An introduction to Physics of Seismic Sources

SP-3: Source representation cd.

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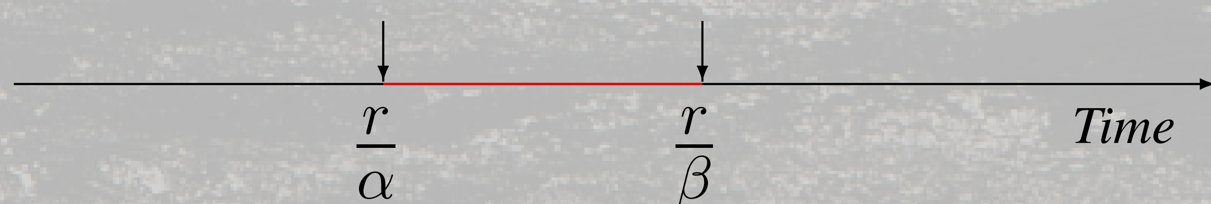
Green's function - solution

For infinite homogeneous isotropic medium we have got:

$$G_{ij} = \frac{1}{4\pi\rho\, r^3}(3\gamma_i\gamma_j - \delta_{ij}) \int_{r/\alpha}^{r/\beta} \tau\delta(t-\tau)d\tau$$

$$+ \frac{1}{\alpha^2 r}\gamma_i\gamma_j\delta(t - r/\alpha)$$

$$- \frac{1}{\beta^2 r}(\gamma_i\gamma_j - \delta_{ij})\delta(t - r/\beta)$$



Near field

$$\int\limits_{r/\alpha}^{r/\beta} \tau\delta(t-\tau)d\tau = \int\limits_{-\infty}^{\infty} \tau\delta(t-\tau)\left[H(\tau-\tfrac{r}{\alpha}) - H(\tau-\tfrac{r}{\beta})\right]$$

$$= tH(t-\tfrac{r}{\alpha}) \; - \; tH(t-\tfrac{r}{\beta})$$

$$tH(t-\frac{r}{\alpha}) \; = \; \left(t-\frac{r}{\alpha}\right)H(t-\frac{r}{\alpha}) + \frac{r}{\alpha}H(t-\frac{r}{\alpha})$$

Near field cd.

$$\mathbf{u}^{nf} \sim \frac{\mathbf{1}}{\mathbf{r^3}} \left[\left(t - \frac{r}{\alpha}\right) H(t - \frac{r}{\alpha}) - \left(t - \frac{r}{\beta}\right) H(t - \frac{r}{\beta}) \right] \quad \text{near field}$$

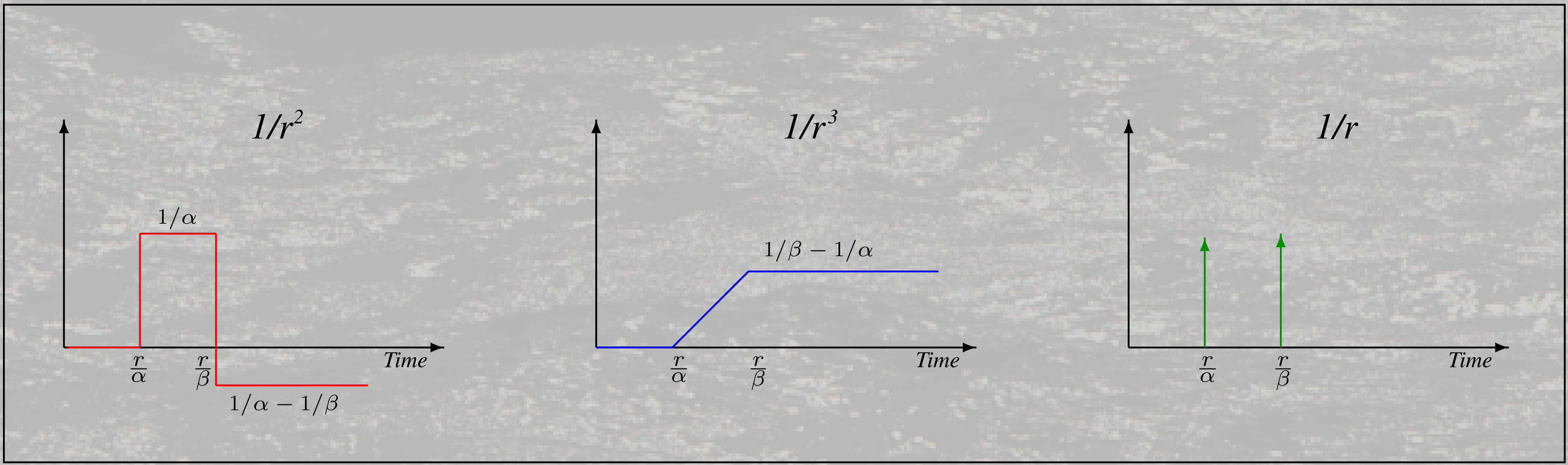
$$+ \frac{\mathbf{1}}{\mathbf{r^2}} \left[\frac{1}{\alpha} H(t - \frac{r}{\alpha}) - \frac{1}{\beta} H(t - \frac{r}{\beta}) \right] \quad \text{intermediate field}$$

The first term is a ramp function of unit slope

The second term is a box function

Both terms have a static part - constant in time for $t > r/\beta$

Green function - time structure



Green's function - complex media

The infinite, homogeneous, isotropic (non-attenuating, non-dispersive) medium case considered previously well illustrates general methods and beauty and complexity of used mathematics but it is not very practical. For realistic problems we need to use more realistic media models. How to calculate than Green's function:

- Lamb's (1904) model: homogeneous half-space . Model includes free-surface and Rayleigh waves Green's function :

$$G_r^R \approx 2i\sqrt{\frac{2\pi}{\omega\, r\, c_R}}\,\frac{k_R\sqrt{(k_R^2-k_\beta^2)}}{\beta^2 F(k_R)}\exp\left(i\left[k_R(r-c_R t)-h\sqrt{k_R^2-k_\beta^2}+\frac{\pi}{4}\right]\right)$$

Green's function - complex media cd.

- Layered (plane or spherical) models: reflectivity method, normal mode method

  Model consists of homogeneous isotropic layers in which plane waves are propagating, transmitting or reflecting and converting at layers interfaces. Solution is found by summing up all type of waves (including Rayleigh and Love)

- High frequency approximation (ray tracing)

- Numerical methods (finite difference, spectral elements, finite element, ...

Boundary conditions

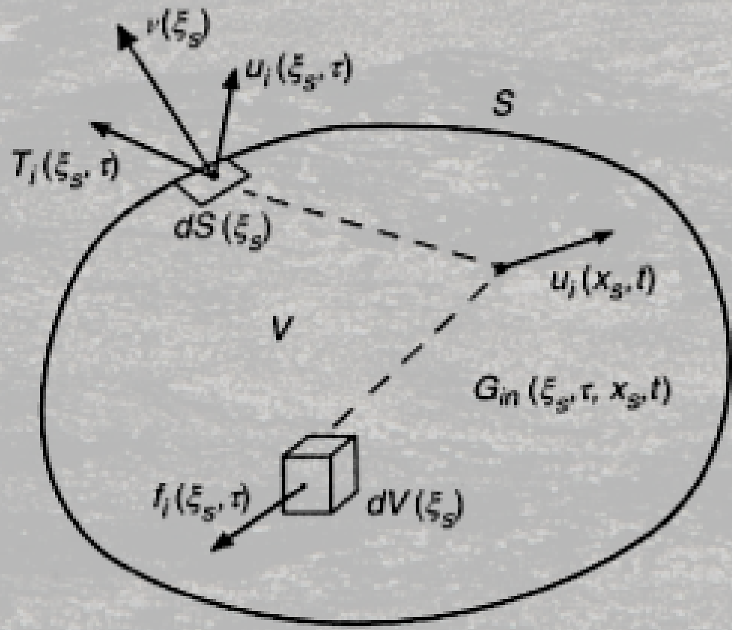
As we have already seen Green's function includes initial condition (e.g. assuring causality of the solution).

It also carries out information about boundaries of a body arising from finite size and geometry of the body, contact interface properties if body consists of different blocks, etc.

Let us now discuss a case of "dynamic" boundary conditions when external forces are supplied to surface of the body The problem can be analyzed starting from the wave equation and imposing appropriate conditions on obtained solutions. However, more elegant is to use the Green-Volterra (Betti's) reciprocity theorem

Representation theorem - imposing boundary condition

Green's function describes the response of the medium to the point-like impulsive source of given direction. Let us use it to find a solution with given boundary condition: an external traction over the surface of the body. trac



Representation theorem cd.

Equation of motion

$$(*) \quad \int\limits_S T_i(\mathbf{x},t)dS + \int\limits_V F_i(\mathbf{x},t)dV = \int\limits_V \rho \frac{\partial^2 u_i}{\partial t^2} dV$$

Let us consider

- two independent body force systems: $\mathbf{f}$, $\mathbf{g}$ and traction $\mathbf{T}^u$, $\mathbf{T}^w$
- corresponding displacements $\mathbf{u}$ and $\mathbf{w}$
- radiation initial condition: $\mathbf{u} = \mathbf{w} = 0$ for $t < 0$

Representation theorem cd.

Using (*) for governing equation for **u**and **w**, multiplying,subtracting and integrating over time and body volume

$$\int_T dt \int_V \rho \left(u_i \ddot{w}_i - w_i \ddot{u}_i\right) = \int_T dt \int_V \left(u_i g_i - w_i f_i\right) dV + \int_T dt \int_S \left(u_i T_i^w - w_i T_i^u\right) dS$$

This is Green-Volterra representation theorem. For radiation initial conditions left term vanish

$$\int_T dt \int_V \left(u_i g_i - w_i f_i\right) dV + \int_T dt \int_S \left(u_i T_i^w - w_i T_i^u\right) dS = 0$$

Representation theorem cd.

- $\mathbf{g} = \delta(\mathbf{x}_s - \mathbf{x}')\delta(t - t')\delta_{in}$
- $T_i^w = \partial\tau_{ij}/\partial x_j$
- $\mathbf{w}$ - Green's function
- Hook's law ($\tau_{ij} = C_{ijkl}e_{kl}$)

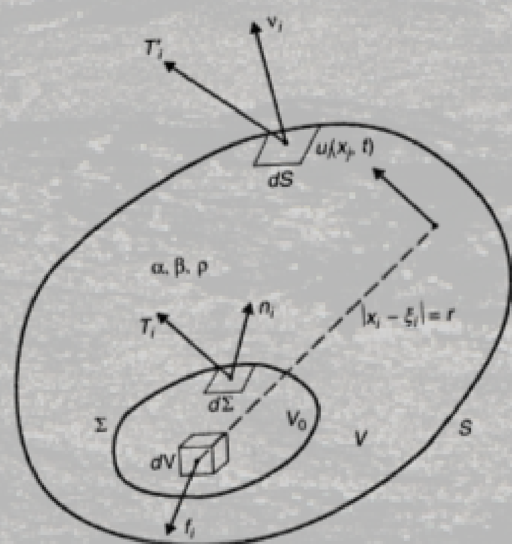
$$\int_T dt \int_V (u_i\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')\delta_{in} - G_{ni}f_n)\,dV = \int_T dt \int_S (G_{ni}T_i - u_iC_{ijkl}G_{nk,l}n_j)\,dS = 0$$

Representation theorem cd.

$$
\begin{aligned}
u_n(\mathbf{x}, t) &= \int_T dt' \int_V f_i(\mathbf{x}', t')\, G_{ni}(\mathbf{x}', t', \mathbf{x}, t) dV' \\
&+ \int_T dt' \int_S G_{ni}(\mathbf{x}', t', \mathbf{x}, t)\, T_i(\mathbf{x}', t') dS' \\
&- \int_T dt' \int_S u_i C_{ijkl}\, u_i(\mathbf{x}', t')\, G_{nk,l}(\mathbf{x}', t', \mathbf{x}, t) n_j(\mathbf{x}') dS'
\end{aligned}
$$

Focal region

Focal region (source region, source foci) is a finite volume (V_o) within a body where energy (in case of earthquake it is an elastic energy due to deformation of the medium) is released due to undergoing inelastic processes.



How to include such inelastic, finite volume source into theory of elasticity?

Let assume that there is no body forces f_i outside V_o

Source representation

$$
\begin{aligned}
u_n(\mathbf{x},t) &= \int\limits_T dt' \int\limits_{V_o} f_i(\mathbf{x}',t')\, G_{ni}(\mathbf{x}',t',\mathbf{x},t) dV' \\
&+ \int\limits_T dt' \int\limits_{\Sigma} \left(G_{ni}(\mathbf{x}',t',\mathbf{x},t)\, T_i(\mathbf{x}',t') - C_{ijkl} u_i G_{nk,l} n_j\right) dS' \\
&+ \int\limits_T dt' \int\limits_{\S} \left(G_{ni}(\mathbf{x}',t',\mathbf{x},t)\, T_i'(\mathbf{x}',t') - C_{ijkl} u_i G_{nk,l} \nu_j\right) dS'
\end{aligned}
$$

If medium is infinite last term vanish

$$
\begin{aligned}
u_n(\mathbf{x},t) &= \int\limits_T dt' \int\limits_{V_o} f_i(\mathbf{x}',t')\, G_{ni}(\mathbf{x}',t',\mathbf{x},t) dV' \\
&+ \int\limits_T dt' \int\limits_{\Sigma} \left(G_{ni}(\mathbf{x}',t',\mathbf{x},t)\, T_i(\mathbf{x}',t') - C_{ijkl} u_i G_{nk,l} n_j\right) dS'
\end{aligned}
$$

Source representation

Since most of earthquakes are due to shear fault movies we can assume that the focal area is much thiner than other fault/body dimensions and deform the foci region into a double-layer with surface σ. Discontinuity of displacement across both faces

$$\Delta u_i = u_i^+ - u_i^-$$

$$(**) \quad u_n(\mathbf{x}, t) = \int_T dt \int_{V_o} f_i\, G_{ni} dV' + \int_T dt \int_\Sigma C_{ijkl}\, \Delta u_i\, G_{nk,l} n_j dS' + \int_T dt' \int_\Sigma G_{ni}\, T_i dS'$$

We represent the complex inelastic fault source by condition on fault surface $(\Delta u_i,\ T_i)$

Forward and inverse source problems

Two different way of using the equation (**)

- **Forward problem** - calculation of seismogram (displacement field) for assumed fault condition
- **Inverse problem** - estimation source parameters (fault condition) from observed seismograms. Having estimated $(\Delta u_i,\ T_i)$ over Σ one can examine different physical models of source processes.

Three different (equivalent) source representations methods

$$u_n(\mathbf{x},t) = \int_T dt \int_{V_o} F_i\, G_{ni} dV' \qquad \text{equivalent forces}$$

$$u_n(\mathbf{x},t) = \int_T dt \int_\Sigma C_{ijkl}\, \Delta u_i\, G_{nk,l} n_j dS' \qquad \text{kinematical model}$$

$$u_n(\mathbf{x},t) = \int_T dt' \int_\Sigma G_{ni}\, T_i dS' \qquad \text{dynamic model}$$

END