Advanced statistical methods and Bayesian inference in scientific research

Lecture 4

W. Dębski

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Random variable transformation

Let transform X to Y

$$X \Longrightarrow Y$$

Consistency requirement:

$$\forall_{\mathcal{A}} \; P_X(\mathcal{A}) = P_Y(\mathcal{B})$$

$$\mathbf{p}(y) = \mathbf{p}(x) \left| \frac{\partial x}{\partial y} \right|$$

Bayes theorem - single variable X

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes “theorem”

$$P(B|A) = P(A|B) \times P(B)/P(A)$$

Bayes theorem - application

- random variable X is described by $\rho(x)$
- we perform another measurement of X

Question: how performed measurement allows us to constraint (update) the pdf distribution $p(x)$ describing X?

Bayes theorem - single variable X

$$P(B|A) = P(A|B) \times P(B)/P(A)$$

Interpretation:

A	$\rightarrow$	measured values of X
B	$\rightarrow$	values of X drawn from $\rho(x)$
P(A)	$\rightarrow$	"normalization term - see next slide"
P(B)	$\rightarrow$	"initial" pdf $\rho(x)$
$P(B\|A)$	$\rightarrow$	thought $p(x)$
$P(A\|B)$	$\rightarrow$	how possible values of x relates to measured x_o expressed by $l(x_o\|x)$

$$p^{(1)}(x) = \kappa \, l(x_o|x) \times \rho(x)$$

when more measurements becomes available

$$p^{(n)}(x) = \kappa \, l(x_o.x_1, \ldots x_{n-1}|x) \times p^{(n-1)}(x)$$

Bayes theorem - single variable X

Normalization condition (symbolic):

$$\int P(B|A)\mathrm{d}B = 1$$

$$P(A) \ = \ \int P(A|B) \times P(B)$$

Problematic contradiction in interpretation of $P(A)$ and $P(B)$

Mathematics of inference - Inference Space

$$(\mathcal{P}, \Sigma, \wedge)$$

where

$\mathcal{P}$ - parameter space

Σ - space of all probability distributions over $\mathcal{P}$

$\wedge(\cdot, \cdot)$ - joining operator: $\Sigma \times \Sigma \to \Sigma$

Joining information according to Tarantola

Two distributions describing **different** pieces of information about the same object

1. $p(x)$
2. $q(x)$

$$\zeta(x) \;=\; \mathbf{p} \wedge \mathbf{q}(\mathbf{x}) = \frac{\mathbf{p}(\mathbf{x})\,\mathbf{q}(\mathbf{x})}{\mu(\mathbf{x})}$$

$\mu(x)$ - non-informative probability

Non-informative distribution

$$q(x) \; = \; \mu(x)$$

$$\mathbf{p} \wedge \mathbf{q}(\mathbf{x}) = \frac{\mathbf{p}(\mathbf{x}) \; \mu(\mathbf{x})}{\mu(\mathbf{x})} \; = \; \mathbf{p}(\mathbf{x})$$

Essentially, $\mu(\cdot)$ can be arbitrary but usually is taken as *volumetric pdf*

Using Inference space approach - example I

- random variable X is described by $\rho(x)$
- we perform another measurement of X

Question: how performed measurement allows us to constraint (update) the pdf distribution $p(x)$ describing X?

Assumption: noisy measurement with known noise characteristic $\psi(\cdot)$

$$x_o = x_o^{true} + n$$

$$X_o \Longrightarrow p(x) = \psi(x - x_o)$$

Using Inference space approach - example I

$$p \wedge q(x) = \frac{p(x)\ \mu(x)}{\mu(x)} = p(x)$$

$$p^{(1)}(x) = \frac{1}{Z} \frac{\rho(x)\psi(x - x_o)}{\mu(x)}$$

$$Z = \int \rho(x)\psi(x - x_o)/\mu(x)dx$$

Comment:
Z (evidence) measure to what extend measurement is compatible with "apriori" $\rho(x)$

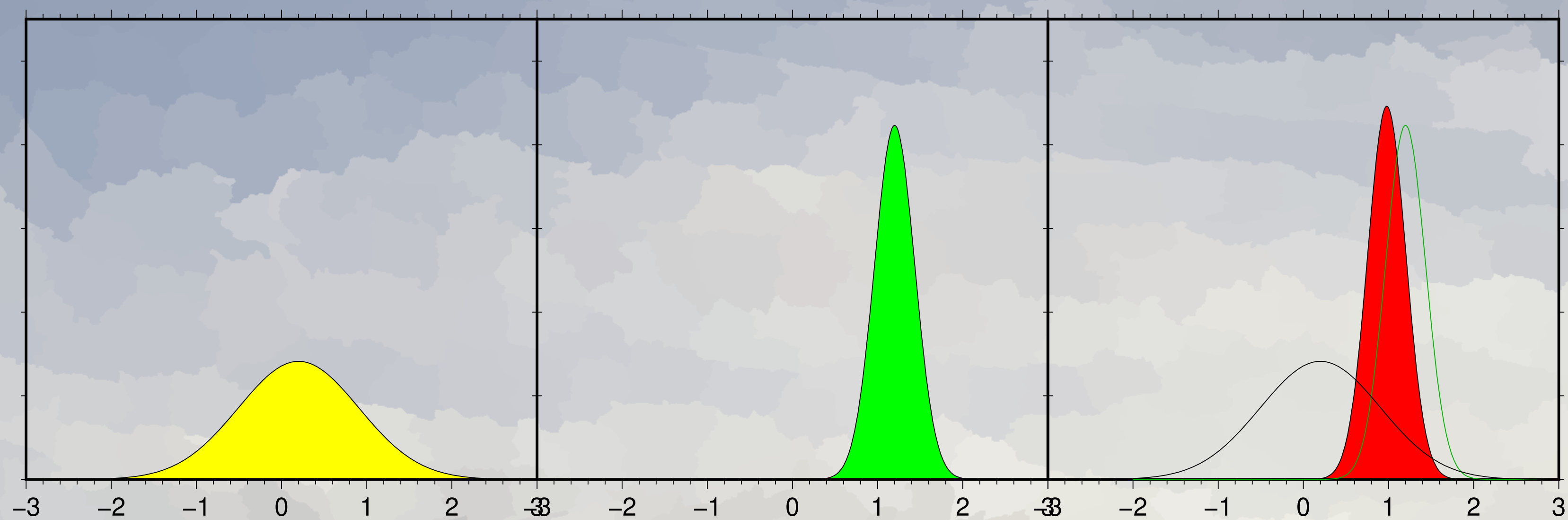
Using Inference space approach - example I

If another (independent) measurement is available

$$p^{(2)}(x) = \frac{1}{Z'} \frac{p^{(1)}(x)\psi(x - x_o)}{\mu(x)}$$

$$p^{(n)}(x) = \frac{1}{Z} \frac{p^{(n-1)}(x)\psi(x - x_o^{(n)})}{\mu(x)}$$

Using Inference space approach - example I



Using Inference space approach - example II

- random variable X is described by $\rho(x)$
- we perform a measurement of Y
- measurement errors are characterized by $\psi(y - y_o)$ distribution
- we know that X and Y are related as $Y = G(X) + \epsilon$
  and the relation is subjected to errors ϵ described by

$$\zeta(X, Y) = \zeta(Y - G(X))$$

Question:

how performed measurement constraints (update) knowledge of X ?

Using Inference space approach - example II

New “vectorized” random variable

$$X, Y \Longrightarrow Z := \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\mathcal{R} \times \mathcal{R} \Longrightarrow \mathcal{R}^2$$

$$\begin{aligned} \rho(x) \quad &\rightarrow \quad \rho'(z) \; = \; \rho(x)\mu(y) \\ \psi(y - y_o) \quad &\rightarrow \quad \psi'(z) \; = \; \psi(y - y_o)\mu(x) \\ \zeta(x, y) \quad &\rightarrow \quad \zeta'(z) \end{aligned}$$

Using Inference space approach - example II

$$p \wedge q(x) = \frac{p(x)\ q(x)}{\mu(x)}$$

$$p(z) = a\,\frac{\psi'(z)\zeta'(z)}{\mu(z)}$$

$$\sigma(z) = a\,\frac{p(z)\rho'(z)}{\mu(z)}$$

$$\sigma(z) = \frac{1}{Z}\,\frac{\zeta(z)\,\psi'(z)\,\rho'(z)}{\mu^2(z)}$$

Using Inference space approach - example II

Taking marginal integrals

$$\sigma(x) = \int_Y \sigma(z)\mathrm{d}y$$

$$\sigma(x) = \frac{1}{Z}\,\rho(x) \int_Y \psi(y - y_o)\zeta(y - G(x))\mathrm{d}y$$

$$Z = \int_X \int_Y \rho(x)\psi(y - y_o)\zeta(y - G(x))\mathrm{d}y\mathrm{d}x$$

Using Inference space approach - example II

If the relation between X and Y is exactly known

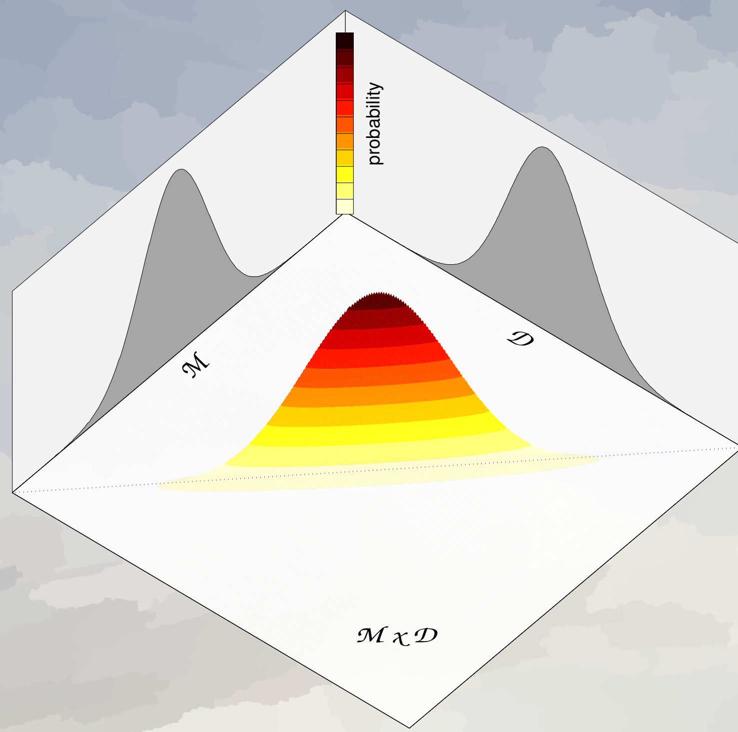
$$\zeta(y - G(x)) \; = \; \delta(y - G(x))$$

$$\sigma(x) = \frac{1}{Z}\,\rho(x)\,\psi(y_o - G(x))$$

$$p^{(1)}(x) \; = \; \frac{1}{Z}\,\frac{\rho(x)\psi(x_o - x)}{\mu(x)}$$

Using Inference space approach - example II

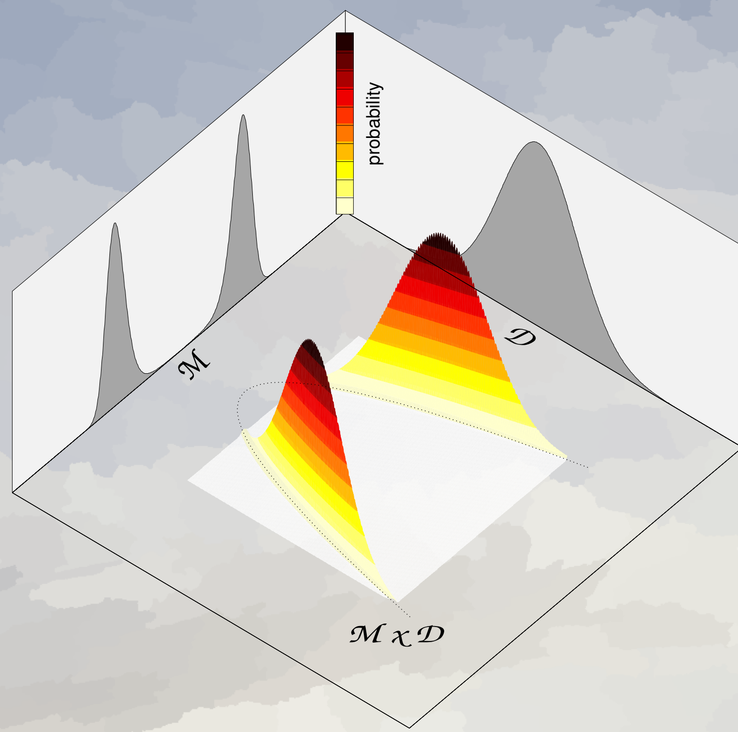
Exact linear relation $y = G \cdot x$



$$\sigma(x) = \frac{1}{Z}\,\rho(x)\psi(y_o - G \cdot x)$$

Using Inference space approach - example II

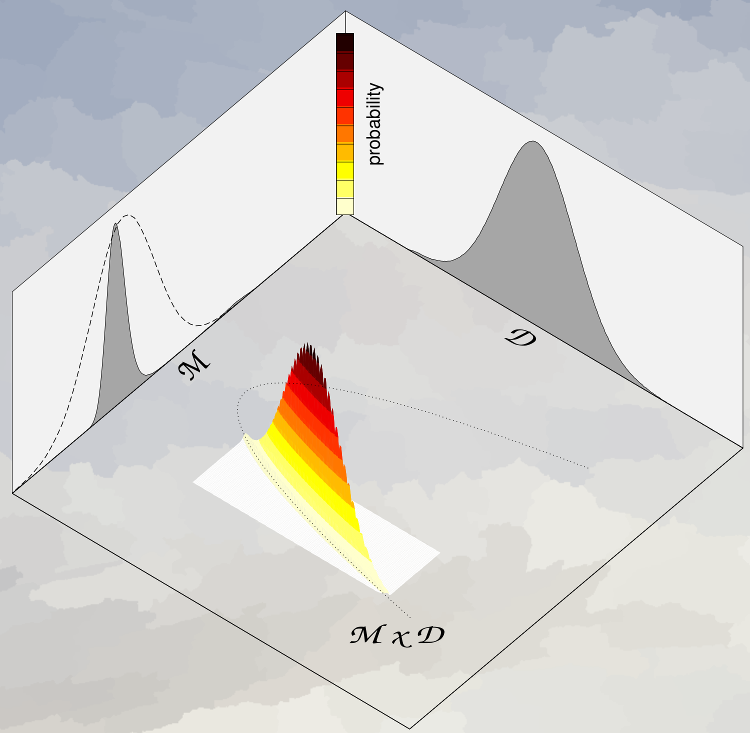
Exact non-linear relation $y = G \cdot x^2$



$$\sigma(x) = \frac{1}{Z}\,\rho(x)\psi(y_o - G \cdot x^2)$$

Using Inference space approach - example II

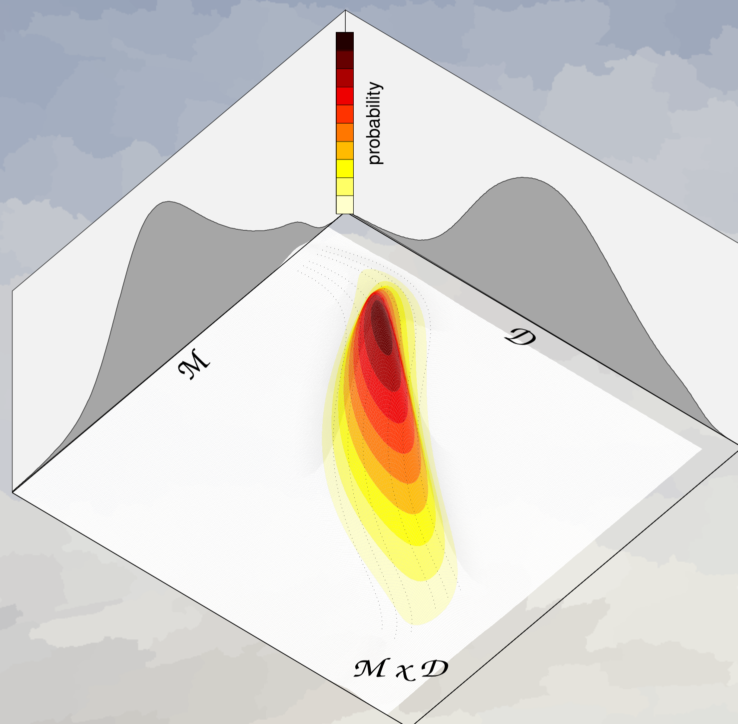
Exact non-linear relation $y = G \cdot x^2$ and "*a priori*" term



$$\sigma(x) = \frac{1}{Z}\,\rho(x)\psi(y_o - G \cdot x^2)$$

Using Inference space approach - example II

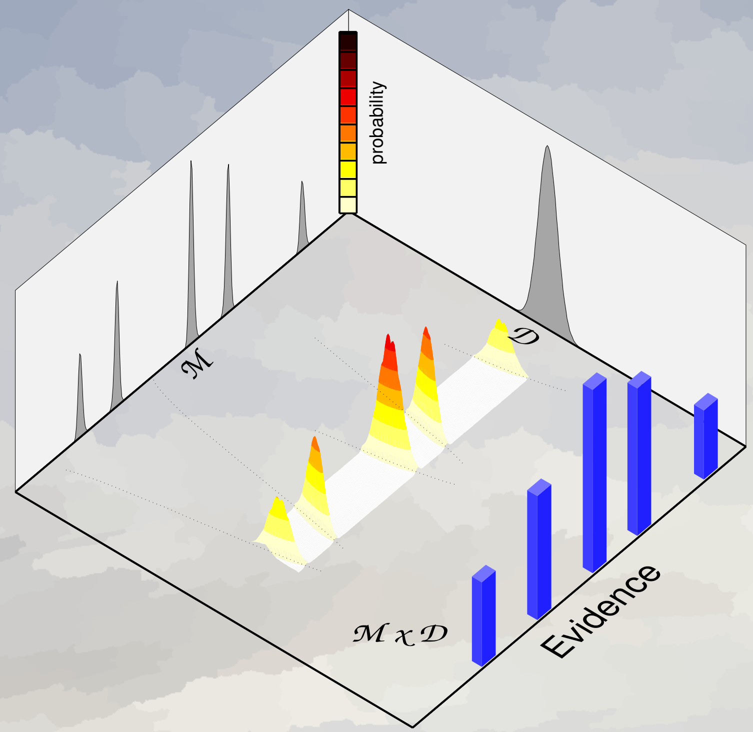
Approximate theory



$$\sigma(x) = \frac{1}{Z}\,\rho(x) \int\limits_{Y} \psi(y - y_o)\zeta(y - G(x))\mathrm{d}y$$

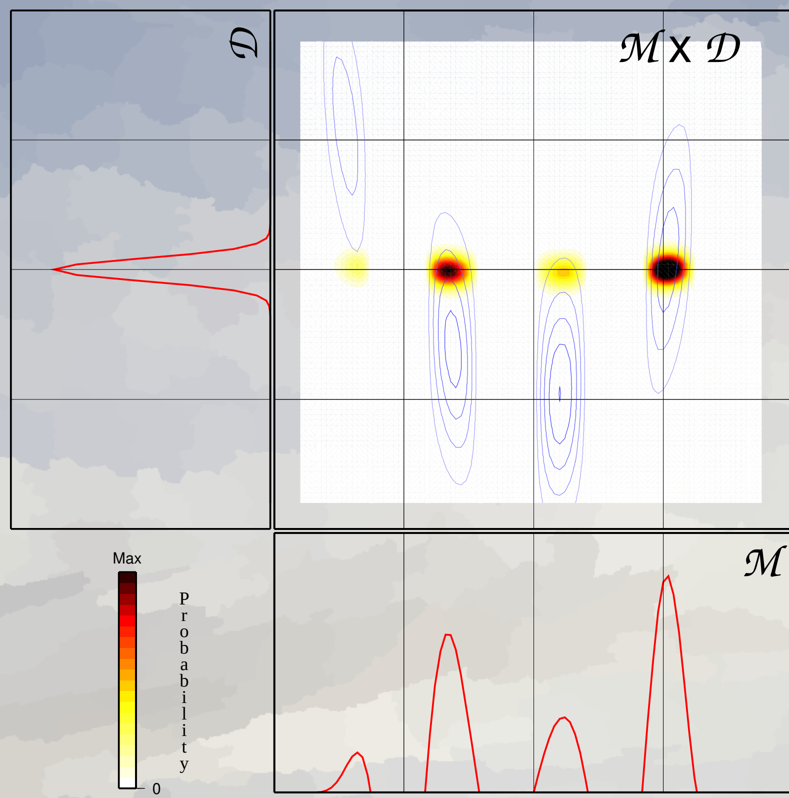
Using Inference space approach - example II

Comparing various theories



Using Inference space approach - example II

Comparing various theories



Evidence

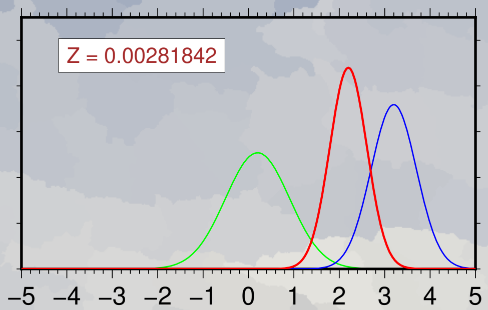
$$Z = \int_X \int_Y \rho(x)\psi(y - y_o)\zeta(y - G(x))\mathrm{d}y\mathrm{d}x$$

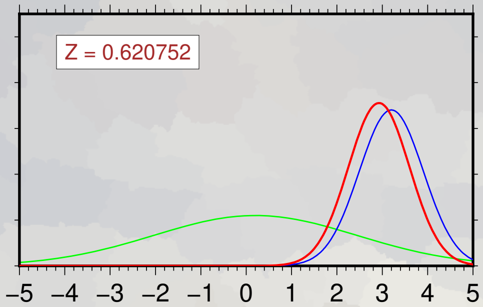
$$\mathcal{L}(x) = \int_Y \psi(y - y_o)\zeta(y - G(x))\mathrm{d}y$$

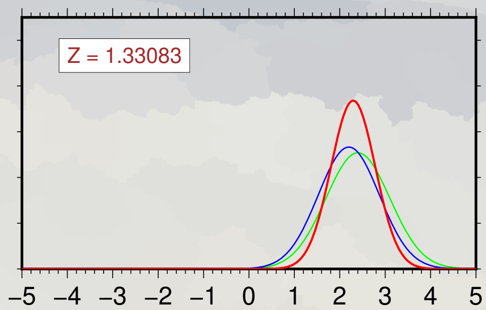
$$Z = \int_X \rho(x)\mathcal{L}(x)\mathrm{d}x$$

Evidence

- $\rho(x)$ - green
- $\mathcal{L}(x)$ - blue
- $\sigma(x) = \rho(x)\mathcal{L}(x)$ - red







See you April, 15th