An introduction to Physics of Seismic Sources

SP-2: Source representation I

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Basic assumptions

Information on energy release from the seismic source area (foci) within a given body is made available for us by measuring seismic (acoustic) waves at surface of the body.

- We need a method to relate the observed elastic (acoustic) waves to processes within the foci.
- Wave equation - relates processes at the source to elastic displacement of seismic (acoustic) waves.
- In the first approximation our body (Earth, rock sample, etc.) is considered as continuous deformable, elastic medium.
- In this approximation the term point of the medium (or particle) means infinitesimal small part of the medium having no dimension, no internal structure - geometrical point at given location. (no e.g. rotations)
- Properties of the medium (density, velocity, etc.) are continuous function of space

Elastic medium

$\mathbf{u}(\mathbf{x}, t)$ - displacement at point $\mathbf{x}$ at time t

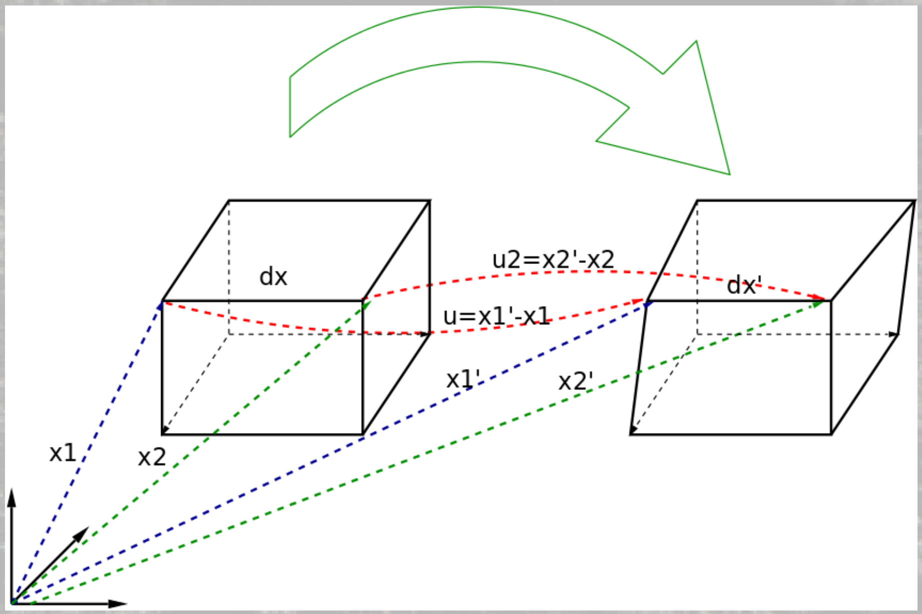
Infinitesimal small deformation can be described by strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Properties: symmetric, linear with respect to $\mathbf{u}$

Strain (deformation) tensor



$$d\mathbf{u}(\mathbf{x}) = d\mathbf{x}' - d\mathbf{x}$$

$$d\mathbf{x}' = d\mathbf{x} + d\mathbf{u}(\mathbf{x})$$

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j + O(d\mathbf{x}^2)$$

$$dl'^2 = d\mathbf{x}'^2 = (d\mathbf{x} + d\mathbf{u})^2 = dl^2 + 2\tilde{e}_{ij} dx_i dx_j$$

$$\tilde{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Small deformation - linear elasticity

$$\tilde{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Last term can be omited if $||d\mathbf{u}/d\mathbf{x}||$ is small

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Volume change

$$dV' = dV\left[1 + Tr(e_{ij})\right]$$

Strain tensor

Any deformation of an elastic body by external forces creates internal forces (stresses) which react against them assuring stability of the body structure. Force balance for any virtual element of the body stipulates (under assumption of short-range molecular interaction in the body) the form of internal stress density f_i :

$$f_i = \frac{\partial \sigma_{ij}}{\partial x_j}$$

Total internal force acting over an arbitrary volume V

$$F_i \; = \int\limits_V f_i dV \; = \int\limits_V \frac{\partial \sigma_{ij}}{\partial x_j} dV = \oint\limits_S \sigma_{ij} dS_k$$

σ_{ij} - stress tensor - generalized “preasure” Units: $[Pa]$

Strain tensor and Hook's law

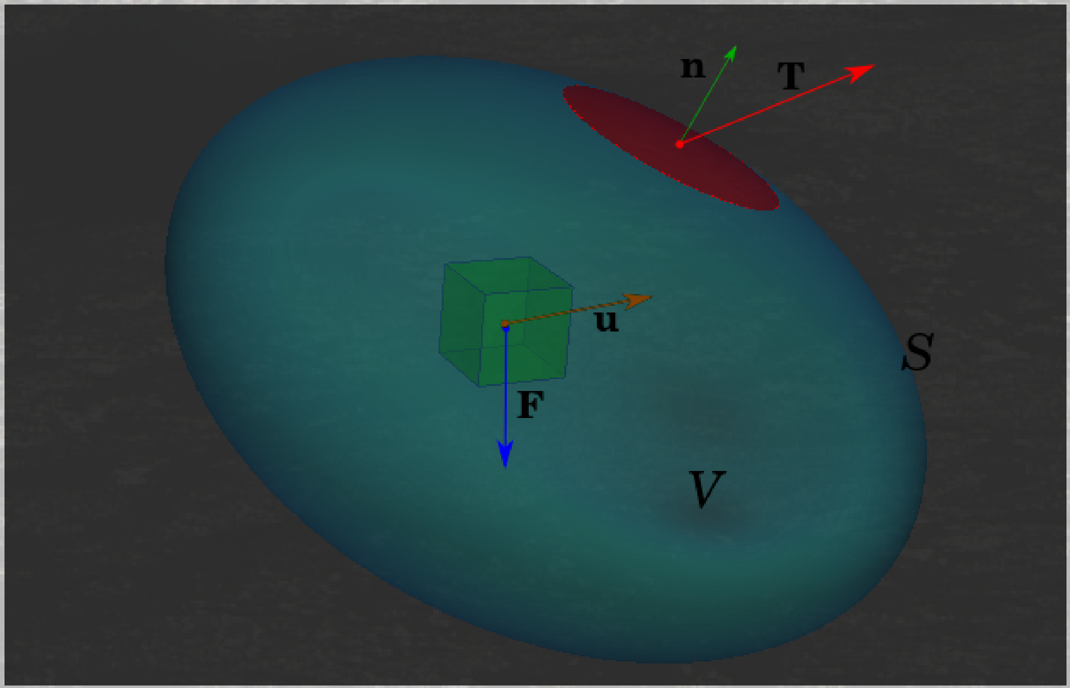
Hook's law

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

For isotropic materials

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

Dynamic of elastic body - momentum balance



$$\int_S T_i(\mathbf{x},t)dS + \int_V F_i(\mathbf{x},t)dV = \frac{d}{dt}\int_V \rho v_i(\mathbf{x},t)dV$$

Dynamic of elastic body - momentum balance

$$\int_S \sigma_{ij} n_j(\mathbf{x},t) dS + \int_V F_i(\mathbf{x},t) dV \approx \int_V \rho \frac{\partial^2 v_i}{\partial t^2} dV$$

Using Gauss's theorem for infinite medium

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j} = S_j \qquad \sigma_{ij} = C_{ijkl}\, e_{kl}$$

Homogeneous, isotropic medium

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(\lambda \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i})\right) = S_i$$

$$\rho \ddot{u}_i - (\lambda + \mu) u_{k,ki} - \mu u_{i,jj} = S_i$$

$$\rho \ddot{\mathbf{u}} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla^2 \mathbf{u} = \mathbf{S}$$

P and S waves

Introducing: $\alpha^2 = (\lambda + 2\mu)/\rho; \qquad \beta^2 = \mu/\rho$

$$\ddot{\mathbf{u}} - \alpha^2 \nabla(\nabla \cdot \mathbf{u}) - \beta^2 \nabla \times (\nabla \times \mathbf{u}) = \tilde{\mathbf{S}}$$

for plane wave

$$\mathbf{u} = \mathbf{A} \exp -i(\omega t - \mathbf{k} \cdot \mathbf{r})$$

longitudinal (P) and perpendicular (S) waves

$$-\omega^2 \ddot{\mathbf{u}} + \alpha^2 \mathbf{k}(\mathbf{k} \cdot \mathbf{u}) + \beta^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{u}) = 0$$

Green's function

Since the wave equation is the linear inhomogeneous (S term) PDE solution can be represented by the sum of general solution of the homogeneous ($\mathbf{L}\,\mathbf{u}(\mathbf{x},t)^{hom}) = 0$) and particular solution of inhomogeneous wave equation. This way we can alwas fulfil given initial and/or boundary conditions

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t)^{hom} + \mathbf{u}(\mathbf{x},t)^{in}$$

Green's function

$$\mathbf{L}\,\mathbf{G}^{in}(\mathbf{x},t) = \mathbf{e}\,\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

Green's function

Solution for given source S_i

$$u_i(\mathbf{x},t) = \int_T \int_V G_{ik}(\mathbf{x}-\mathbf{x}',t-t')S_k(\mathbf{x}',t')dV'dt'$$

Explicit form of Green's function equation:

$$\boxed{\rho\ddot{G}_{ni} - C_{ijkl}G_{nk,lj} = \delta_{ni}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')}$$

for static "displacement-like" force (Somigliana's tensor)

$$-C_{ijkl}G_{nk,lj} = \delta_{ni}\delta(\mathbf{x}-\mathbf{x}')$$

Static solution - “Unit” force representation

Any vector fields can be decomposed into rotation-free and divergent-free parts. For our “unit” force

$$\delta(\mathbf{x})\mathbf{e} = \nabla\Phi + \nabla \times \mathbf{\Psi}$$

We can represnt Φ and Ψ by vector potential

$$\Phi = -\nabla \cdot \mathbf{W}; \quad \mathbf{\Psi} = \nabla \times \mathbf{W}$$

With this representation the “unit” force is described by

$$\nabla^2\mathbf{W} = -\delta(\mathbf{x})\mathbf{e}$$

Static solution - Somigliana's potentials

Solution:

$$\mathbf{W}(\mathbf{x}) = \frac{1}{4\pi r}\mathbf{e} \qquad r = ||\mathbf{x}||$$

Assume $\mathbf{e} = (1, 0, 0)$

$$\Phi(\mathbf{x}) = -\nabla \cdot \mathbf{W} = -\frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)$$

$$\mathbf{\Psi}(\mathbf{x}) = \nabla \times \mathbf{W} = \frac{1}{4\pi}\left(0, \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right), -\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right)$$

Static solution - infinite homogeneous isotropic medium

$$\alpha^2\nabla(\nabla\cdot\mathbf{u}) - \beta^2\nabla\times\nabla\times\mathbf{u} = \frac{1}{\rho}\delta(\mathbf{x})\mathbf{e}$$

Again

$$\mathbf{u} = \nabla\phi + \nabla\times\vec{\psi}$$

Two Poison-type equations

$$\nabla^2\phi = -\frac{\Phi}{\alpha^2\rho}$$

$$\nabla^2\vec{\psi} = -\frac{\mathbf{\Psi}}{\beta^2\rho}$$

Static solution - $\mathbf{e} = (1, 0, 0)$

General solution

$$\phi(\mathbf{x}) = \frac{1}{4\pi\rho\alpha^2} \int_V \frac{\Phi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV'$$

Substituting previous form of “unit” force potentials

$$\phi(\mathbf{x}) = \frac{-1}{16\pi^2\rho\alpha^2} \int_V \frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial}{\partial x_1'} \left(\frac{1}{|\mathbf{x}'|} \right) d^3\mathbf{x}'$$

Integral is evaluated in spherical coordinates with center at $\mathbf{x}$ and change veriable $r' = \alpha\tau$

Static solution - cd.

After performing surface (angular) integrations we obtain

$$\phi(\mathbf{x}) = \frac{-1}{4\pi\rho} \frac{\partial}{\partial x_1} \left(\frac{1}{r}\right) \int_0^{r/\alpha} \tau d\tau$$

and

$$\phi(\mathbf{x}) = \frac{\gamma_1}{8\pi\rho\alpha^2}$$

where

$$\gamma_i = \frac{\partial r}{\partial x_i} \qquad \frac{\partial \gamma_i}{\partial x_j} = -\frac{1}{r}(\gamma_i\gamma_j - \delta_{ij})$$

In a similar way we proceed to obtain $\vec{\psi}$

Static solution - cd.

Finally, for $\mathbf{e} = (1,0,0)$ unit force

$$u_i^{(1)} = \frac{1}{8\pi\rho r}\left[\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\gamma_1\gamma_i + \left(\frac{1}{\beta^2} + \frac{1}{\alpha^2}\right)\delta_{1,i}\right]$$

and in general the full Somigliana's tensor

$$S_{ij} = \frac{1}{8\pi\rho r}\left[\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\gamma_i\gamma_j + \left(\frac{1}{\beta^2} + \frac{1}{\alpha^2}\right)\delta_{ij}\right]$$

It describes the static deformation in the elastic medium due to a “unit” force at center of coordinate system

Green function

$$\ddot{G}_{ij} - \alpha^2 \nabla(\nabla \cdot G_{ij}) - \beta^2 \nabla \times (\nabla \times G_{ij}) = \frac{1}{\rho}\delta_{ij}\delta(t)\delta(\mathbf{x})$$

Again we represent the rhs term by the potential $W(\mathbf{x}, t)$ (see Somigliana's case). Now decompose the displacement $\mathbf{u}$

$$\mathbf{u} = \nabla\phi + \nabla \times \vec{\psi}$$

and get two Helmholtz equations for potentials

$$\ddot{\phi} - \alpha^2\nabla^2\phi = -\frac{\Phi}{\rho}$$

$$\ddot{\vec{\psi}} - \beta^2\nabla^2\vec{\psi} = -\frac{\mathbf{\Psi}}{\rho}$$

Green function cd

Retarded (casual) Green's function for the scalar Helmholtz equation

$$\ddot{G} - \alpha^2 \nabla^2 G = \delta(\mathbf{x})\delta(t)$$

$$G(\mathbf{x}, t) = \frac{1}{4\pi\rho\alpha^2 \; ||\mathbf{x}||} \delta(t - ||\mathbf{x}||/\alpha)$$

(sketch how to calculate it)

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\rho\alpha^2} \int_V \frac{\Phi(t - ||\mathbf{x} - \mathbf{x}'||/\alpha)}{||\mathbf{x} - \mathbf{x}'||} d\mathbf{x}'$$

and similar for vectorial part $\vec{\psi}$

Green function cd.

After calculations similar to the static case ($r = ||\mathbf{x}||$) for $\mathbf{e} = (1,0,0)$ case

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\rho r^2}\gamma_1 \int_0^{r/\alpha} \tau\delta(t-\tau)d\tau$$

$$\vec{\psi}(\mathbf{x},t) = \frac{1}{4\pi\rho r^2}(0,\gamma_3,-\gamma_2) \int_0^{r/\beta} \tau\delta(t-\tau)d\tau$$

$$\begin{aligned} G_{ij} = & \frac{1}{4\pi\rho\, r^3}(3\gamma_i\gamma_j - \delta_{ij}) \int_{r/\alpha}^{r/\beta} \tau\delta(t-\tau)d\tau \\ + & \frac{1}{\alpha^2 r}\gamma_i\gamma_j\delta(t-r/\alpha) \\ - & \frac{1}{\beta^2 r}(\gamma_i\gamma_j - \delta_{ij})\delta(t-r/\beta) \end{aligned}$$

END