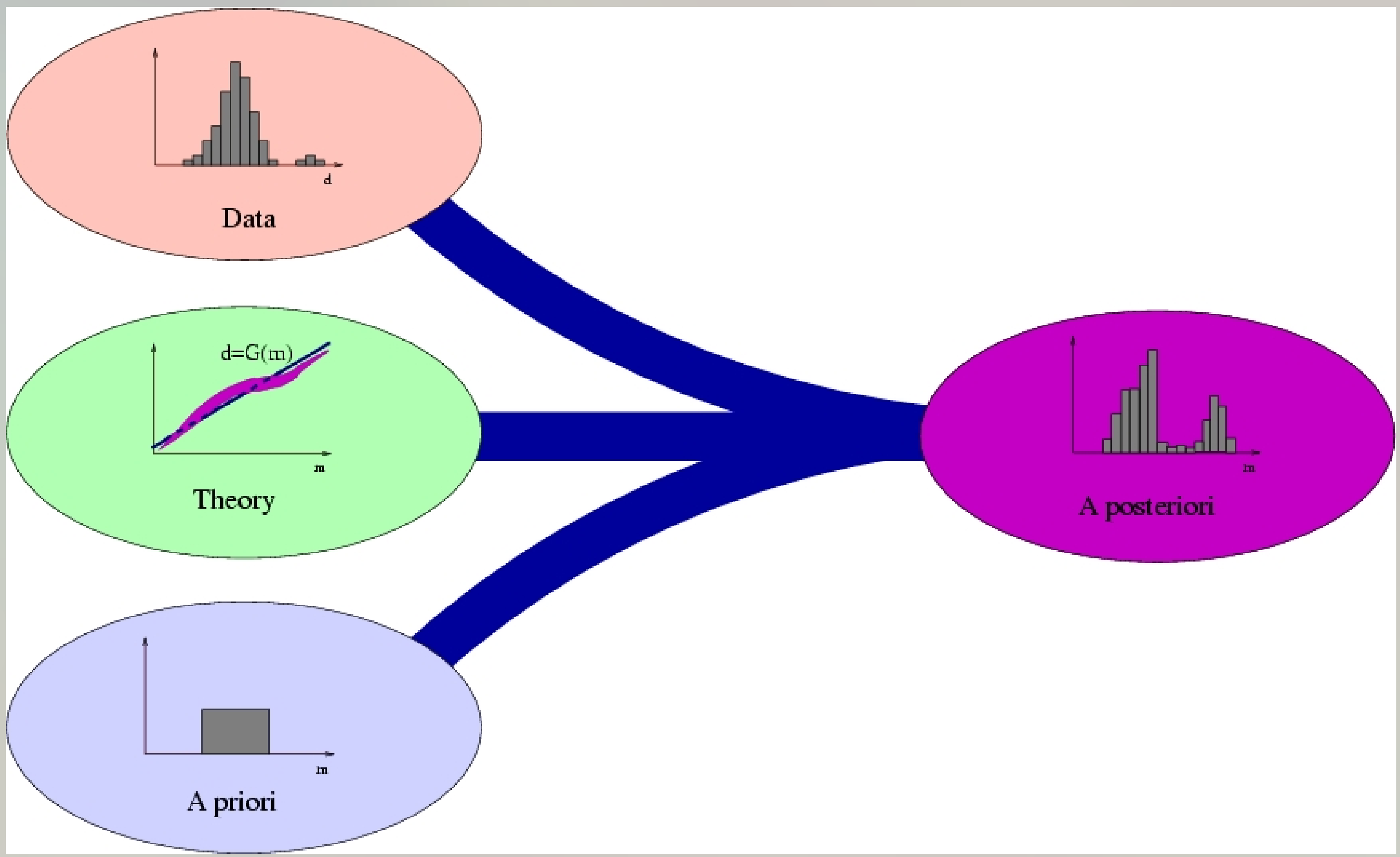
- Inverse problem $\Longleftrightarrow$ indirect measurement
- Output of measurement $\Longrightarrow$ probability distribution

$$\downarrow$$

Solution of inverse problem $\equiv$ probability distribution

Interpretation: $\sigma(\mathbf{m})$ – probability that $\mathbf{m} = \mathbf{m}^{true}$

Source of *a posteriori* errors



Bayes Theorem

$$p_{\mathbf{m}|\mathbf{d}}(\mathbf{m}|\mathbf{d}) = \frac{p_{\mathbf{d}|\mathbf{m}}(\mathbf{d}|\mathbf{m})p_{\mathbf{m}}(\mathbf{m})}{p_{\mathbf{d}}(\mathbf{d})}$$

Bayes Theorem - interpretation

$$\mathbf{d} \Longrightarrow \mathbf{d}^{obs}$$

$$p_{\mathbf{m}|\mathbf{d}}(\mathbf{m}|\mathbf{d}^{obs}) = \frac{p_{\mathbf{d}|\mathbf{m}}(\mathbf{d}^{obs}|\mathbf{m})p_{\mathbf{m}}(\mathbf{m})}{p_{\mathbf{d}}(\mathbf{d}^{obs})}$$

Probabilistic solution

$$\sigma(\mathbf{m}) = f(\mathbf{m})\, L(\mathbf{m}, \mathbf{d}^{obs})$$

$$L(\mathbf{m}, \mathbf{d}^{obs}) \;=\; \int_{d} \rho_{th}(\mathbf{d} - G(\mathbf{m}))\, \rho_o(\mathbf{d} - \mathbf{d}_o)\, d\mathbf{d}$$

Probabilistic approach - classic

inverse problem

$\Updownarrow$

indirect measurement

(parameter estimation)

Tarantolas' approach

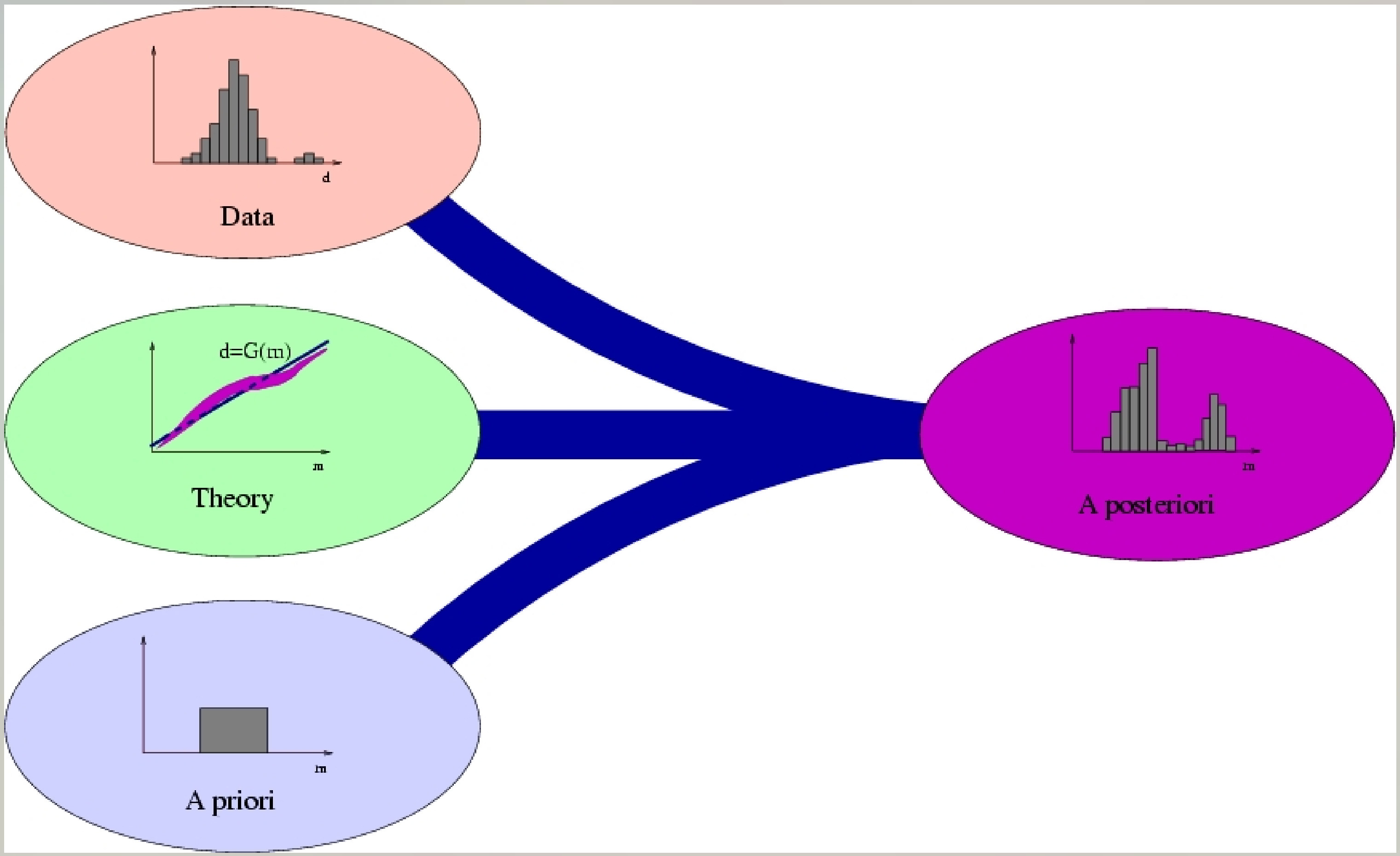
Probabilistic approach - generalization

inverse problem

$\Updownarrow$

inference process

Inversion - Source of information



Inversion - reanalysis of foundations

question

$\Downarrow$

observational data

$\Downarrow$

inference

$\Downarrow$

output ....

Conclusion ...

inverse problem

$$\Updownarrow$$

inference process

$$\Updownarrow$$

joining available information

Bayes interpretation of probability

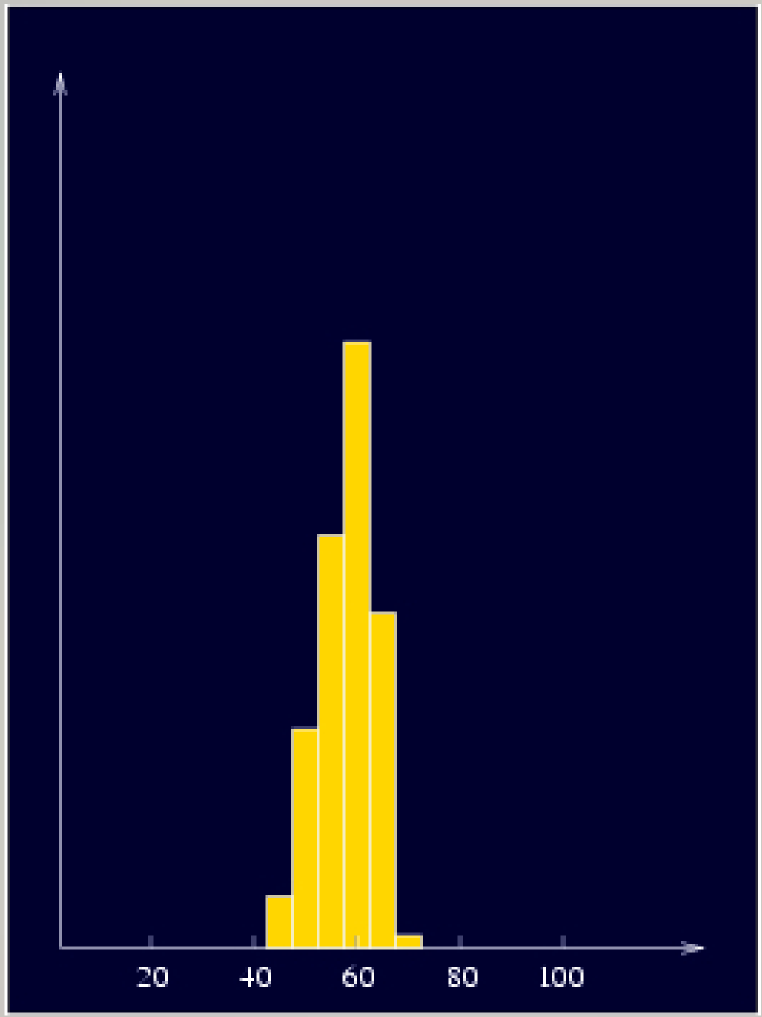
available information $\Longleftrightarrow$ probability distribution.

Sir Bayes:

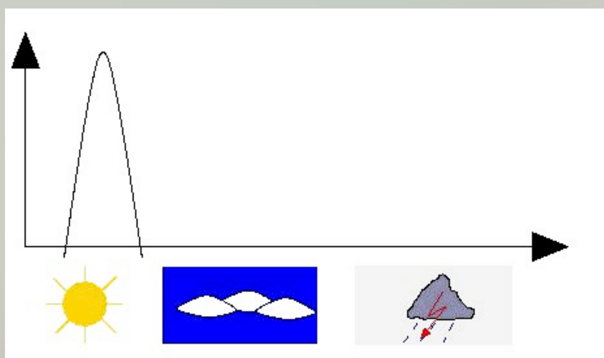
Any piece of information can be quantitatively describe by a probability distribution and vice versa each probability represents a piece of information

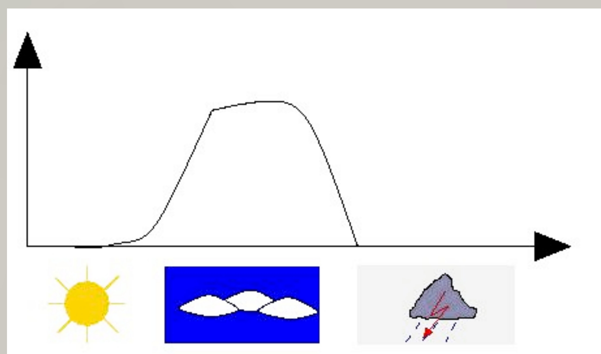
Information and probability distributions

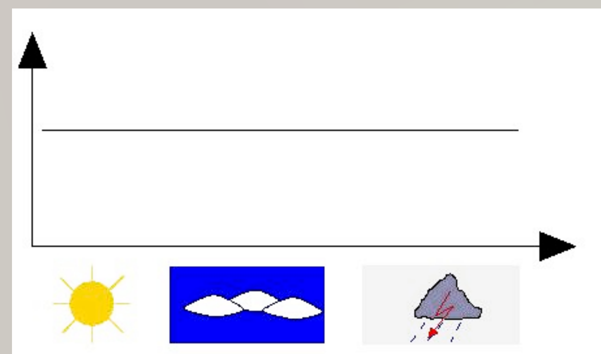
Information and probability distributions



Bayesian interpretation of probability - example







Joining information according to Tarantola

Two distributions describing different pieces of information about the same object

1. $p(x)$
2. $q(x)$

$$\mathbf{p} \wedge \mathbf{q}(\mathbf{x}) = \frac{\mathbf{p}(\mathbf{x})\,\mathbf{q}(\mathbf{x})}{\mu(\mathbf{x})}$$

$\mu(x)$ - non-informative probability

Mathematics of inference - Inference Space

$$(\mathcal{P}, \Sigma, \wedge(\cdot,\cdot))$$

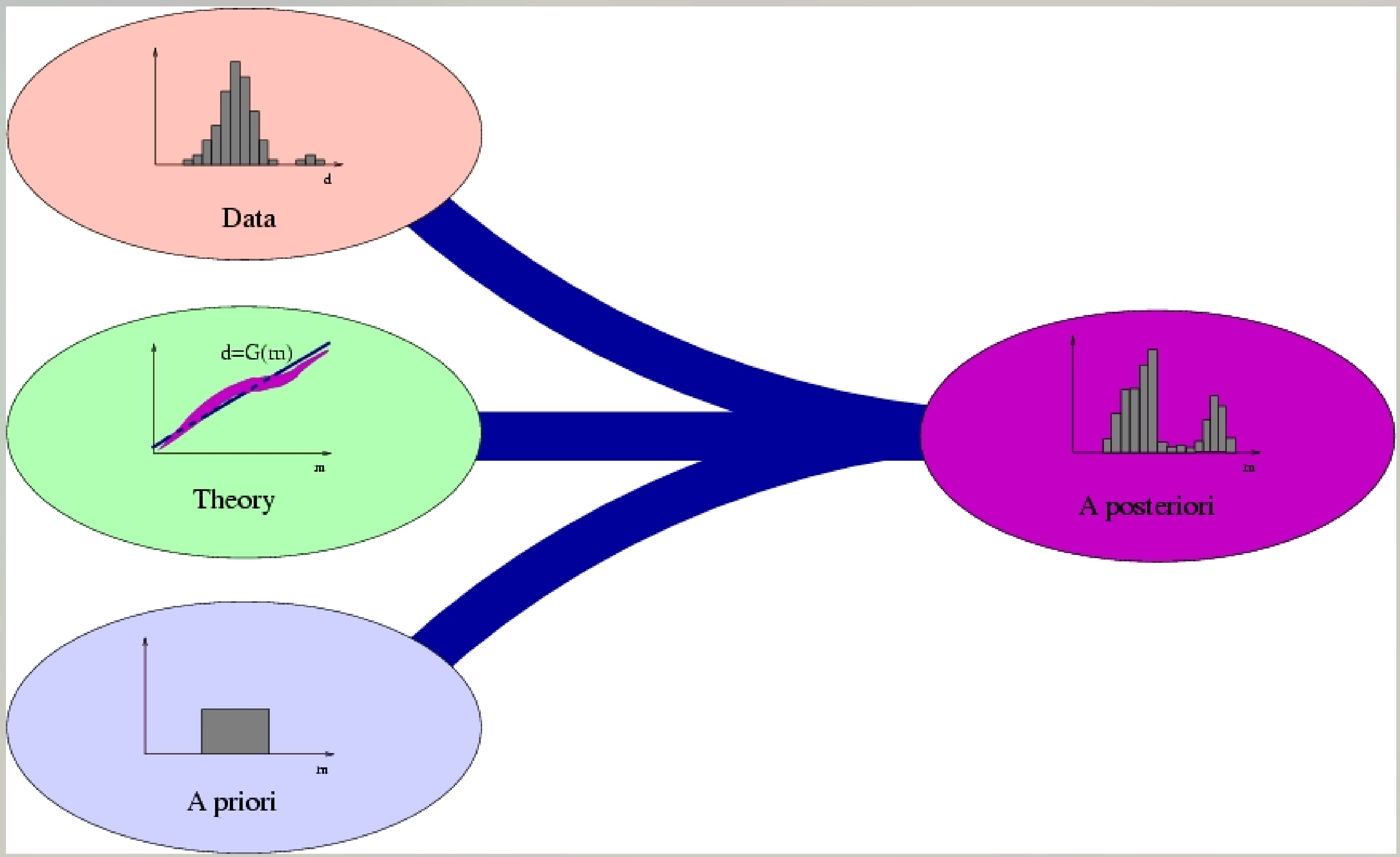
where

$\mathcal{P} = (\mathbf{M}, \mathbf{D})$ - parameter space

Σ - space of all probability distributions over $\mathcal{P}$

$\wedge(\cdot,\cdot)$ - joining operator: $\Sigma \times \Sigma \rightarrow \Sigma$

Inversion - Source of information



Solution of inverse problem

1. observation: $p(\mathbf{d})$
2. theory : $q(\mathbf{m}, \mathbf{d})$
3. *a priori* : $f(\mathbf{m})$

$$(p \wedge q)(\cdot) = \frac{p(\cdot)\, q(\cdot)}{\mu(\cdot)}$$

A posteriori pdf over $\mathcal{P} = \mathbf{M} \times \mathbf{D}$

1. Defining all distribution over $\mathbf{M} \times \mathbf{D}$ space

$$p(\mathbf{d}) \Longrightarrow \bar{p}(\mathbf{m},\mathbf{d}) \ = \ \mu(\mathbf{m})p(\mathbf{d})$$
$$f(\mathbf{m}) \Longrightarrow \bar{f}(\mathbf{m},\mathbf{d}) \ = \ f(\mathbf{m})\mu(\mathbf{d})$$

2. Creation the likelihood function:

$$\mathcal{L}(\mathbf{m},\mathbf{d}) = (p \wedge q)(\mathbf{m},\mathbf{d}) \ = \ \frac{p(\mathbf{m},\mathbf{d})\, q(\mathbf{m},\mathbf{d})}{\mu(\mathbf{m},\mathbf{d})}$$

3. Adding *a priori* information:

$$\sigma(\mathbf{m},\mathbf{d}) = (\mathcal{L} \wedge f)(\mathbf{m},\mathbf{d}) \ = \ \frac{\mathcal{L}(\mathbf{m},\mathbf{d})\, f(\mathbf{m},\mathbf{d})}{\mu(\mathbf{m},\mathbf{d})}$$

A posteriori pdf

$$\sigma(\mathbf{m},\mathbf{d}) = \frac{p(\mathbf{m},\mathbf{d})\, q(\mathbf{m},\mathbf{d})\, f(\mathbf{m},\mathbf{d})}{\mu^2(\mathbf{m},\mathbf{d})}$$

Marginal*A posteriori* distribution

$$\sigma_m(m) = \int\limits_D \sigma(m, d) dD$$

$$\sigma_d(d) = \int\limits_M \sigma(m, d) dM$$

A posteriori pdf

$$\sigma_m(m) = f(m) \cdot L(m, d^{obs})$$

$$L(\mathbf{m}, \mathbf{d}^{obs}) = \int_D p(\mathbf{d}, \mathbf{d}^{obs}) \frac{q(\mathbf{m}, \mathbf{d})}{\mu(\mathbf{m}, \mathbf{d})} \mathrm{d}\mathbf{d}$$

Example - exact theory

$$q(\mathbf{m},\mathbf{d}) \ = \ \delta(\mathbf{d}-\mathbf{G}(\mathbf{m})\mu_M(\mathbf{m})$$

$$p(\mathbf{d},\mathbf{d}^{obs}) \ = \ \rho(||\mathbf{d}-\mathbf{d}^{obs}||)$$

then

$$\sigma(\mathbf{m}) = f(\mathbf{m})\,\frac{p(\mathbf{d}^{obs}-\mathbf{G}(\mathbf{m}))}{\mu_D(\mathbf{d})}$$

likelihood function - observational errors

Example - exact measurements

$$p(\mathbf{d}, \mathbf{d}^{obs}) \; = \; \delta(\mathbf{d} - \mathbf{d}^{obs})$$

then

$$\sigma(\mathbf{m}) = f(\mathbf{m}) \, \frac{q(\mathbf{m}, \mathbf{d}^{obs})}{\mu(\mathbf{m}, \mathbf{d}^{obs})}$$

likelihood function - modelling errors

Example - missing theoretical information

role of *a priori*

$$q(\mathbf{m},\mathbf{d}) \;=\; \mu(\mathbf{m},\mathbf{d})$$

$$L(\mathbf{m}) = \int_D p(\mathbf{d},\mathbf{d}^{obs})\frac{q(\mathbf{m},\mathbf{d})}{\mu(\mathbf{m},\mathbf{d})}\mathrm{d}\mathbf{d} = \int_D p(\mathbf{d})\mathrm{d}\mathbf{d} = const.$$

$$\sigma(\mathbf{m}) = f(\mathbf{m})$$

Example - missing observational data

role of *a priori*

$$p(\mathbf{d}) \;=\; \mu_D(\mathbf{d})$$

then

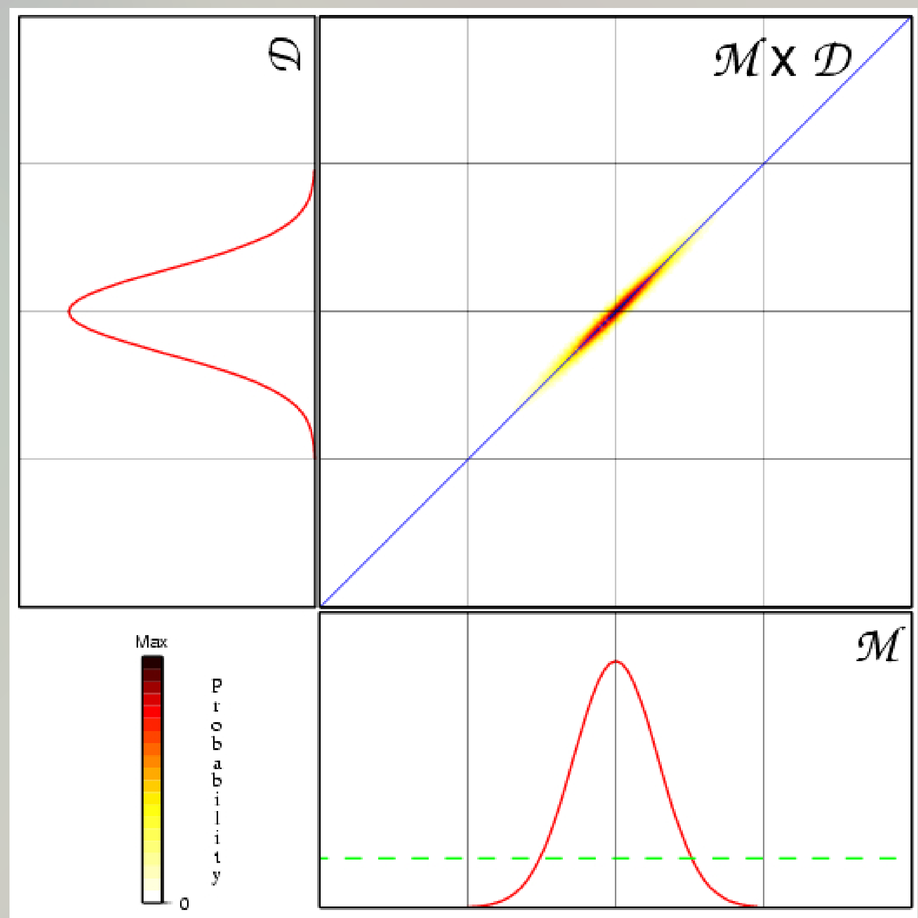
$$L(\mathbf{m}) = \int\limits_D p(\mathbf{d})/\mu_D(\mathbf{d})\frac{q(\mathbf{m},\mathbf{d})}{\mu_M(\mathbf{m})}\mathrm{d}\mathbf{d}$$

if

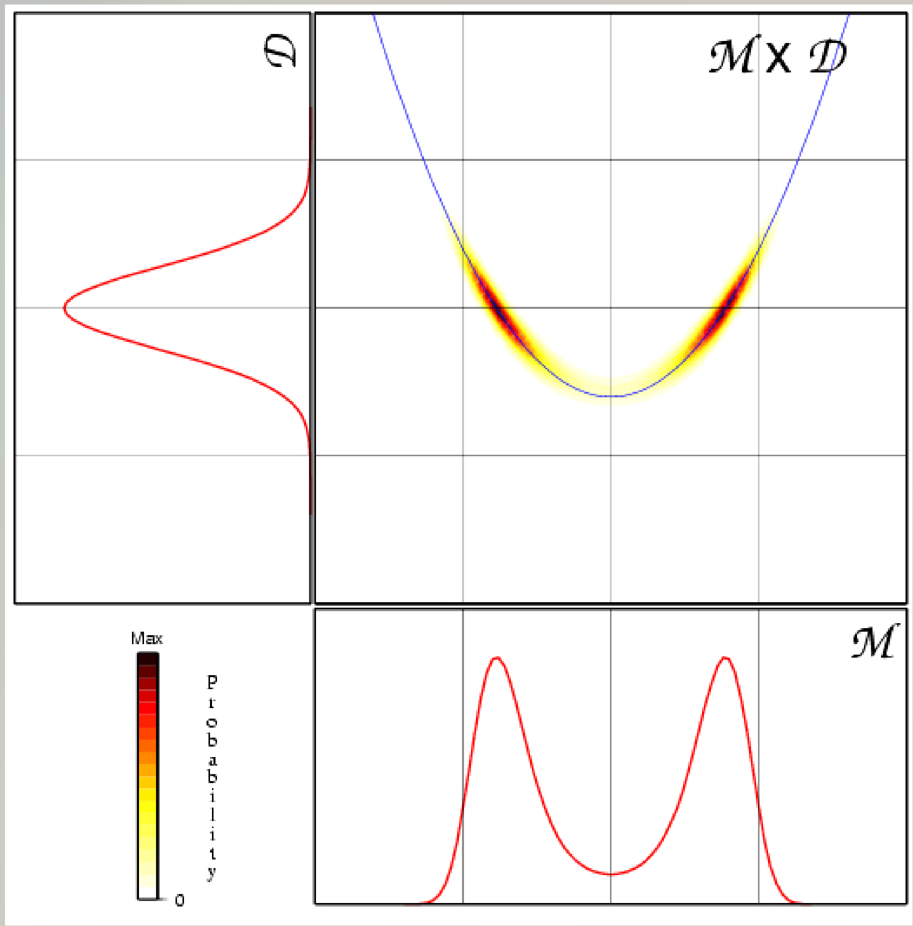
$$q(\mathbf{m},\mathbf{d}) = \mu_M(\mathbf{m})\rho(\mathbf{d} - G(\mathbf{m}))$$

$$\sigma(\mathbf{m}) = f(\mathbf{m})$$

Example - exact theory (linear)

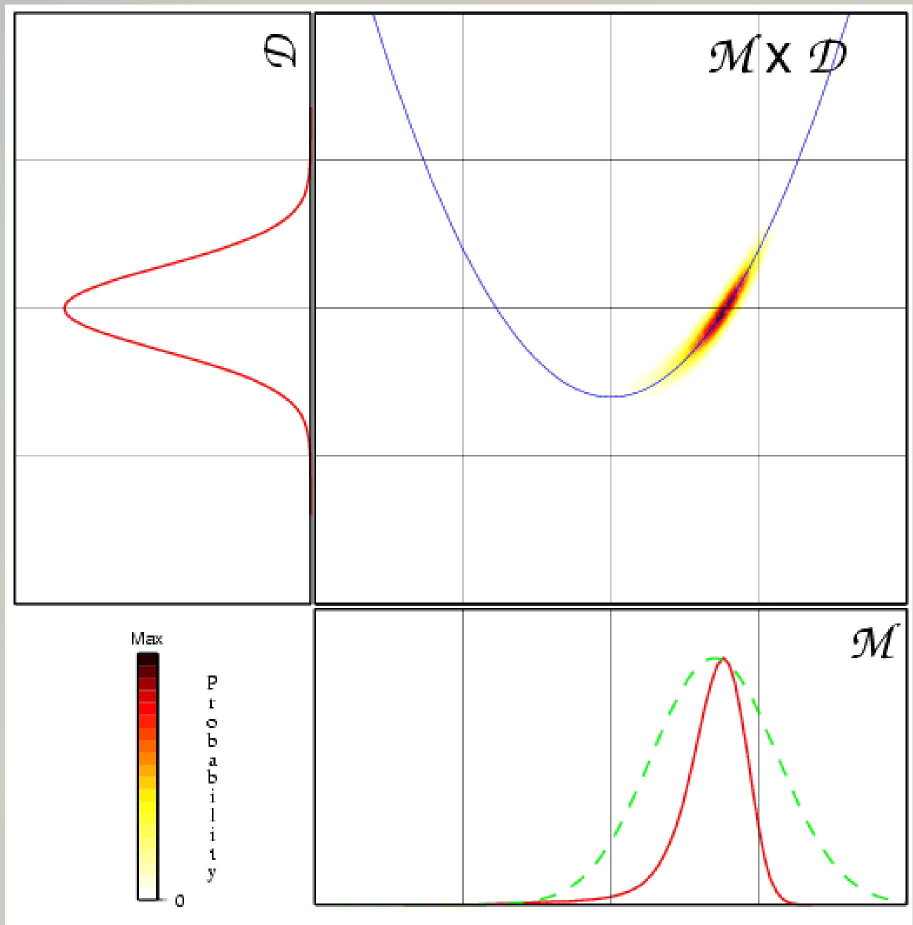


Example - exact theory (non-linear)

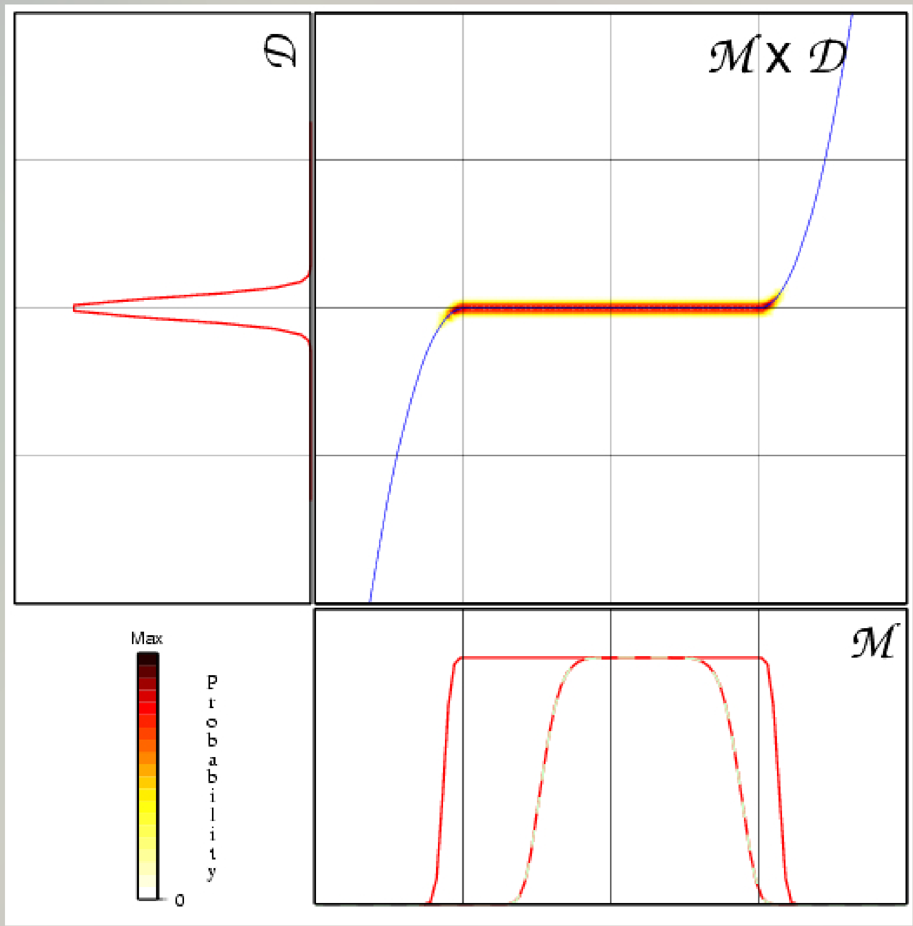


Example - exact theory (non-linear)

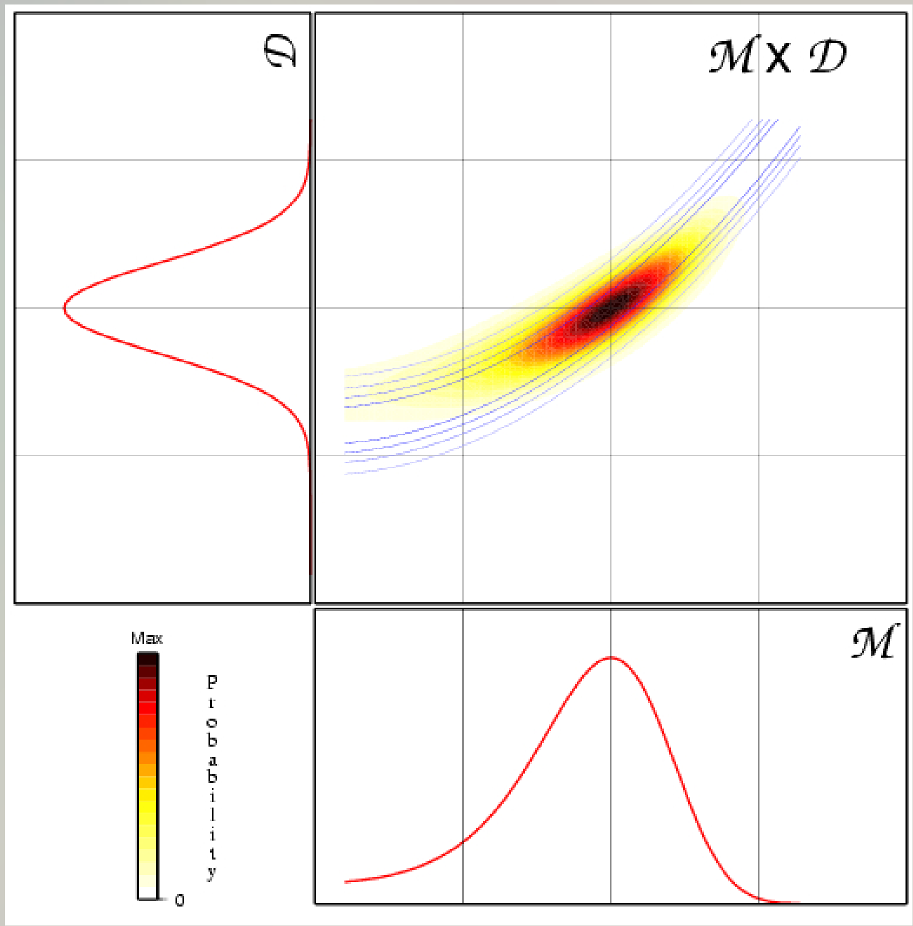
role of *a priori*



Example - exact theory (null space)

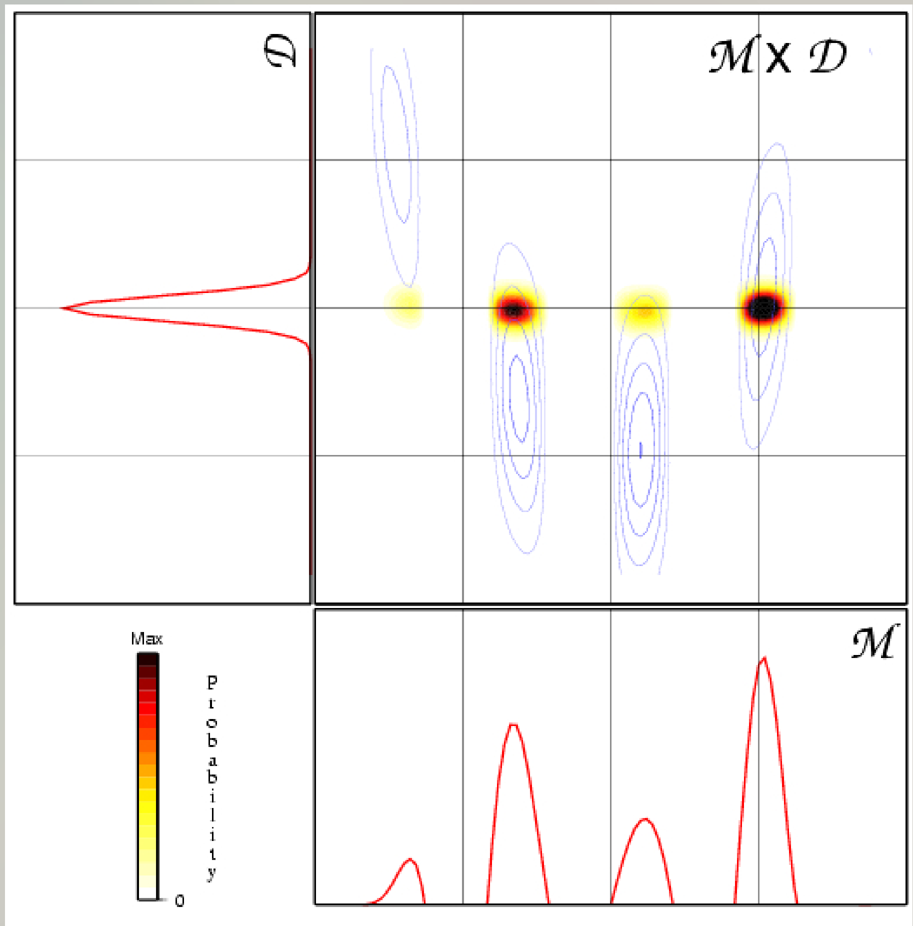


Example - approximate theory

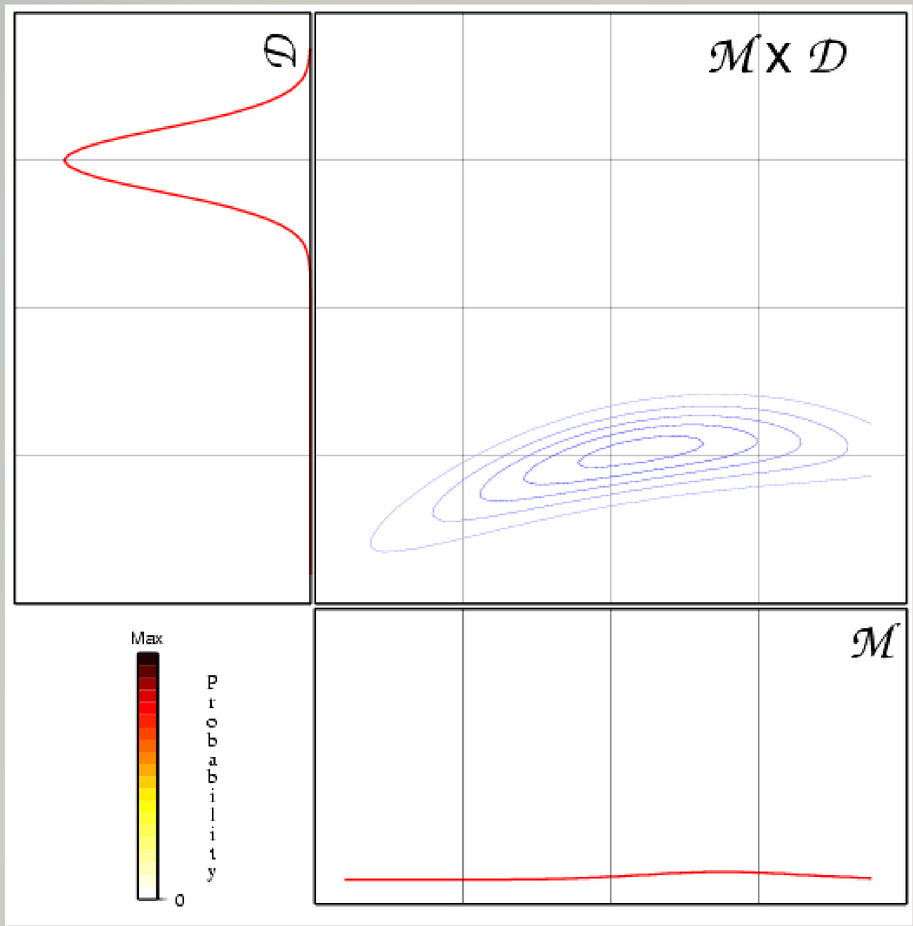


Example - “pathological” theory

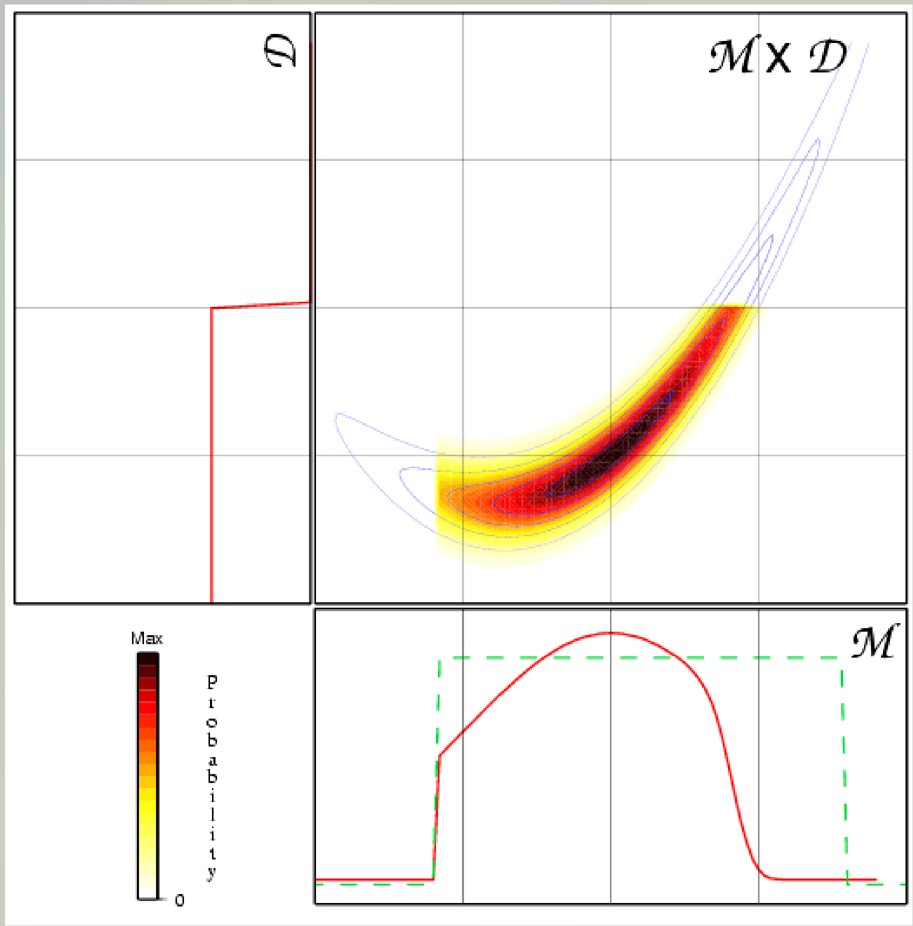
multi-modal (non-unique) solution



Example - theory inconsistent with data



Example - large data errors (range of values)



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