

Probabilistic Inverse Theory

Lecture 3

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Linear Invers Problem

$$\mathbf{m} = (m_1, m_2, \dots, m_M) \in \mathcal{M}$$

$$\mathbf{d} = (d_1, d_2, \dots, d_N) \in \mathcal{D}$$

$$\mathbf{d}(\mathbf{m}) = \mathbf{G} \cdot \mathbf{m}$$

$$d_i = \sum_j g_{ij} m_j$$

Problem: How to find $\mathbf{m}$ provided $\mathbf{d}$ is known ?

Linear problem

$$\mathbf{d} = \mathbf{G} \cdot \mathbf{m} \quad / \times \mathbf{G}^T$$

$$\mathbf{G}^T \cdot \mathbf{d} = \mathbf{G}^T \mathbf{G} \cdot \mathbf{m}$$

$$\mathbf{G}^T \mathbf{G} \Rightarrow \mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}$$

$$\mathbf{m}^{est} = (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot \mathbf{d}^{obs}$$

Linear problem

$$\mathbf{m} \sim \mathbf{m}^{apr}; \quad \bar{\mathbf{d}} = \mathbf{G} \cdot \mathbf{m}^{apr}$$

$$\mathbf{d} = \mathbf{G} \cdot \mathbf{m} \quad / \quad - \bar{\mathbf{d}}$$

$$\mathbf{d} - \bar{\mathbf{d}} = \mathbf{G} \cdot (\mathbf{m} - \mathbf{m}^{apr})$$

$$\mathbf{G}^T \cdot (\mathbf{d} - \bar{\mathbf{d}}) = \mathbf{G}^T \mathbf{G} \cdot (\mathbf{m} - \mathbf{m}^{apr})$$

$$\mathbf{G}^T \mathbf{G} \implies \mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}$$

$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

Linear problem

λ - arbitrary parameter $\implies$ subjective solution

$$\mathbf{m}^{est} = \mathbf{m}^{est}(\lambda)$$

1) $\lambda \rightarrow \infty$

$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda} \mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

2) $\lambda \rightarrow 0$ (if exists)

$$\mathbf{m}^{est} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \cdot (\mathbf{d}^{obs})$$

Linear problem - resolution analysis

$$\mathbf{m} = (m_1, m_2, \dots, m_M)$$

- ◆ how accurately one can estimate m_i ?
- ◆ are m_i, m_j correlated ?

$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

$$\mathbf{d}^{obs} = \mathbf{d}^{true}; \quad \mathbf{d}^{true} = \mathbf{G} \cdot \mathbf{m}^{true}$$

$$\mathbf{m}^{est} - \mathbf{m}^{apr} = \mathbf{R}(\lambda) \cdot (\mathbf{m}^{true} - \mathbf{m}^{apr})$$

Linear problem - resolution analysis

definition : $\frac{\mathbf{A}}{\mathbf{B}} = \mathbf{A}\mathbf{B}^{-1}$

$$\mathbf{R} = \frac{\mathbf{G}^T \mathbf{G}}{\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}}$$

$$\mathbf{m}^{est} = \mathbf{R}(\lambda) \cdot \mathbf{m}^{true}$$

$\mathbf{m}^{est}$ are filtered estimators of $\mathbf{m}^{true}$

Linear problem - well defined problem

$$\det(\mathbf{G}^T \cdot \mathbf{G}) \neq 0 \longrightarrow \lambda = 0$$

$$\mathbf{R} = \frac{\mathbf{G}^T \mathbf{G}}{\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}}$$

$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{m}^{est} = \mathbf{m}^{true}$$

Linear problem - under-determined problem

$$\det(\mathbf{G}^T \cdot \mathbf{G}) = 0 \longrightarrow \lambda \neq 0$$

$$\mathbf{R} = \frac{\mathbf{G}^T \mathbf{G}}{\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}}$$

$$\mathbf{R} \neq \mathbf{I}$$

$$\mathbf{m}^{est} \neq \mathbf{m}^{true}$$

Linear problem - regularization

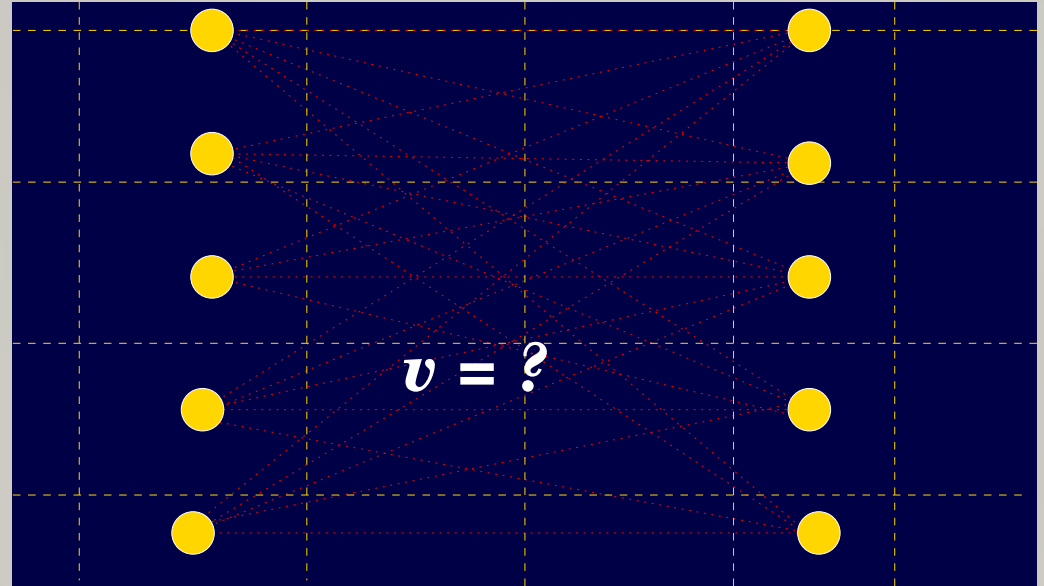
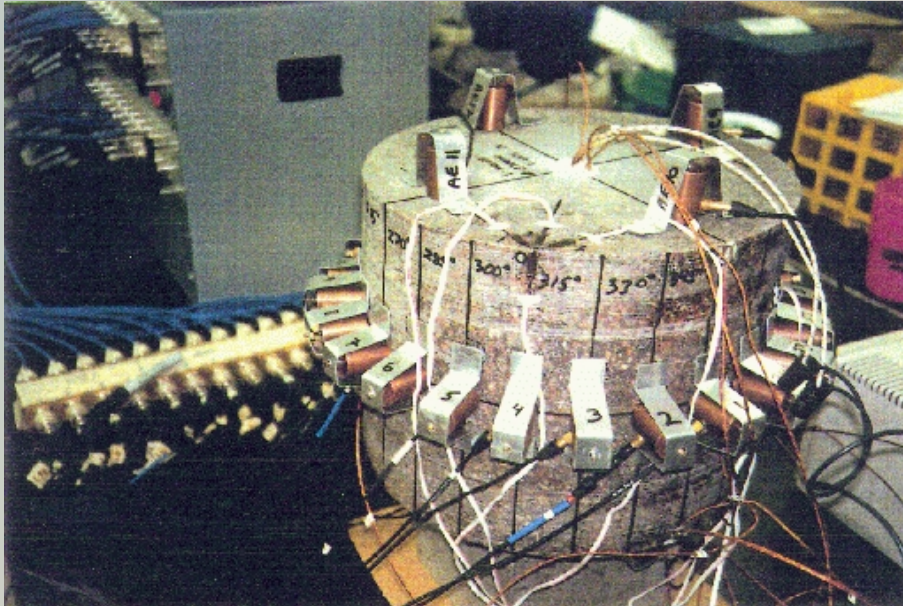
1) $\lambda \rightarrow \infty$

$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda} \mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

2) $\lambda \rightarrow 0$ (if exists)

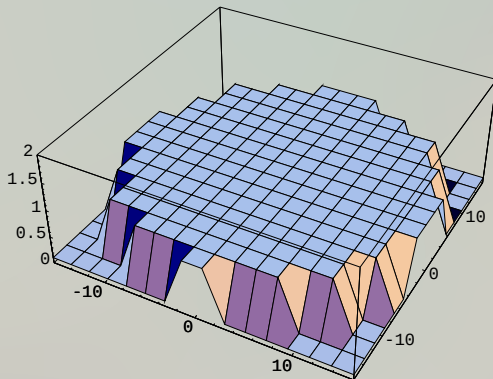
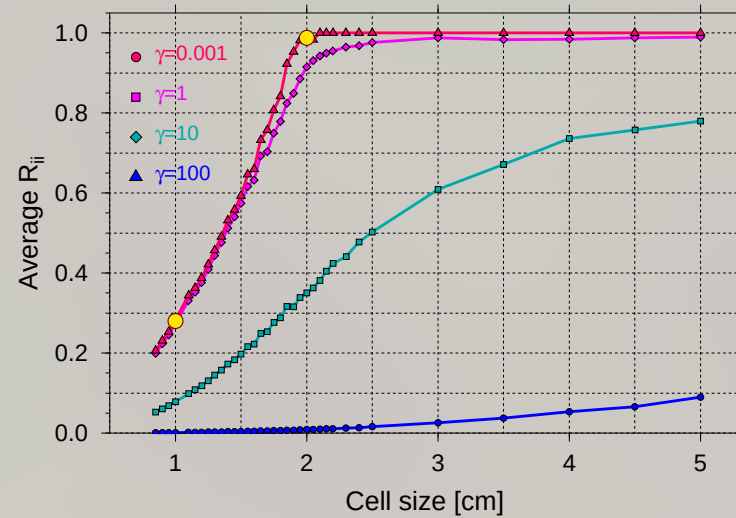
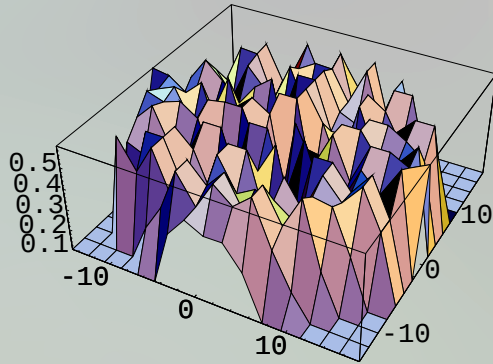
$$\mathbf{m}^{est} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \cdot (\mathbf{d}^{obs})$$

Linear problem - illustration



$$\Delta t = G \cdot s$$

Linear problem - resolution illustration



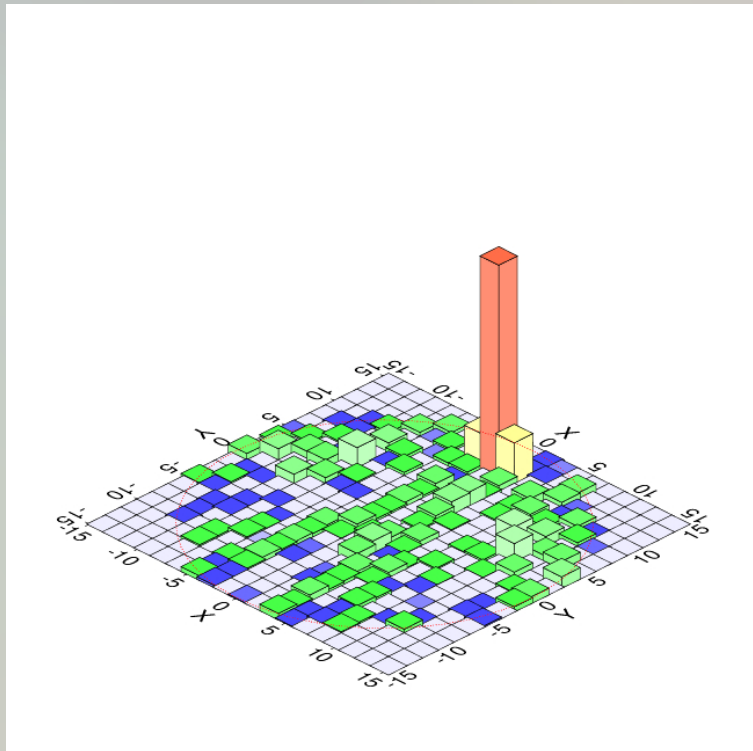
$$\mathbf{R} = \frac{\mathbf{G}^T \mathbf{G}}{\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}} \quad \lambda \rightarrow \infty \quad \mathbf{R} \sim \frac{1}{\lambda} \dots$$

$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda} \mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

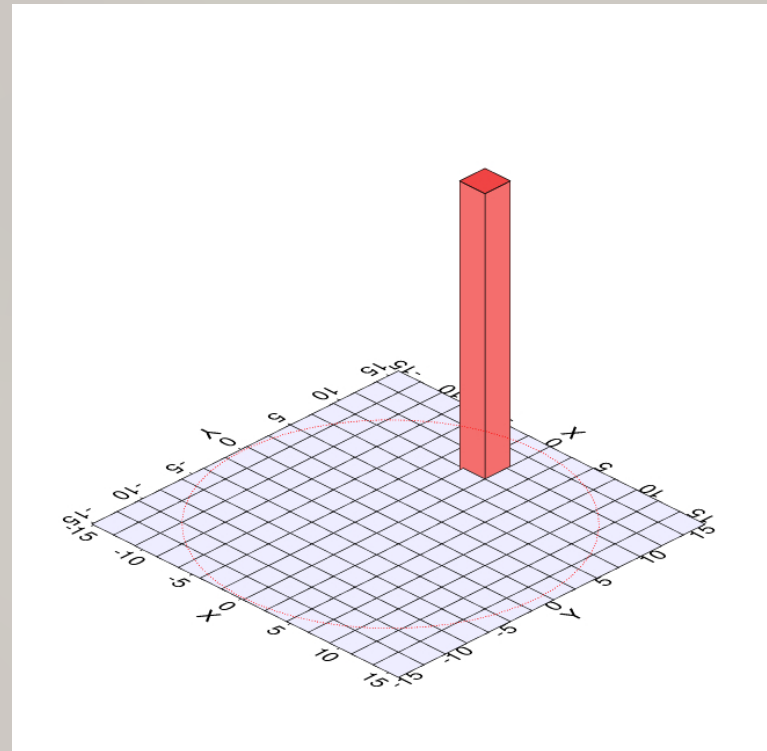
$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot (\dots)$$

Linear problem - correlations illustration

$$\lambda = 100$$



$$\lambda = 1$$



Linear problem - data error sensitivity

$$\mathbf{m}^{apr} = \mathbf{m}^{true}; \quad \mathbf{d}^{obs} = \mathbf{d}^{true} + \delta$$

$$\mathbf{m}^{est}(\lambda) = \mathbf{m}^{true} + (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{d}^{true})$$

$$\mathbf{m}^{est} = \mathbf{m}^{true} + (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot \delta$$

$$\Delta \mathbf{m}^{est} = (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot \delta$$

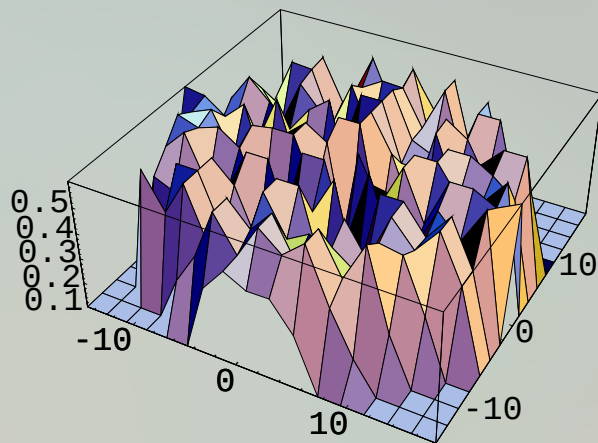
Linear problem - error propagation ($\mathbf{m}^{apr} \neq \mathbf{m}^{true}$)

$$\mathbf{m}^{est}(\lambda) = \mathbf{m}^{apr} + (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot \left\{ \begin{array}{l} (\mathbf{d}^{true} - \mathbf{G} \cdot \mathbf{m}^{apr}) \\ +\delta \end{array} \right.$$

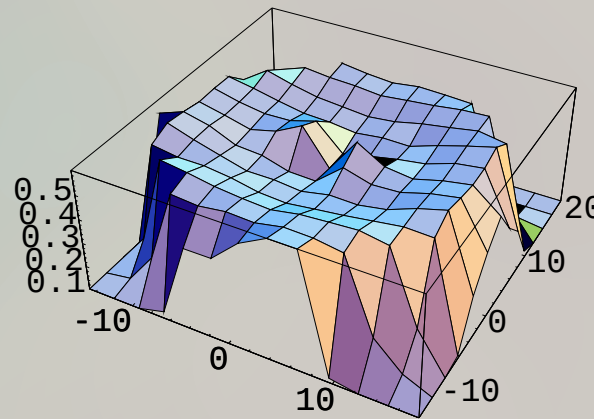
Przykład 1D: $d = gm$

$$\mathbf{m}^{est}(\lambda) = \mathbf{m}^{apr} + \frac{g}{g^2 + \lambda} \cdot \left\{ \begin{array}{l} (d^{true} - g \mathbf{m}^{apr}) \\ +\delta \end{array} \right.$$

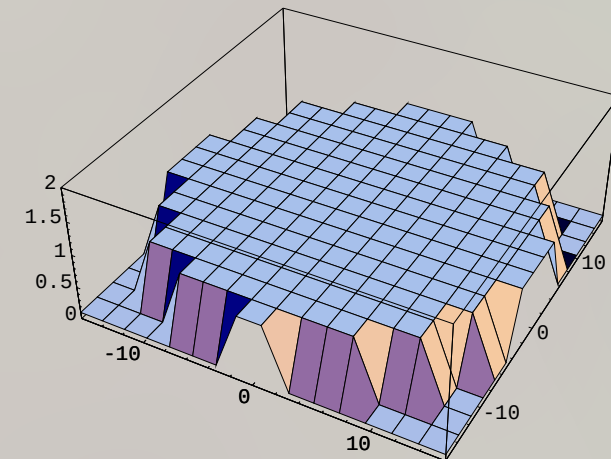
Linear problem - error propagation



$\lambda \sim 0.01$



$\lambda \sim 1$



$\lambda \sim 100$

Linear problem- Residua

$$\mathbf{d}^{est} = \begin{cases} + [\mathbf{I} - \mathbf{G}(\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T] \mathbf{G} \mathbf{m}^{apr} \\ + \mathbf{G}(\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \mathbf{d}^{true} \\ + \mathbf{G}(\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \delta \end{cases}$$

$$\mathbf{d}^{est} - \mathbf{d}^{true} = \begin{cases} + \lambda \mathbf{I} (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T (\mathbf{G} \mathbf{m}^{apr} - \mathbf{d}^t) \\ + \mathbf{G} (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \delta \end{cases}$$

Linear problem - inversion efficiency

Final verification of inversion results: how accurately $\mathbf{m}^{est}$ “reproduces” $\mathbf{d}^{obs}$?

RMS value of data residua: $r_i = \mathbf{d}_i^{obs} - \mathbf{d}_i(\mathbf{m}^{est})$

$$r_{rms} = \frac{1}{N} \sum_i \sqrt{(\mathbf{d}_i^{obs} - \mathbf{d}_i(\mathbf{m}^{est}))^2}$$

Linear problem - inversion efficiency (practical approach)

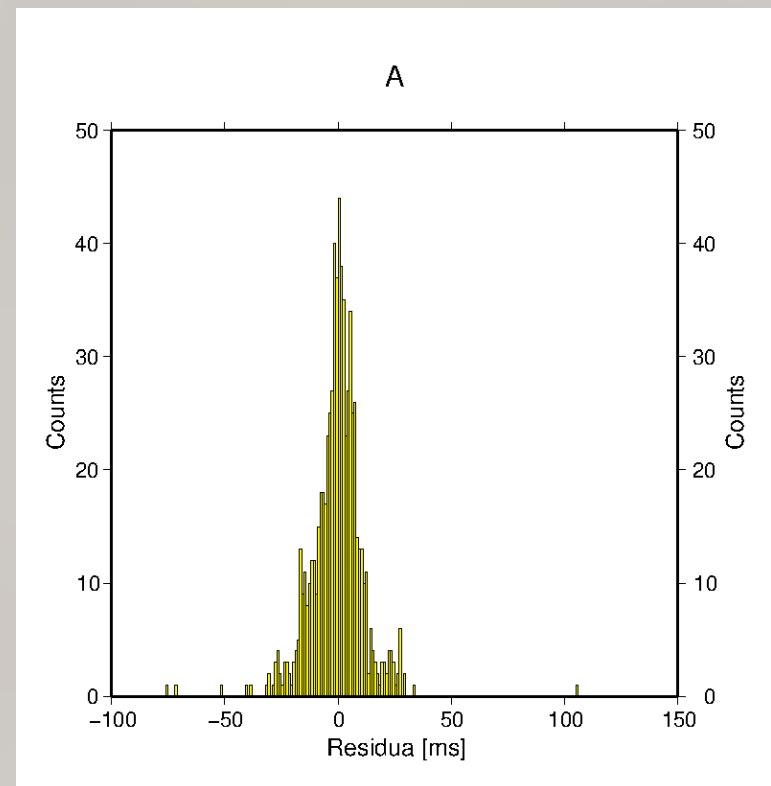
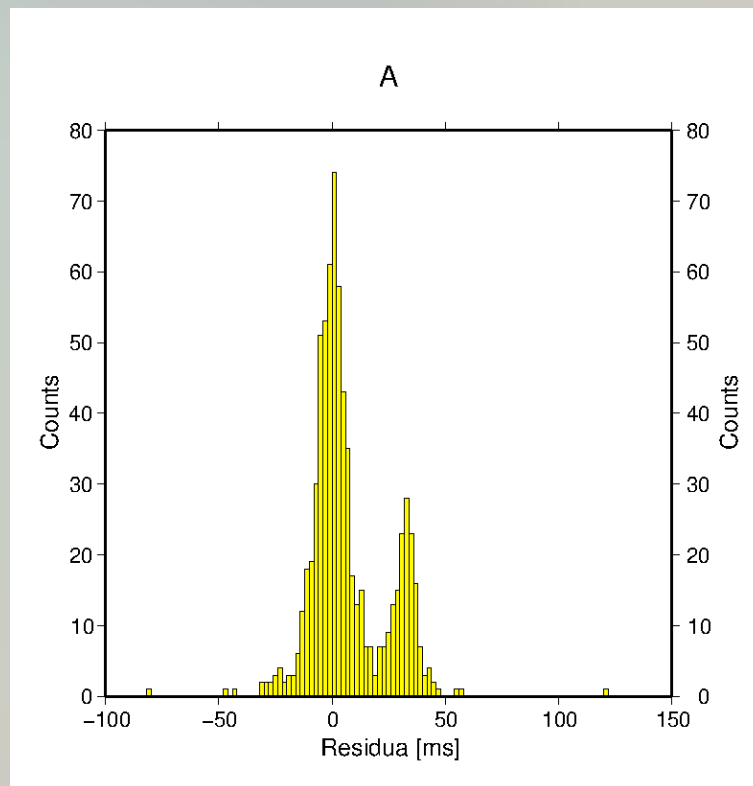
$$\bar{r} = \frac{1}{N} \sum_1^N r_i$$

$$\delta r_i = \sqrt{\frac{1}{N} (r_i - \bar{r})^2}$$

Provides a qualitative measurer of inversion “efficiency” and simultaneously data quality

Linear problem - inversion efficiency (better approach)

compare residua histogram for m^{apr} and m^{est} models



Linear problem - dependences on m^{apr}

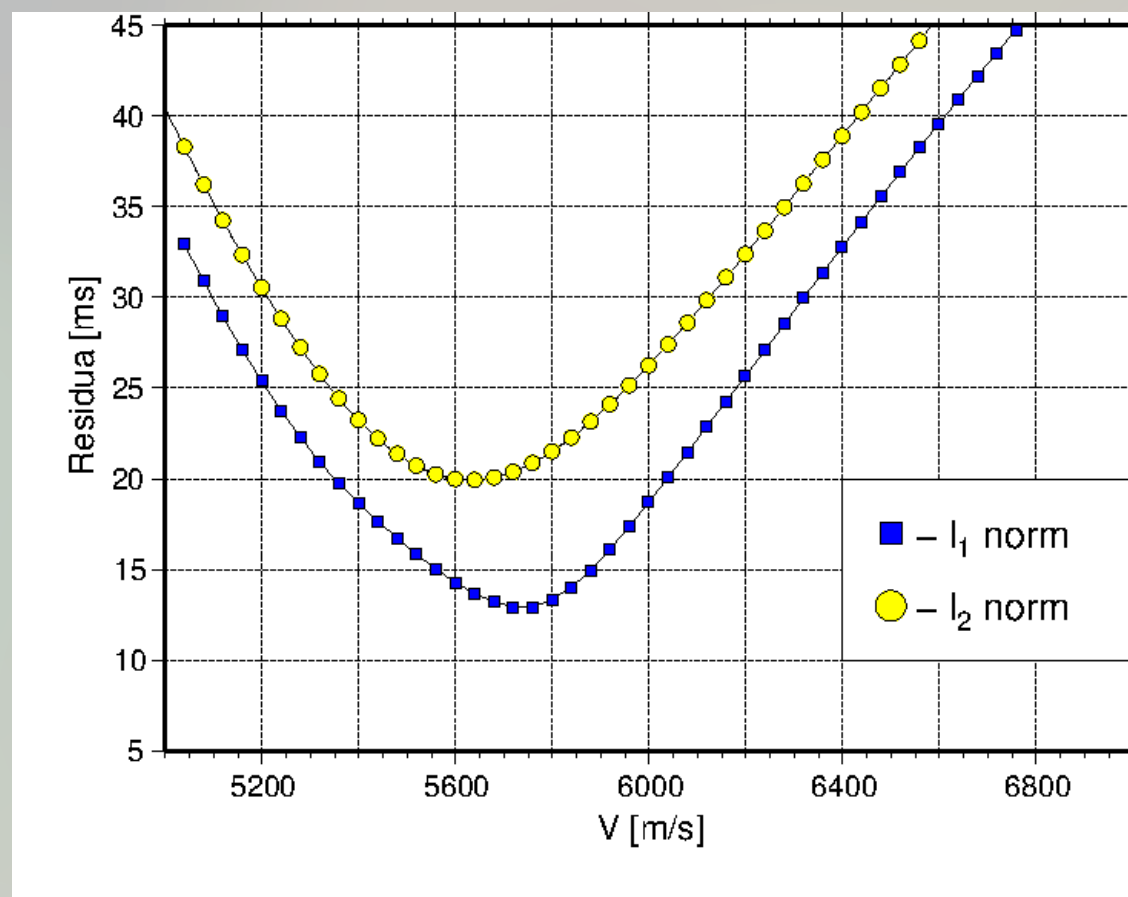
$$d = g \cdot m \quad d^{obs} = d^{true} + \delta$$

$$m^{est}(\lambda) = \frac{\lambda}{g^2 + \lambda} m^{apr} + \frac{g}{g^2 + \lambda} (d^{true} + \delta)$$

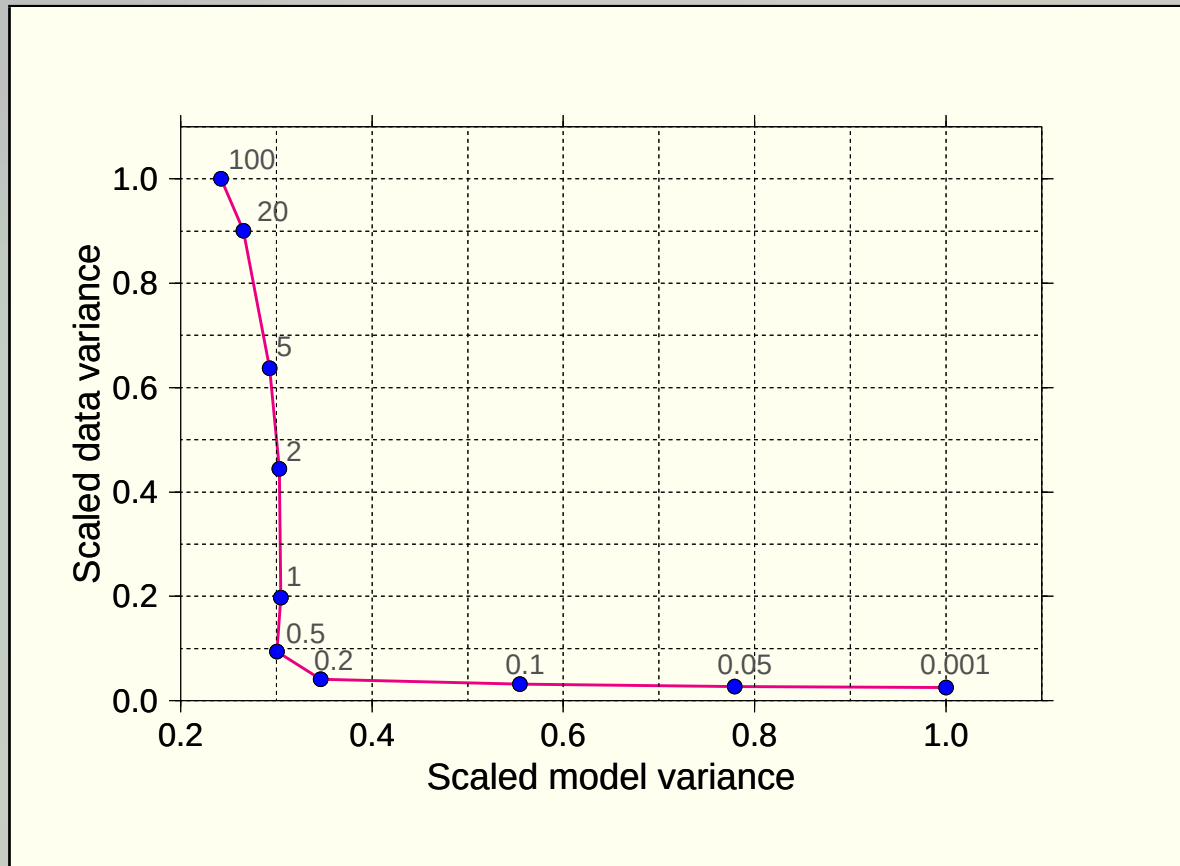
$$d^{est} = d^{true} + \frac{\lambda}{g^2 + \lambda} (g \cdot m^{apr} - d^{true}) + \frac{g^2}{g^2 + \lambda} \delta$$

$$\Delta m^{est} = m^{est} - m^{true} = \frac{g}{g^2 + \lambda} \delta$$

Linear problem - selecting m^{apr}



Linear problem - selecting λ



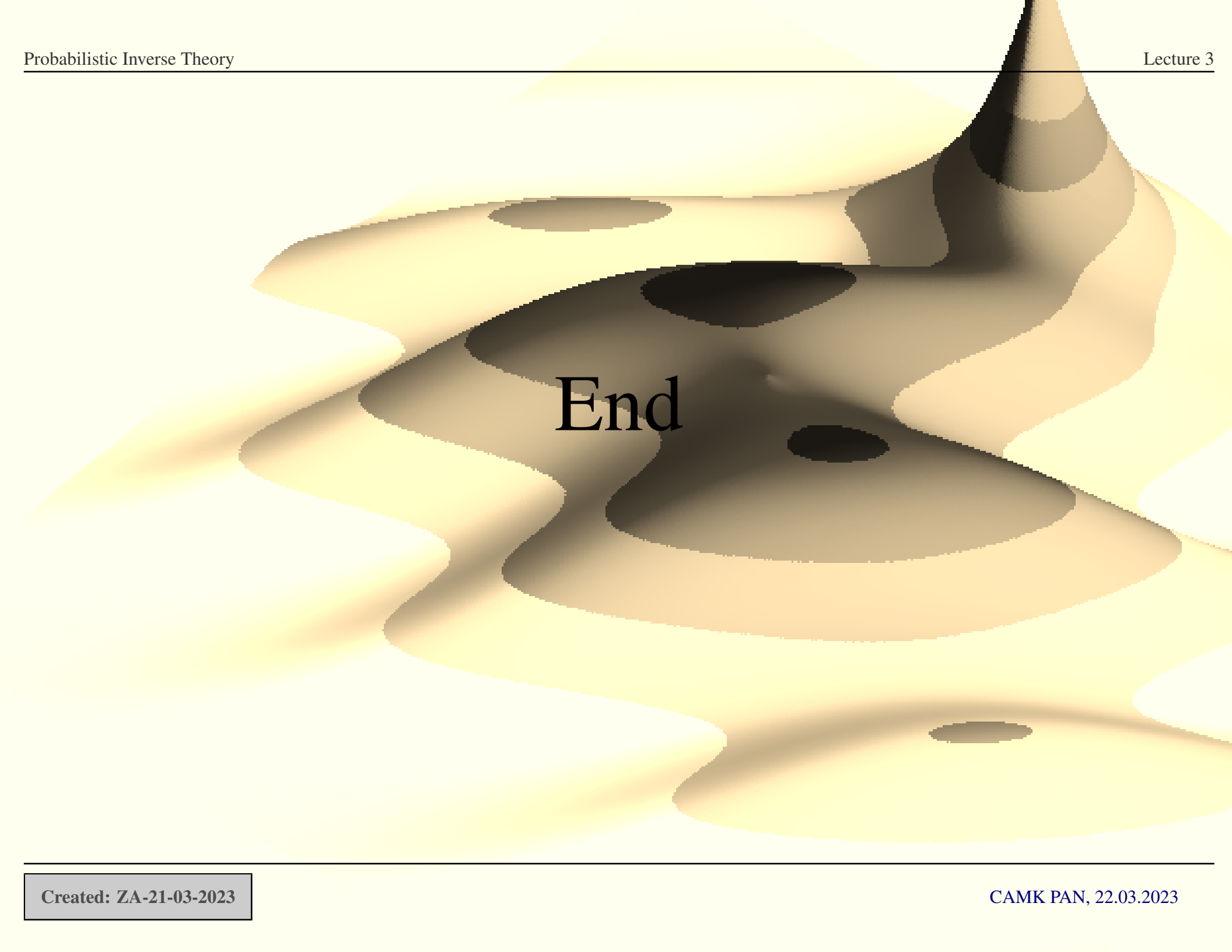
Linear problem - single λ ?

$$\mathbf{G}^T \mathbf{G} \implies \mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}$$

should we use single “global value of λ ?

diagonalize $\mathbf{G}^T \cdot \mathbf{G}$

$$\begin{pmatrix} \cdot & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 0 \\ & & & & & 0 \end{pmatrix}$$



End