Probabilistic Inverse Theory

Lecture 2

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15.03.2023

Forward problems

This is a class of problems when one can try to understand qualitatively some observed phenomena or to predict new ones. The ultimate goal of solving such problems is an ability to predict (calculate) behavior of a system in hand.

Question: why...

Inverse problems

Inverse problems are tasks when one can try to grasp quantitative description of given system in hand or observed processes. The goal is not to provide a general description how the system behaves but to infer information on it allowing its realistic description.

Question: What is ....

Inverse problems - comments

- to solve inverse problems we HAVE TO to be able to solve a corresponding forward problem
- for given forward problem there can be many different inverse problems - we can pose different questions
- solution of inverse problems - some characteristics of studied object/process

Inverse theory - basic mathematical notions

Physical system:

$$p_1, p_2, \cdots p_K$$

parameters:

$$\mathbf{m} = (m_1, m_2, \cdots m_M)$$

predicted (measureable) quantities:

$$\mathbf{d} = (d_1, d_2, \cdots d_N)$$

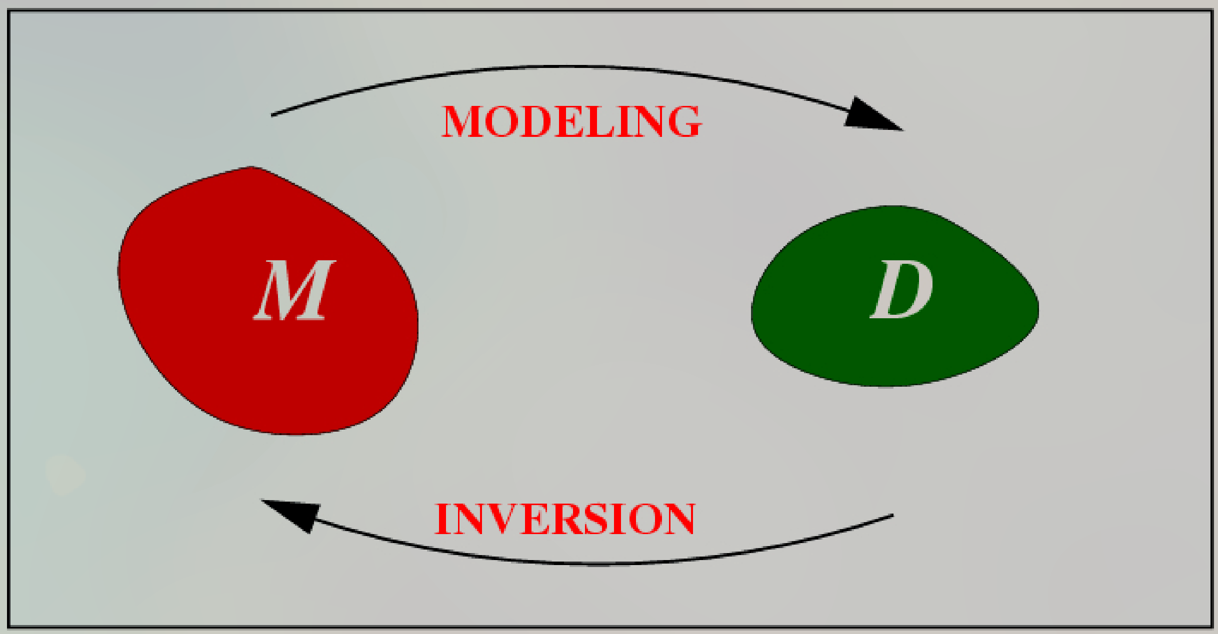
fixed parameters (*a priori* fixed):

$$\mathbf{m}^{fix} = (u_1, u_2, \cdots)$$

Forward modelling: $\mathbf{d}^{th} = f(\mathbf{m}, \mathbf{m}^{fix})$

Inversion : $\mathbf{m} = ... \pm ...$

Inverse theory - cartoon



Inverse problems - different approaches

1. (mathematics) parameter estimation
2. (physics) indirect measurements
3. (general) inference

Parameter estimation tasks

Modelling:

$$\mathbf{m} \longrightarrow \mathbf{d}^{th} = \mathbf{G}(\mathbf{m})$$

Inversion:

$$\mathbf{d}^{obs} \longrightarrow \mathbf{d}^{th}$$

$$\mathbf{d}^{obs} = \mathbf{G}(\mathbf{m}^{est})$$

$$\mathbf{m}^{est} \longleftarrow \mathbf{d}^{obs}$$

Measurements - information about the world

Counting:

- events
- number of elements
- quntified parameters

Unit counting :

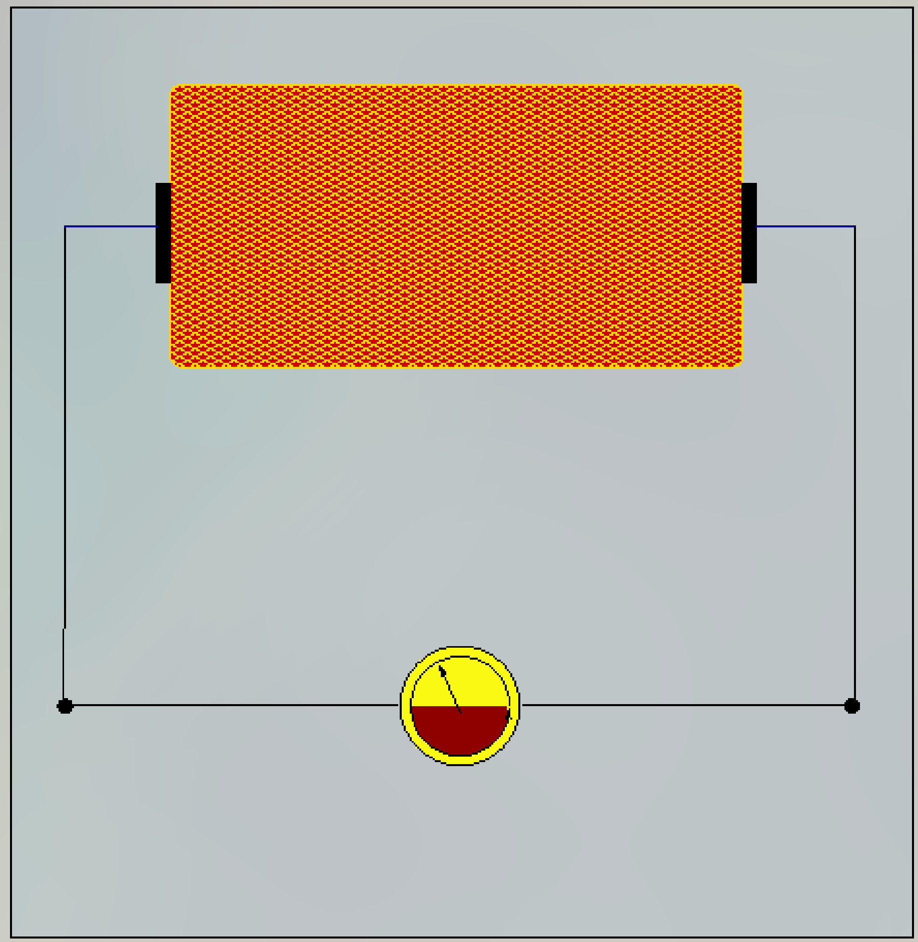
- mass
- spatial position
- temperature

Inverse problem - indirect measurements

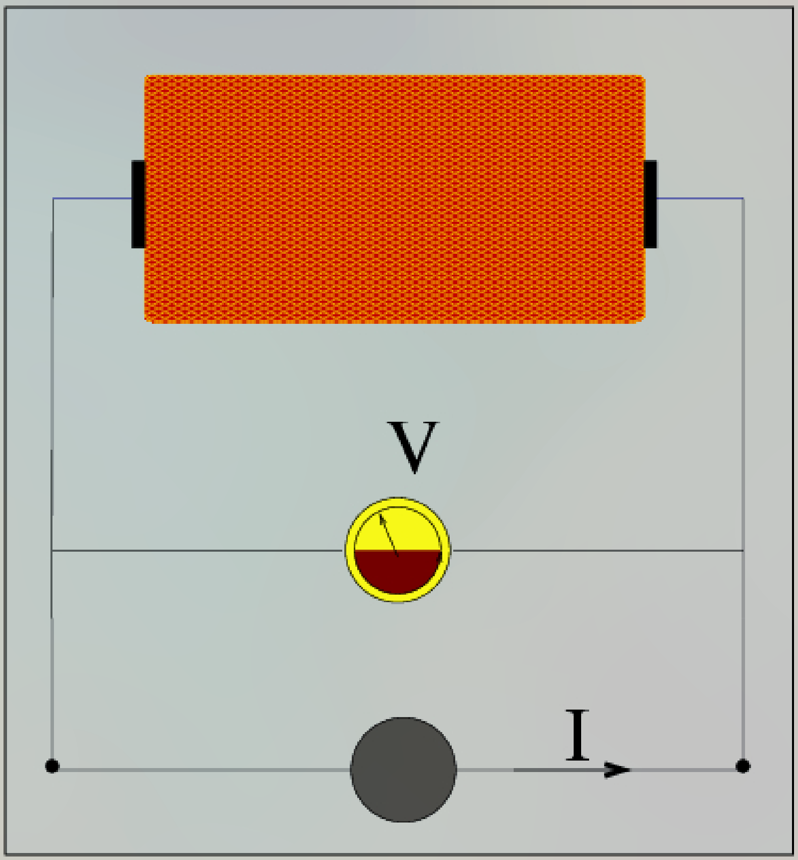
Quantities that cannot be measured directly

- Earth mass
- Temperature in the Earth interior
- Earthquake location
- Randomness of the process
- Missing equipment, ...

Example - direct measurement



Example - lack of appropriate devices



Data: V

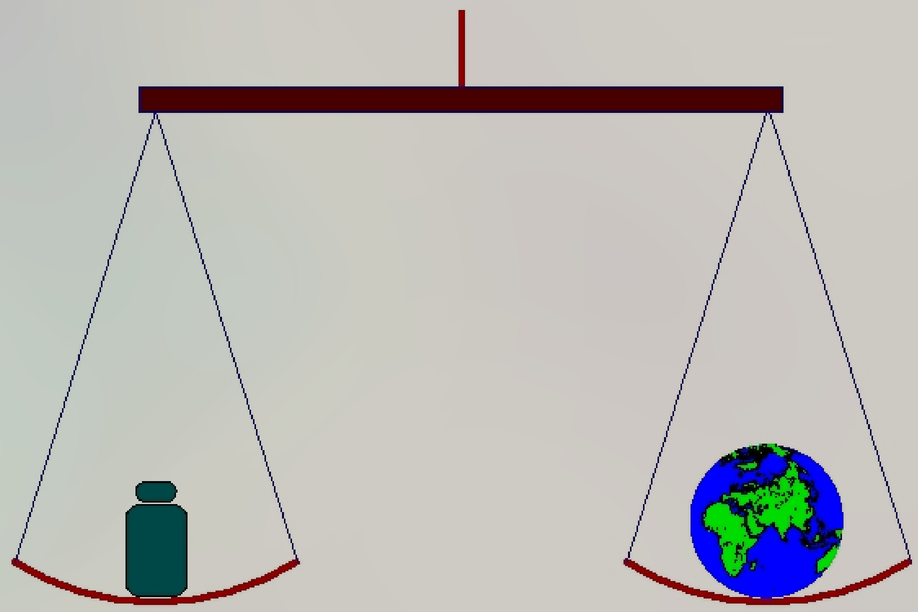
Theory (Ohm law + ...)

$$V = I\,\rho + n(\rho, \dot{\rho})$$

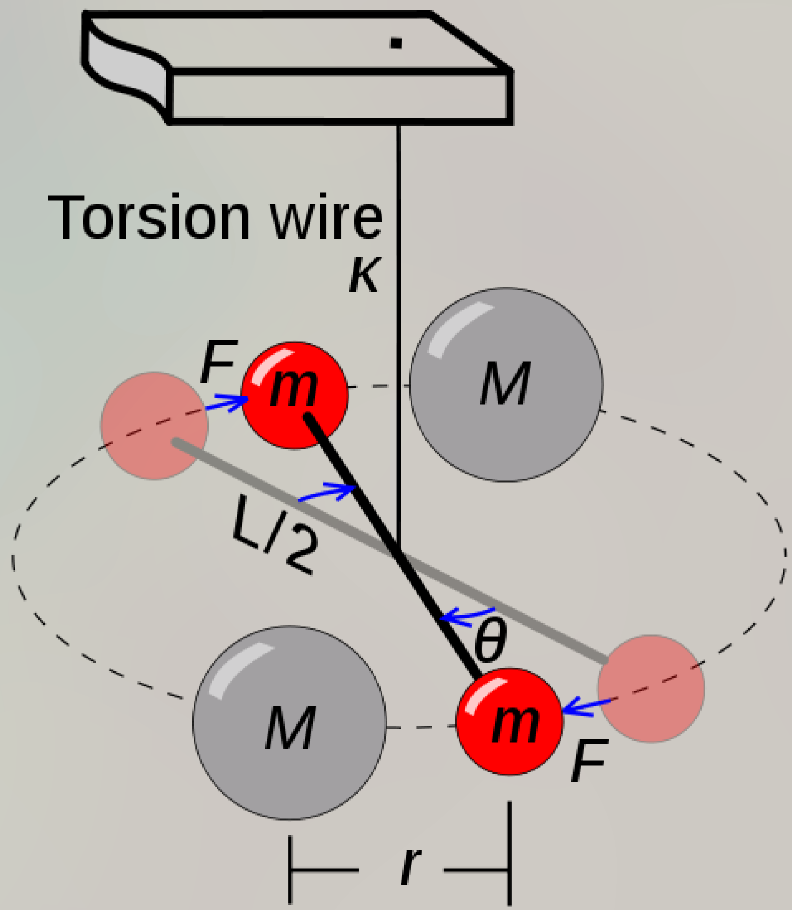
A priori

$$1 < \rho < 10$$

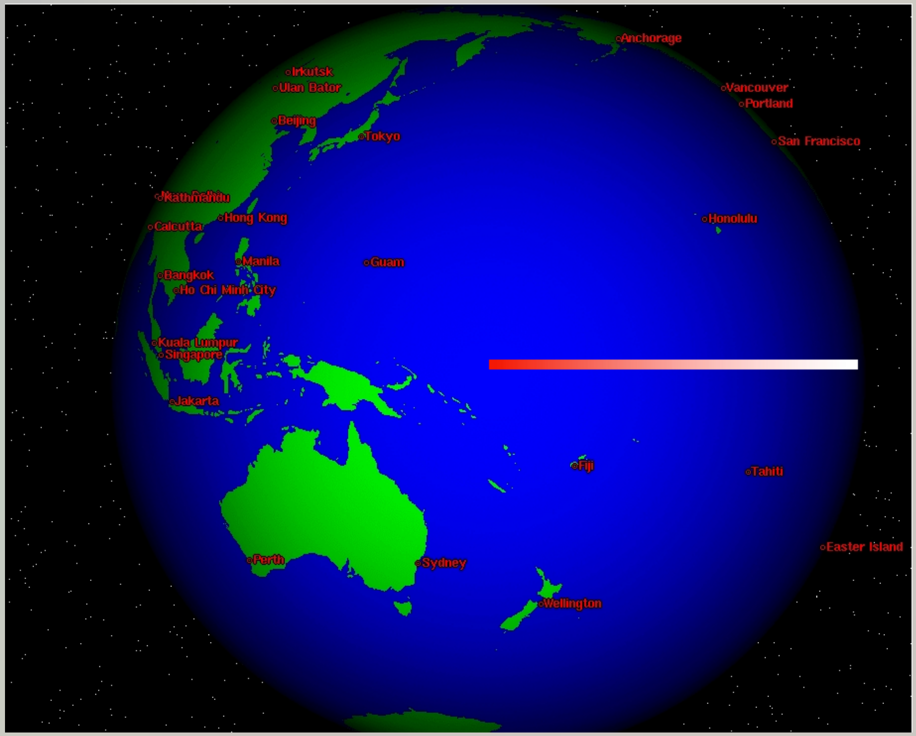
Example - methodological limitations



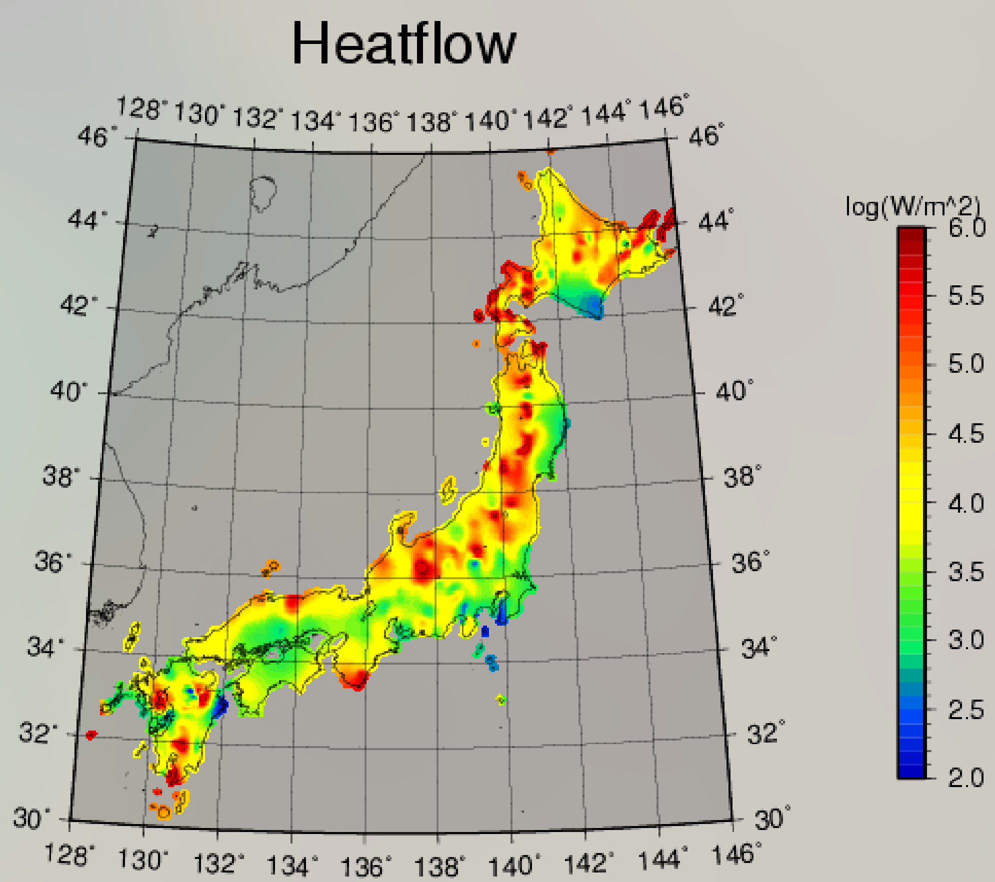
Solution: Cavedish torsinal scales



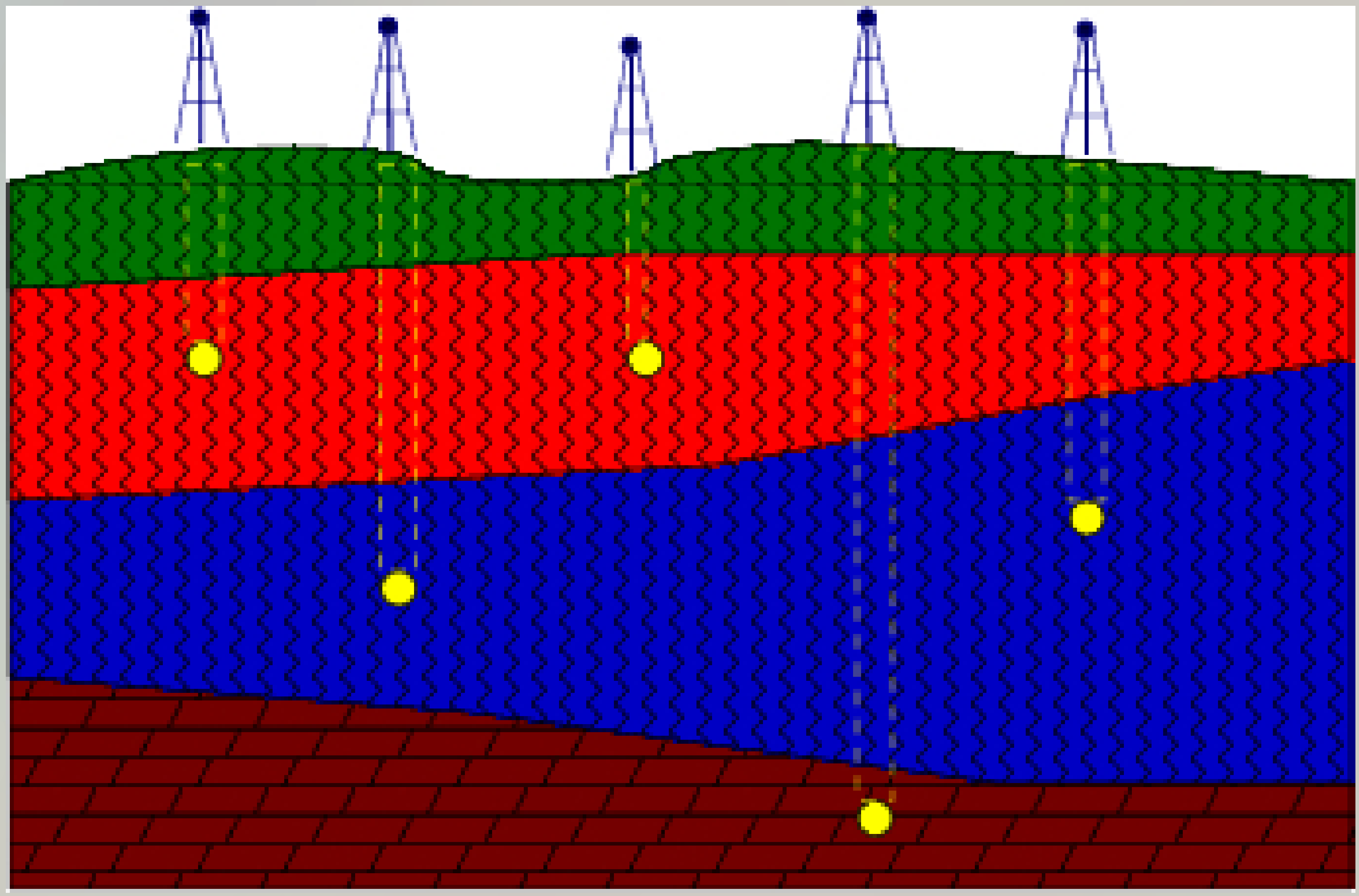
Example - remote location



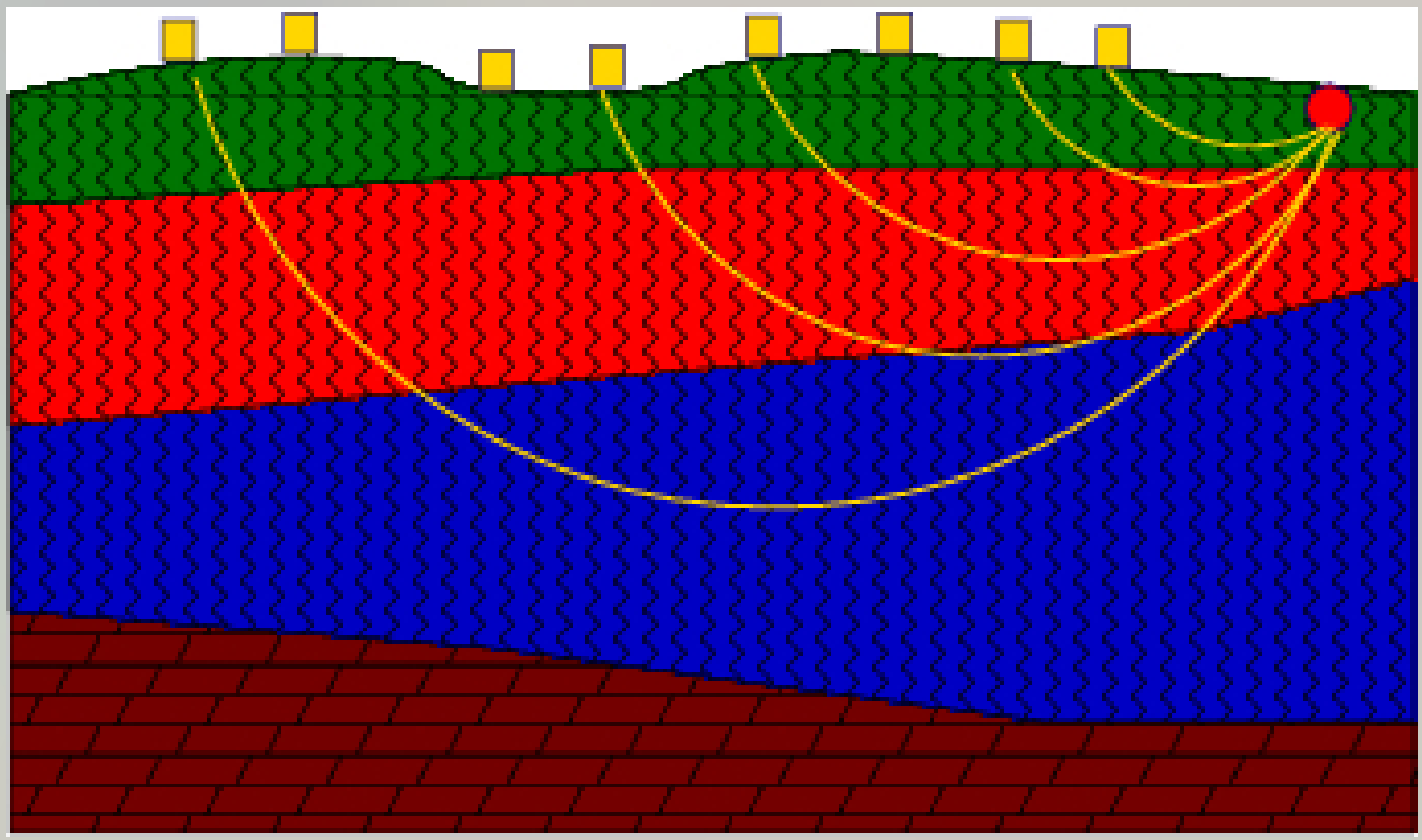
Solution: heat transfer measurement



Example - spatial extension

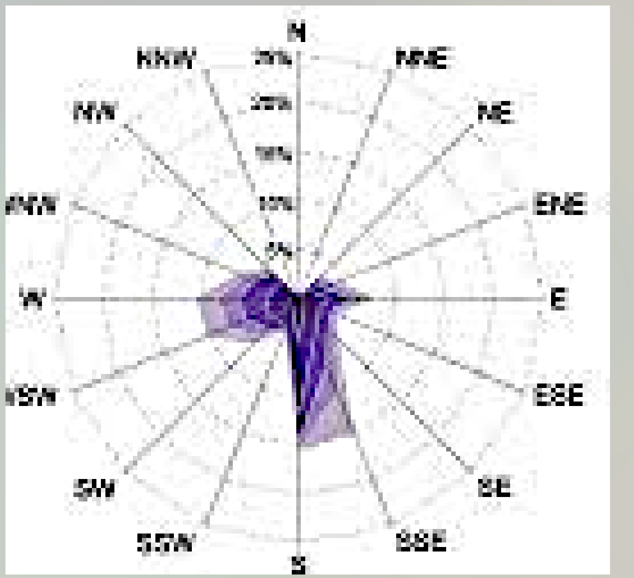


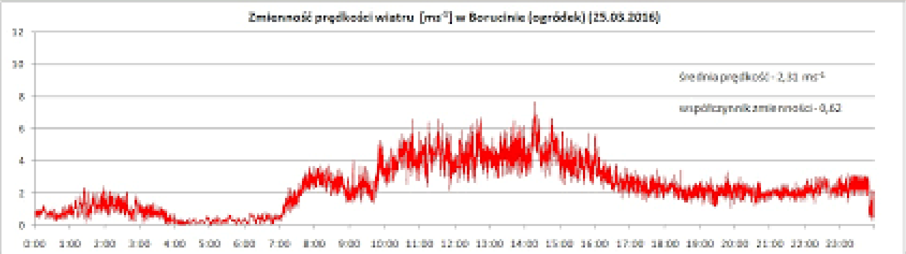
Solution: “continuous sampling”



Example - noisy measurement

Solution: proper data processing





$$\mathbf{d}^{obs} = \mathbf{d} + \mathbf{d}^{n}$$

Solution: proper data processing

Process

$$s(t) = \mathbf{G}(t;\, \mathbf{m}; init\_cond)$$

Initial condition, e.g: $s(t_o) = \mathbf{d}^{obs}$

Prediction $t = t_d > t_o$

$$s_{th}(t) = \mathbf{G}(t,\, |\mathbf{m}; \mathbf{d}^{obs}_{t_o})$$

Verification $(t = t_d)$

$$s^{obs}(t_d) \equiv s_{th}(t_d) \quad ???$$

NO

Solution: data assimilation method

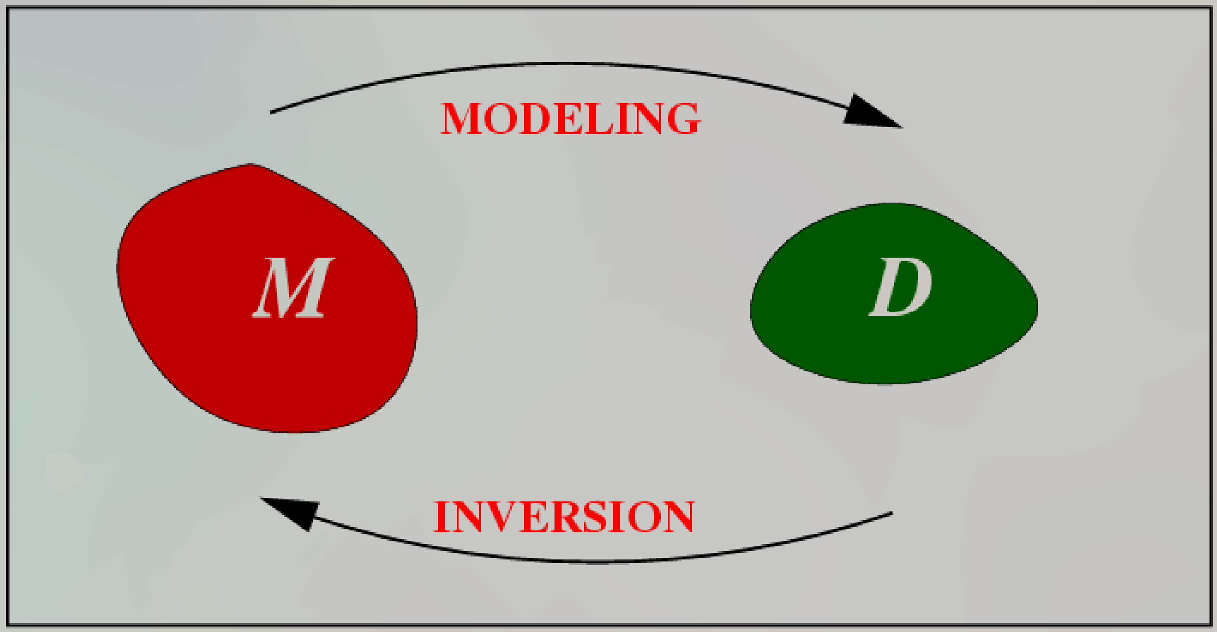
Use

$$r = ||s^{obs}(t_d) - s_{th}(t_d)||$$

to improve prediction for $t > t_d$ (e.g. adjust $\mathbf{m}$)

- Kalman filtering
- inverse filtering
- Bayesian rules
- ....

Abstraction



Inverse problem - mathematical point of view

Modelling:

$$\mathbf{m} \longrightarrow \mathbf{d}^{th} = \mathbf{G}(\mathbf{m})$$

Inversion:

$$\mathbf{d}^{obs} \longrightarrow \mathbf{m}^{est}$$

Inverse problem - physical point of view

inverse problem

$\Updownarrow$

indirect measurement

parameter estimation

Linear Invers Problem

$$\mathbf{m} = (m_1, m_2, \cdots m_M) \in \mathcal{M}$$

$$\mathbf{d} = (d_1, d_2, \cdots d_N) \in \mathcal{D}$$

$$\mathbf{d}(\mathbf{m}) = \mathbf{G} \cdot \mathbf{m}$$

$$d_i = \sum_j g_{ij} m_j$$

Problem: How to find **m** provided **d** is known ?

Comment

The solution of inverse problem for given observational data $\mathbf{d}^{obs}$ will in generall (not always) be denoted as

$$\mathbf{m}^{est}$$

and reefered to as “estimated values”

Thus, for linear case

$$\mathbf{d}^{obs} = \mathbf{G} \cdot \mathbf{m}^{est}$$

Solution - 1D

if $N = M = 1$

$$d = G\, m$$

$$m = G^{-1} d$$

Yes, provided $G \neq 0$

Otherwise d and m are not related each other

Measured d^{obs} does not constrain m !!!

Naive (incorrect) solution

Following 1D case

$$\mathbf{m}^{est} = \mathbf{G}^{-1} \cdot \mathbf{d}^{obs}$$

but $\mathbf{G}$ is in general non-square matrix $(N \neq M)$

$$\dim(\mathbf{G}) = N \times M$$

so $\mathbf{G}^{-1}$ does not exist

Linear problem

$$\mathbf{d} = \mathbf{G} \cdot \mathbf{m} \quad / \times \mathbf{G}^T$$

$$\mathbf{G}^T \cdot \mathbf{d} = \mathbf{G}^T \mathbf{G} \cdot \mathbf{m}$$

$$\mathbf{G}^T \mathbf{G} \implies \mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}$$

$$\mathbf{m}^{est} = (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot \mathbf{d}^{obs}$$

Linear problem

$$\mathbf{m} \sim \mathbf{m}^{apr}; \qquad \bar{\mathbf{d}} = \mathbf{G} \cdot \mathbf{m}^{apr}$$

$$\mathbf{d} = \mathbf{G} \cdot \mathbf{m} \quad / - \bar{\mathbf{d}}$$

$$\mathbf{d} - \bar{\mathbf{d}} = \mathbf{G} \cdot (\mathbf{m} - \mathbf{m}^{apr})$$

$$\mathbf{G}^T \cdot (\mathbf{d} - \bar{\mathbf{d}}) = \mathbf{G}^T\mathbf{G} \cdot (\mathbf{m} - \mathbf{m}^{apr})$$

$$\mathbf{G}^T\mathbf{G} \implies \mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}$$

$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

Linear problem

λ - arbitrary parameter $\Longrightarrow$ subjective solution

$$\mathbf{m}^{est} = \mathbf{m}^{est}(\lambda)$$

1) $\lambda \to \infty$

$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda}\,\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr}\right)$$

2) $\lambda \to 0$

$$\mathbf{m}^{est} \ldots$$

Linear problem non-uniqueness

$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr}\right)$$

Let assume that $\mathbf{G}^T\mathbf{G}$ is diagonal

1) $\det(\mathbf{G}^T\mathbf{G}) \neq 0$

$$\mathbf{m}^{est} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs}\right)$$

2) $\det(\mathbf{G}^T\mathbf{G}) = 0$

$$\mathbf{m}^{est} - \text{undefined}$$

λ - important often for some subset of $\mathbf{m}$ only

See you next week ...