Advanced statistical methods and Bayesian inference in scientific research

Lecture 12

W. Dębski

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Extreme value statistics - introduction

The previously discussed analysis and consequently corresponding numerical methods were aimed at exploring a bulk properties of probability distributions describing problem at hand. This is because, most of statistical inference address questions like what is maximum or expected value of given parameter, its variation, etc. In more complex analysis we may calculate and analyze such quantities like entropy, evidence, Fisher matrix or explore Cramer-Rao bound. In all these cases the answer is determined by bulk structure of corresponding probability distribution.

Extreme value statistics - introduction

However, in some analysis the most important is an asymptotic behavior of probability density rather than its bulk properties. This is for example a case when we ask about maximal possible values of given variables/parameters and probability of its occurrence. Examples of such analysis is flood frequency analysis in hydrology, extreme meteorological events, stock crash (econometrics) to name a few.

Extreme value statistics - introduction

The part of statistical analysis devoted to such issues is called

Extreme value statistics

In this lecture I give a very brief introduction based on the short review: *The three Extreme Value Distributions: An Introductory Review* by A. Hansen
Frontiers in Physics, (2020), doi: 10.3389/fphy.2020.604053

Extreme value statistics - notions

The basic object of interest is probability distribution $p(x)$ and corresponding cumulative distribution

$$P(x) = \int_{-\infty}^{x} p(x')dx'$$

$P(x)$ - probability that random variable takes value smaller or equal x.

Extreme value statistics - formulation

Let us draw N numbers from $p(x)$ and record the largest one

$$u \ = \ \max(x_1, x_2 \cdots x_N)$$

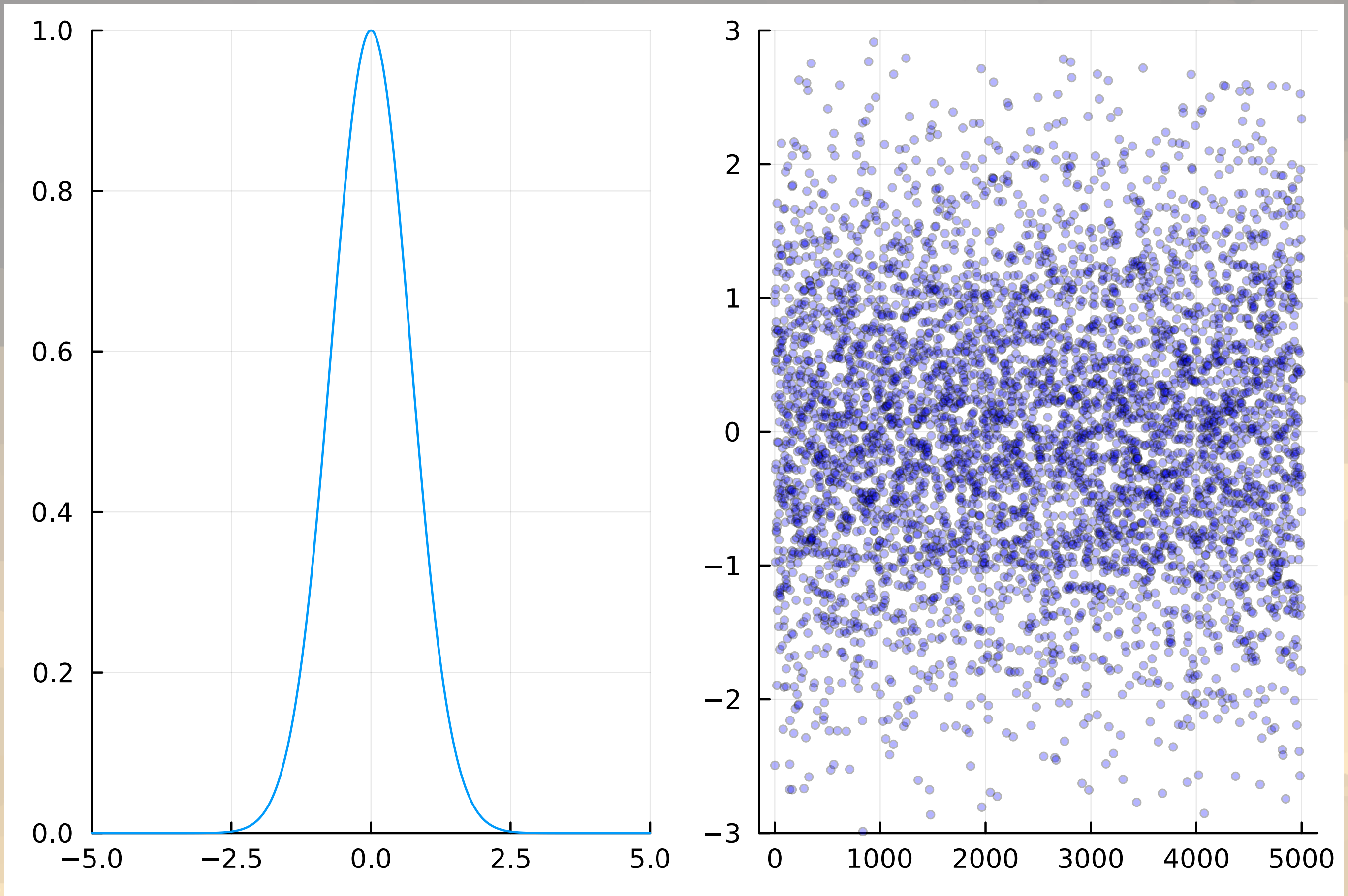
Let repeat it M times obtaining a sequence

$$u : \ (u_1, u_2, \cdots u_M)$$

u is obviously a random variable and let denot its distribution

$$Q^{(M)}(u)$$

Extreme value statistics - illustration



Extreme value statistics - formulation

Obvious questions are:

1. what this distribution looks like in a limit $M \to \infty$,
2. how u is related to x
3. how $Q(u)$ depends on $p(x)$
4. which properties of $p(x)$ determines shape of $Q(u)$

Extreme value statistics - approach

Let drawn N samples from $p(x)$

$$p(x) : x_1, x_2, \cdots x_N$$

Probability that all x_i are smaller than x reads

$$P(x_1 \leq x, x_2 \leq x, \cdots x_N \leq x) = \left[\int_{-\infty}^{x} p(x') dx' \right]^N = P^N(x)$$

Limited range pdf

- Limited range pdf: $p(x) = 0$ for $x > x_0$

$$p(x) = \begin{cases} b\,\alpha(x_o - x)^{\alpha-1} & \text{for } x \to x_o^- \\ 0 & \text{for } x > x_o \end{cases}$$

where $b, \alpha > 0$.

Features:

- If $0 < \alpha < 1)$ than $p(x) \sim 1/(x - x_o)^{|\beta|}$ is divergent
- If $\alpha = 1$ than $p(x) \sim const.$ for $x \precsim x_o^-$
- If $\alpha > 1)$ than $p(x)$ goes smoothly to 0 as $x \to x_o^-$

Limited range pdf

Cumulative distribution

$$P(x) = \begin{cases} 1 - b\,(x_o - x)^\alpha & \text{for } x \to x_o^- \\ 1 & \text{for } x \geq x_o \end{cases}$$

Than cumulative (extreme value) distribution for N samples for $x \approx x_o$ reads

$$P^N(x) = \left[1 - b\,(x_o - x)^\alpha\right]^N$$

Probability that at least one x_i is larger than x: $1 - P^N(x)$

Limited range pdf

$$P^N(x) \;=\; [1 - b\,(x_o - x)^\alpha]^N \qquad \lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N \;=\; e^x$$

Let make transformation

$$x - x_o \;=\; \frac{u}{(bN)^{1/\alpha}}$$

$$P^N(x) \;=\; \left[1 - \frac{(-u)^\alpha}{N}\right]^N$$

$$Q(u) := \lim_{N\to\infty} P^N(x) = e^{-(-u)^\alpha} \text{ for } u < 0$$

Limited range pdf

Cumulative extreme value distribution

$$Q(u) = \begin{cases} e^{-(-u)^\alpha} & \text{for } u < 0 \\ 1 & \text{for } u \geq 0 \end{cases}$$

is *Weibull* cumulative distribution. Corresponding pdf function

$$q(u) = \frac{dQ}{du} = \begin{cases} \alpha(-u)^{\alpha-1} e^{-(-u)^\alpha} & \text{for } u < 0 \\ 0 & \text{for } u \geq 0 \end{cases}$$

Important:

We have got $\mathbf{Q(u)}$ analyzing asymptotic behaviour of p(x) around x_o "boundary" point!

Limited range pdf

Cumulative extreme value distribution in x variable

$$Q(x) = \begin{cases} e^{-Nb(x_o - x)^\alpha} & \text{for } x < x_o \\ 1 & \text{for } x \geq x_o \end{cases}$$

Weibull's extreme value probability density

$$q(x) = \begin{cases} Nb\alpha(x_o - x)^{\alpha-1} e^{-Nb(x_o - x)^\alpha} & \text{for } x < x_o \\ 0 & \text{for } x \geq x_o \end{cases}$$

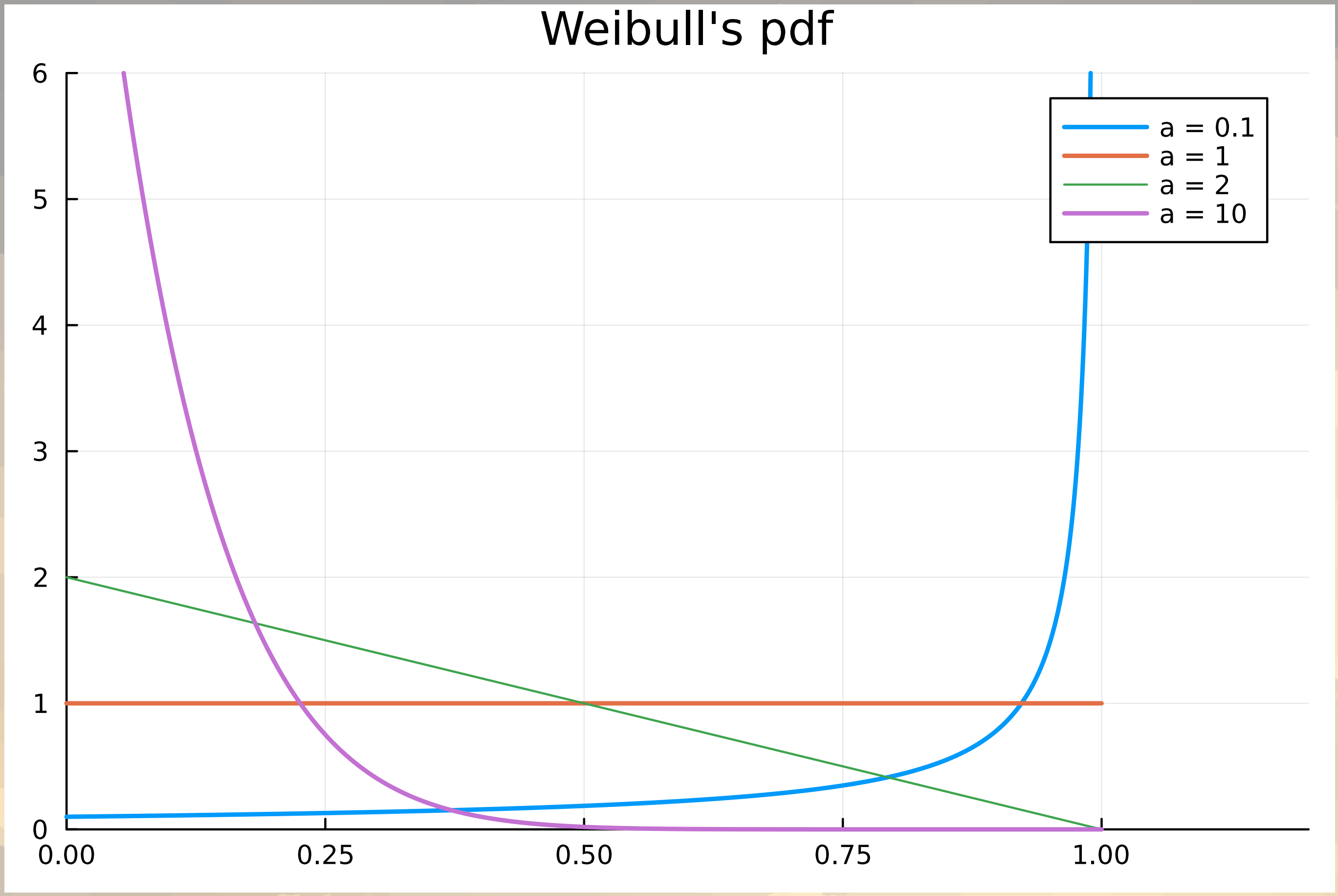
Limited range pdf - example

$$p(x) = \begin{cases} 0 & \text{for } x < 0 \\ \alpha(1-x)^{\alpha-1} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases}$$

cumulative distribution

$$P(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-(1-x)^{\alpha} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

Limited range pdf - example

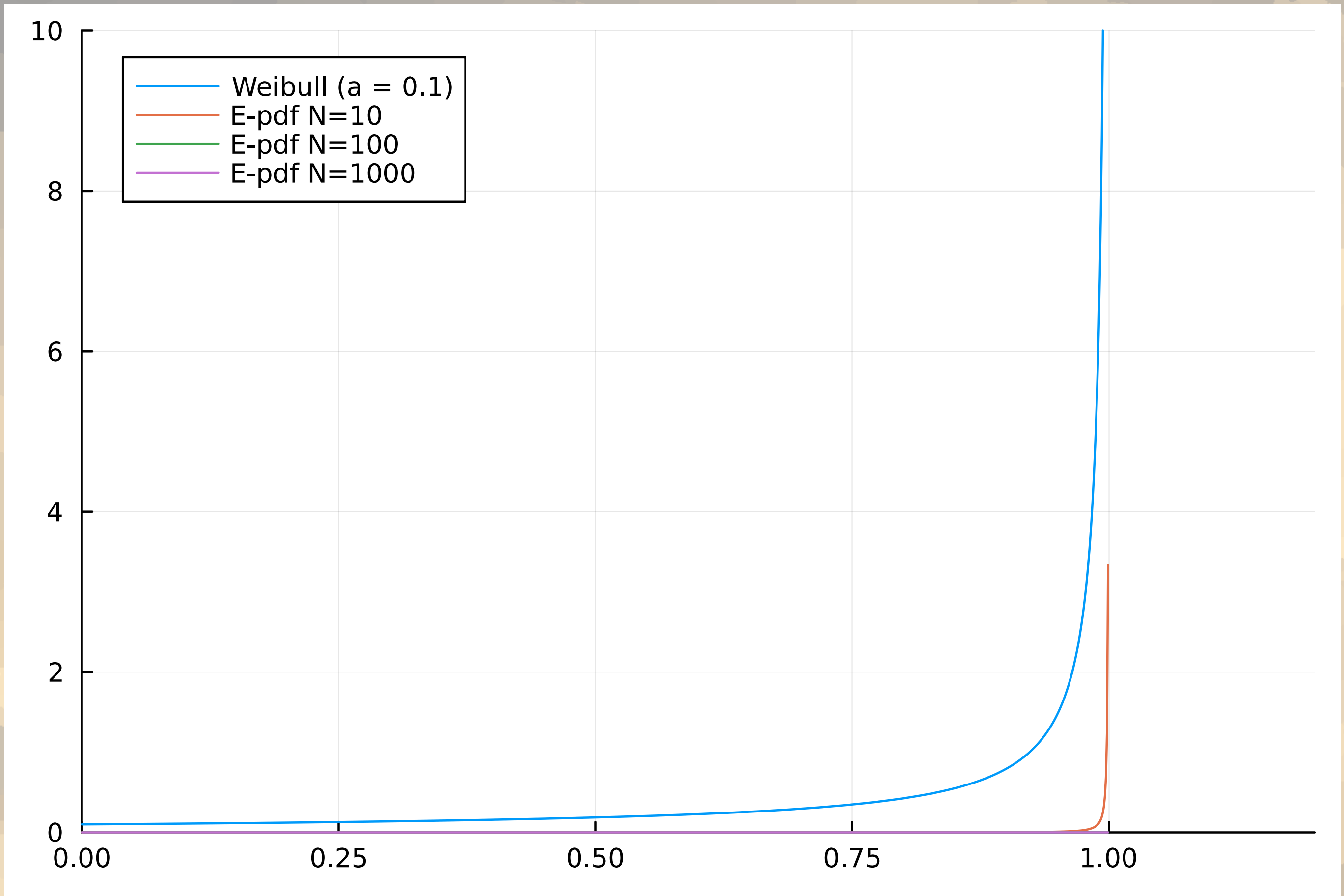


Limited range pdf - example

$$q(x) = \begin{cases} 0 & \text{for } x < 0 \\ N\alpha(1-x)^{\alpha-1}e^{-N(1-x)^{\alpha}} & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x > 1 \end{cases}$$

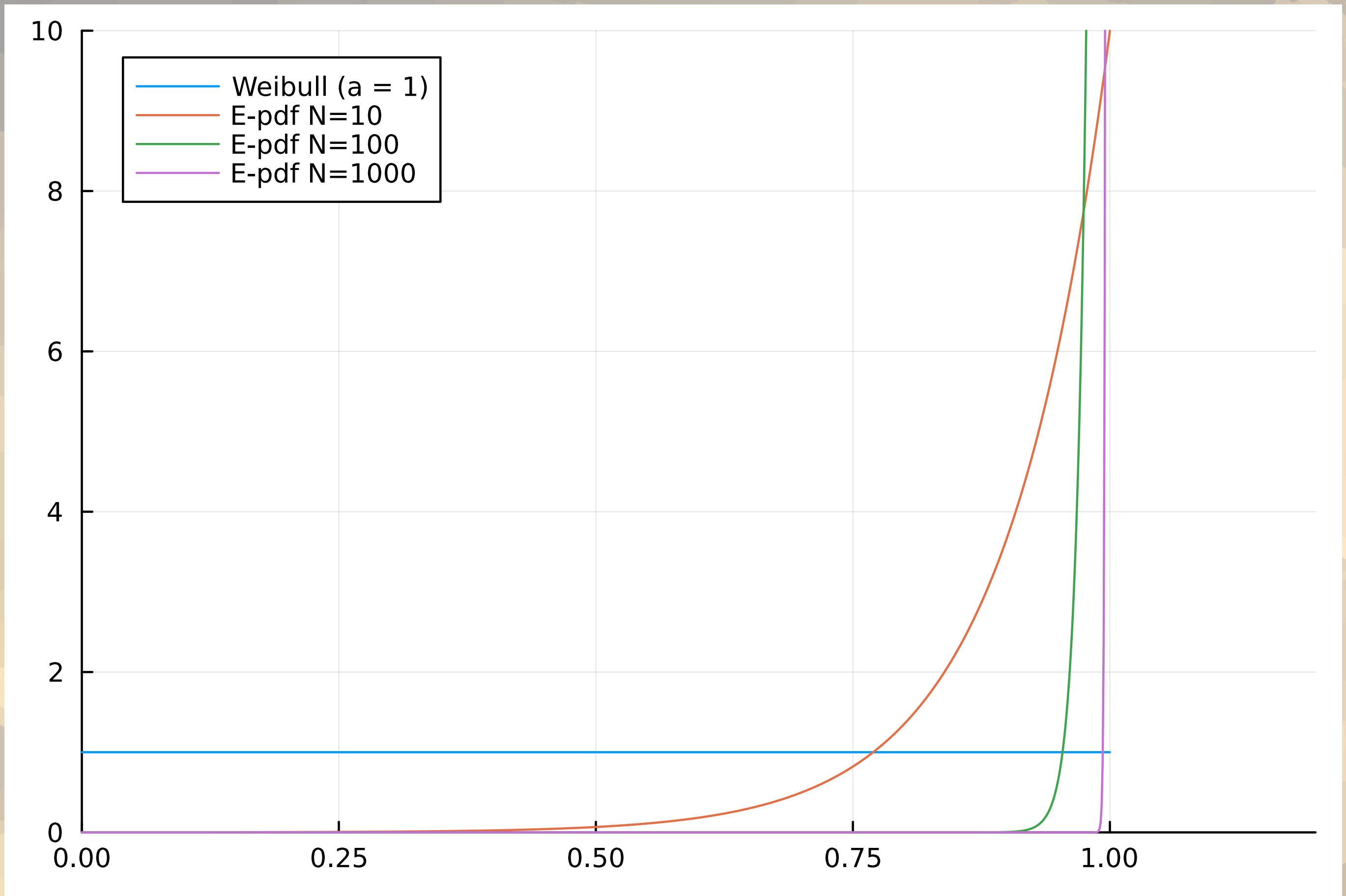
Limited range pdf - example

$$q(x) \;=\; N\alpha(1-x)^{\alpha-1}e^{-N(1-x)^{\alpha}} \qquad \alpha = 0.1$$



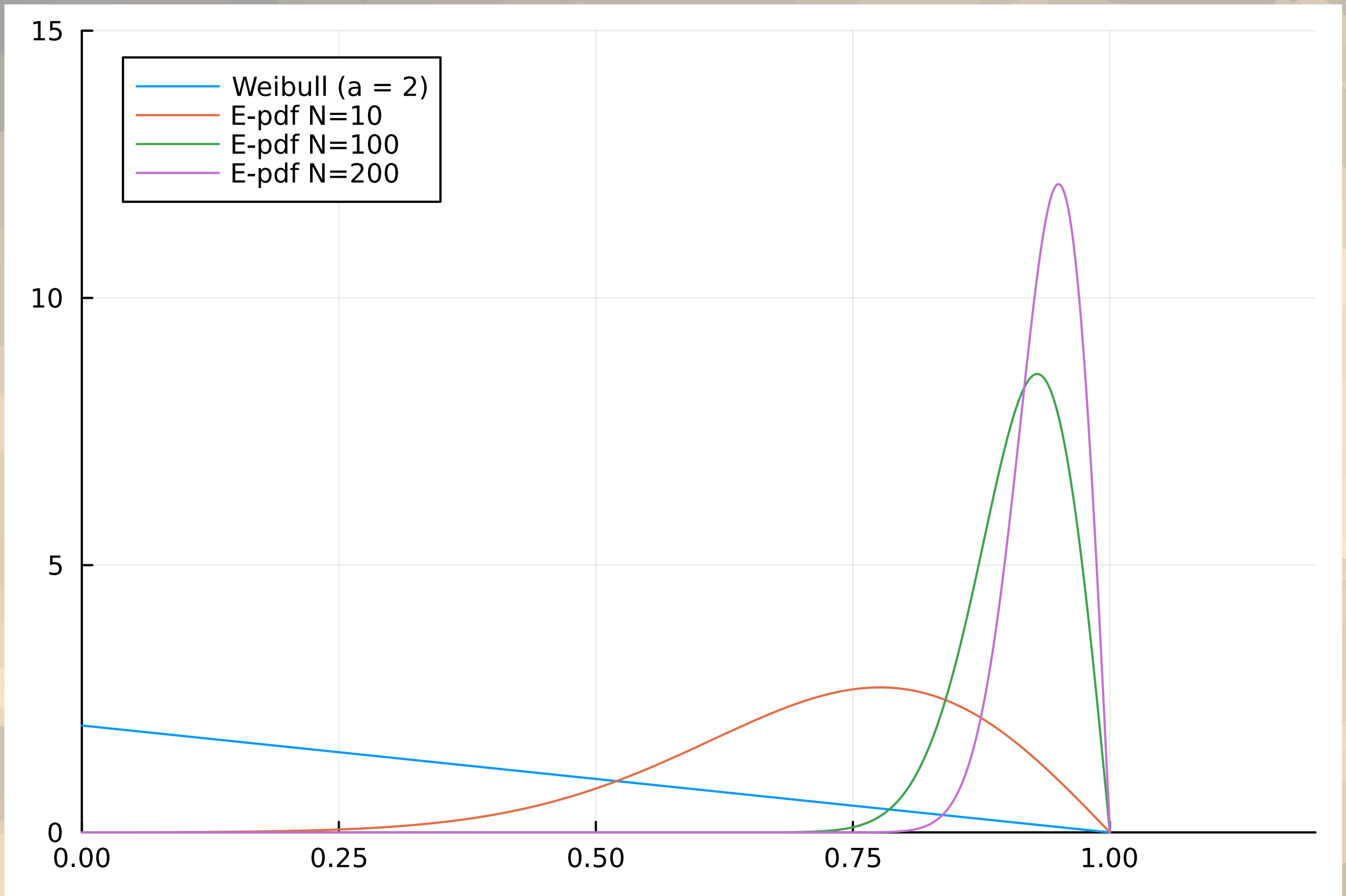
Limited range pdf - example

$$q(x) = Ne^{-N(1-x)} \qquad \alpha = 1$$



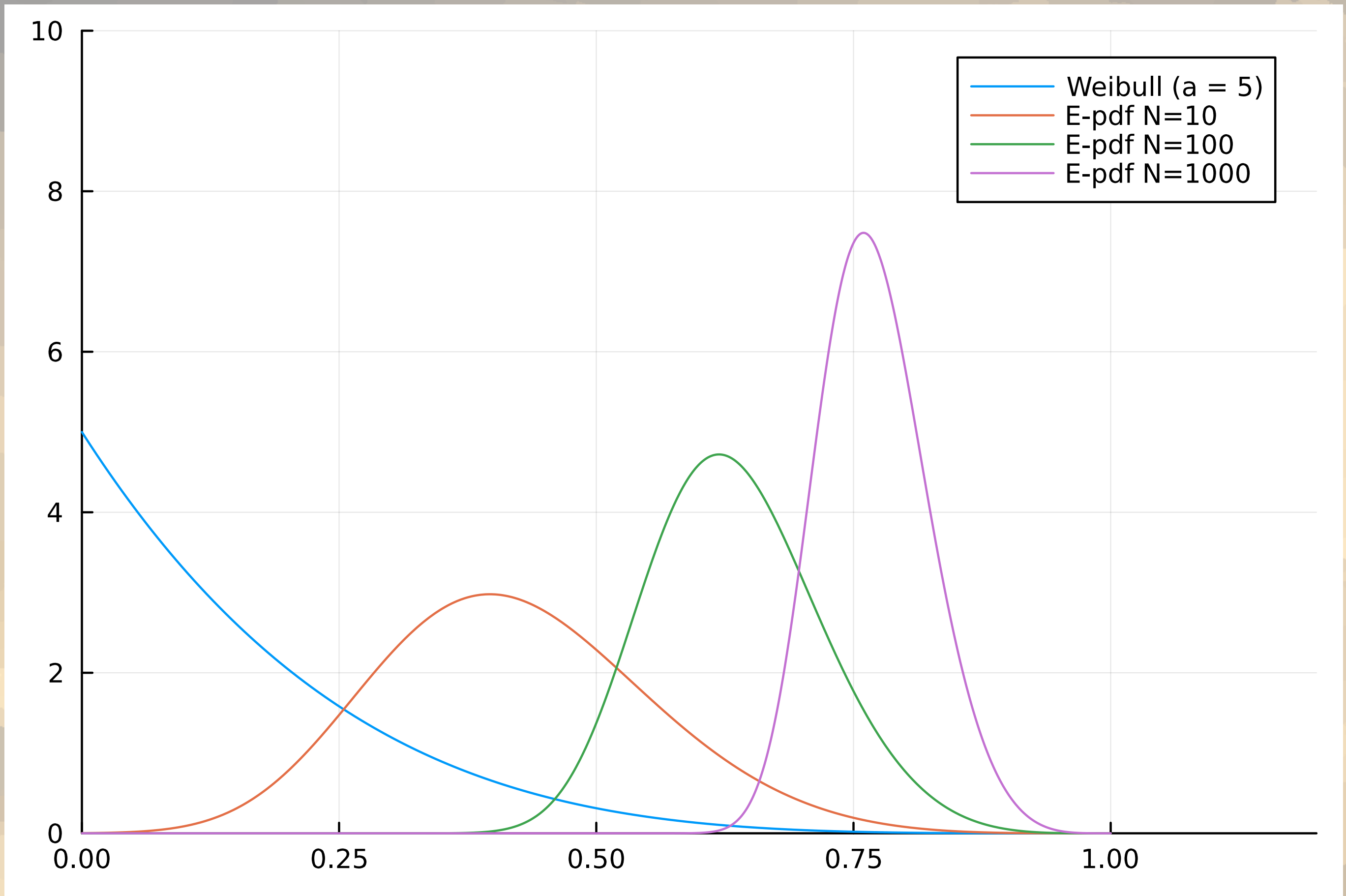
Limited range pdf - example

$$q(x) \;=\; 2N(1-x)e^{-N(1-x)^2} \qquad \alpha = 2$$



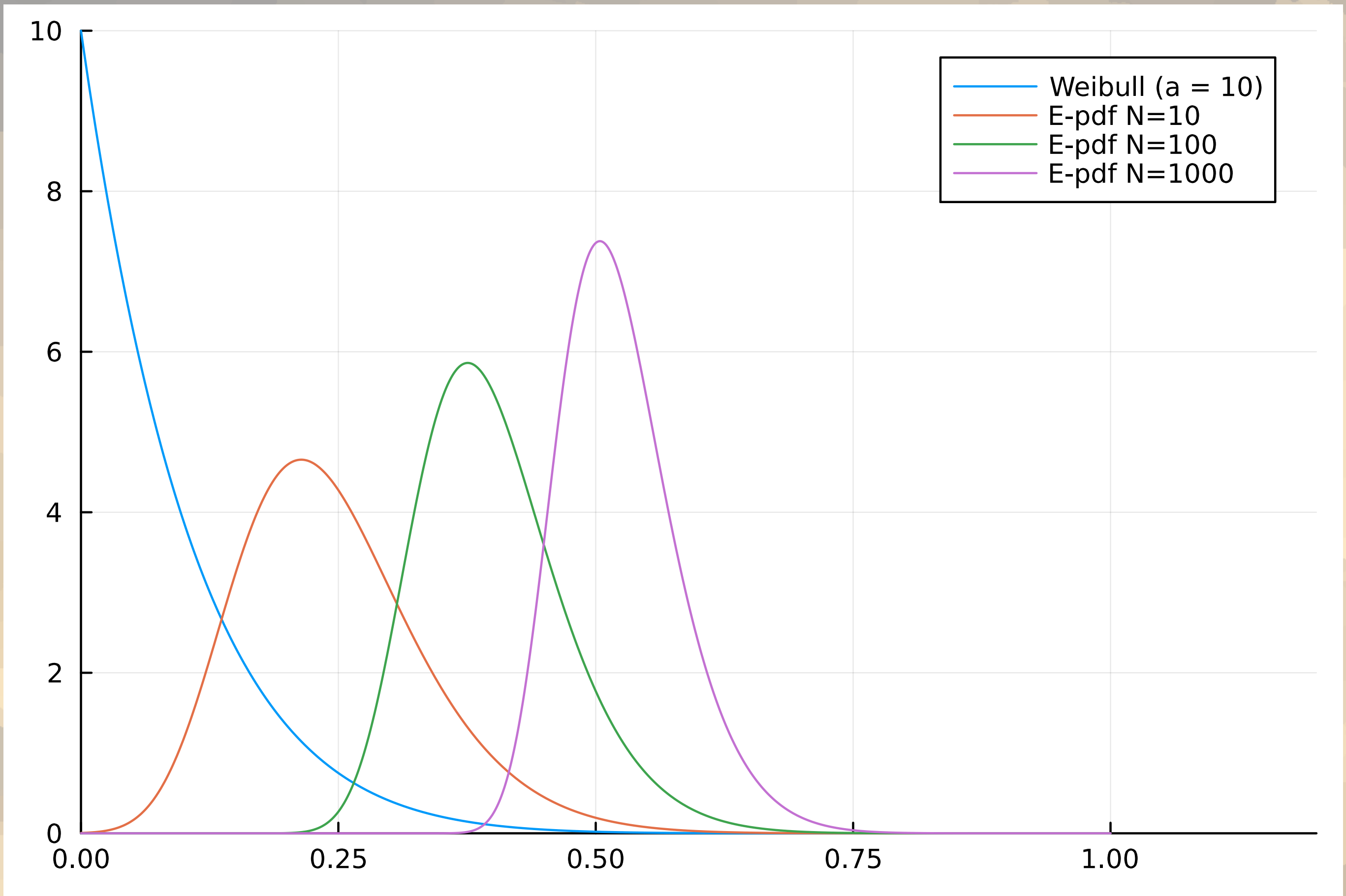
Limited range pdf - example

$$q(x) \;=\; N\alpha(1-x)^{\alpha-1}e^{-N(1-x)^{\alpha}} \qquad \alpha = 5$$



Limited range pdf - example

$$x = 1 - \epsilon \quad q(\epsilon) = N\alpha\epsilon^{\alpha-1}e^{-N\epsilon^{\alpha}} \qquad \alpha = 10$$



Power decay (Fréchet) pdf

- Power decay pdf: $p(x) = 1/x^a$ for $x \to \infty$

$$p(x) \;=\; b\alpha x^{-\alpha-1} \text{ for } x \to \infty$$

Cumulative distribution $(x > 0)$

$$P(x) = 1 - bx^{-\alpha} \quad \text{for} \quad x \to \infty$$

Extreme value cumulative distribution

$$P^N(x) = \left[1 - bx^{-\alpha}\right]^N \Longrightarrow x = (bN)^{1/\alpha}u \Longrightarrow \left[1 - \frac{u^{-\alpha}}{N}\right]^N$$

Fréchet pdf

In the limit $N \to \infty$ and keeping in mind that $x, u > 0$

$$Q(u) = e^{-u^{-\alpha}}$$

and pdf

$$q(u) = \alpha u^{-\alpha-1} e^{-u^{-\alpha}}$$

Fréchet pdf

Cumulative extreme value distribution in terms of x

$$Q(x) \; = \; e^{-Nx^{-\alpha}}$$

probability density

$$q(x) \; = \; N\alpha x^{-\alpha-1}e^{-Nx^{-\alpha}}$$

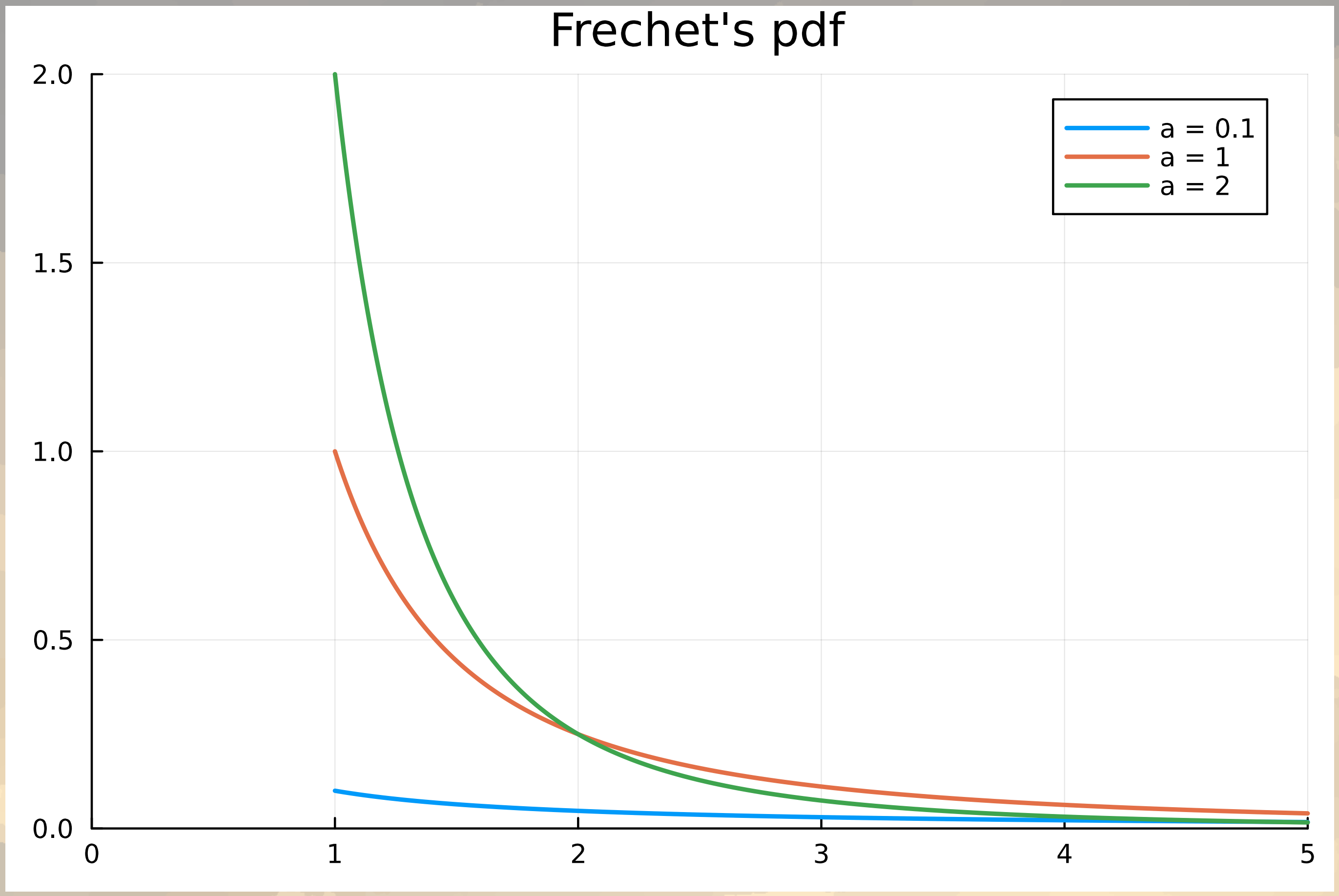
Fréchet pdf - example

$$p(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ \alpha x^{-\alpha-1} & \text{for } x > 1 \end{cases}$$

cumulative distribution

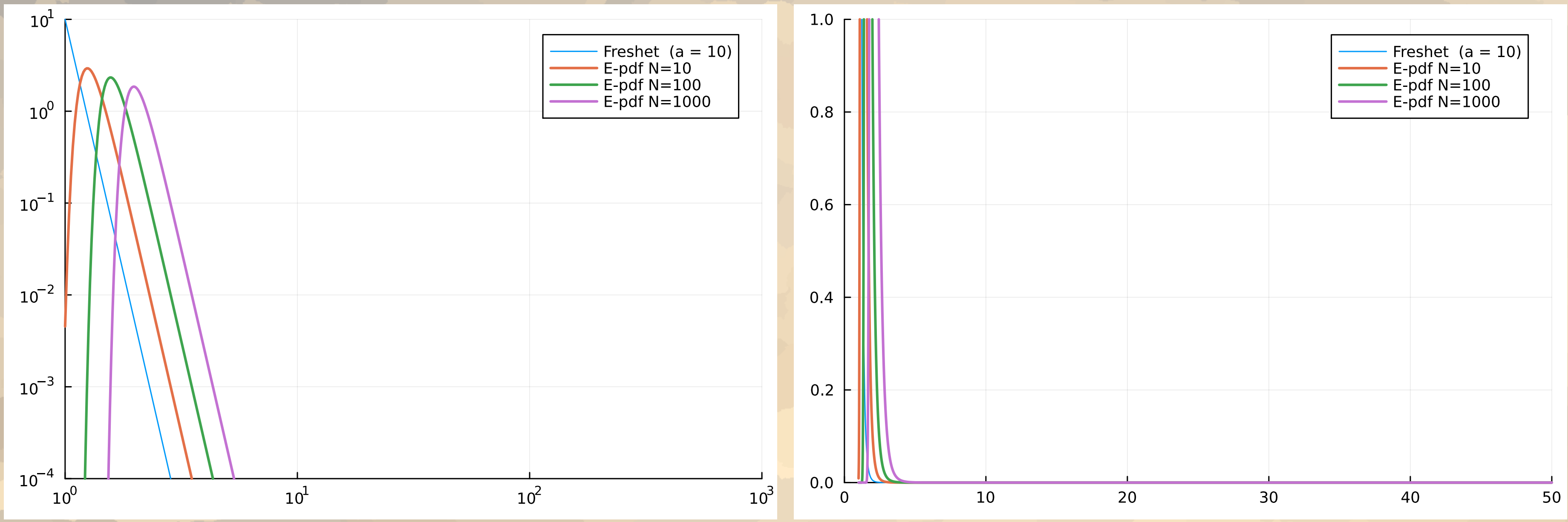
$$P(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ 1 - x^{\alpha} & \text{for } x > 1 \end{cases}$$

Fréshet pdf - example



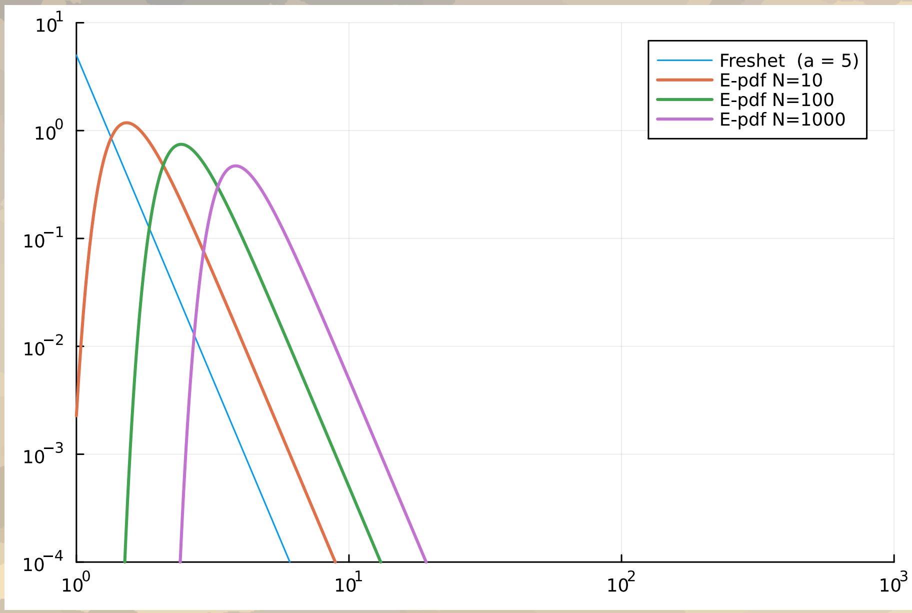
Fréshet pdf - example

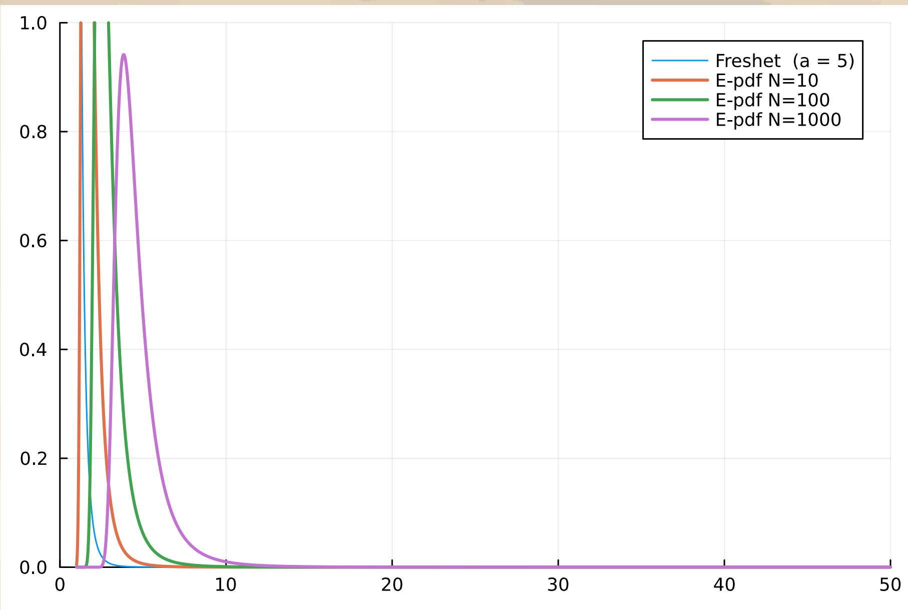
$$p(x) \;=\; \alpha x^{-\alpha-1} \qquad \alpha = 10$$



Fréshet pdf - example

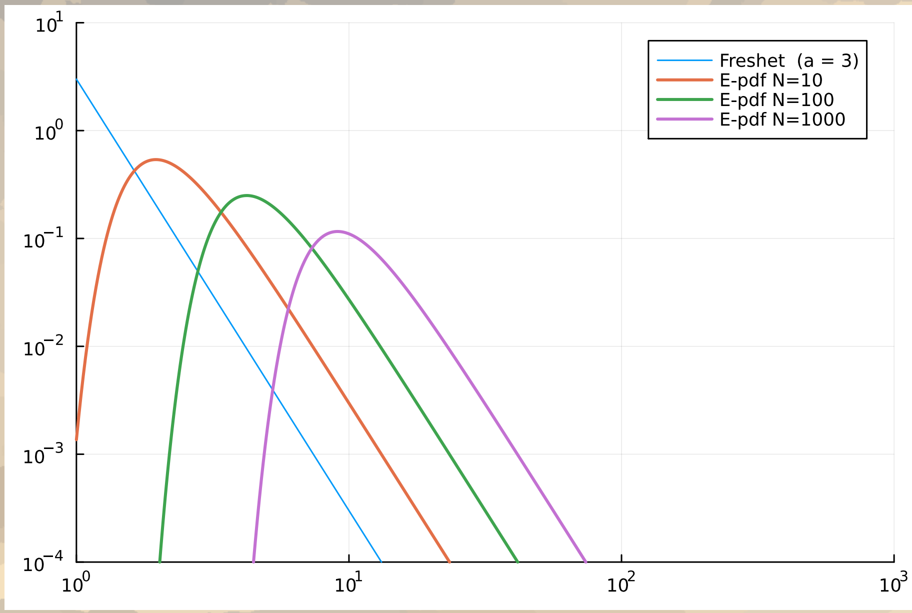
$$p(x) = \alpha x^{-\alpha-1} \qquad \alpha = 5$$

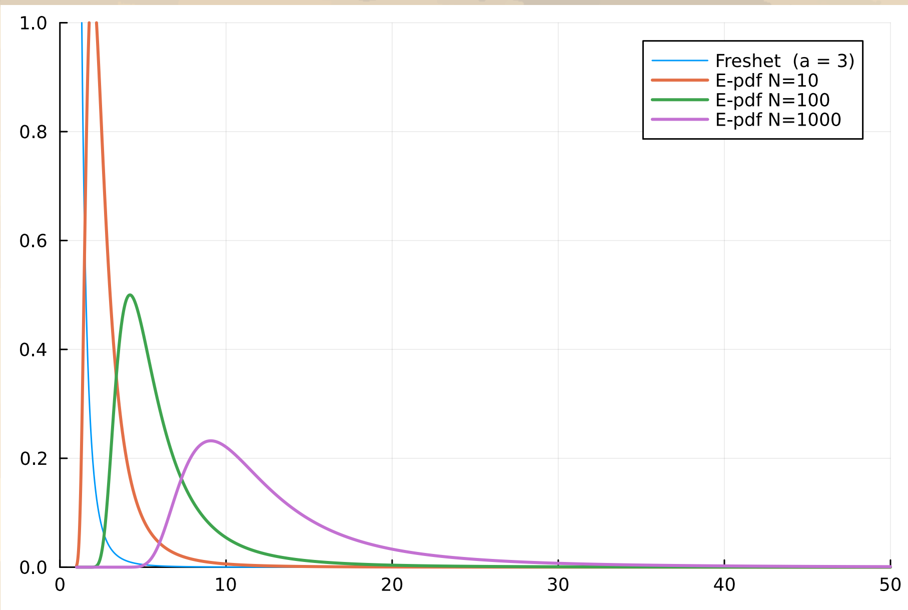




Fréshet pdf - example

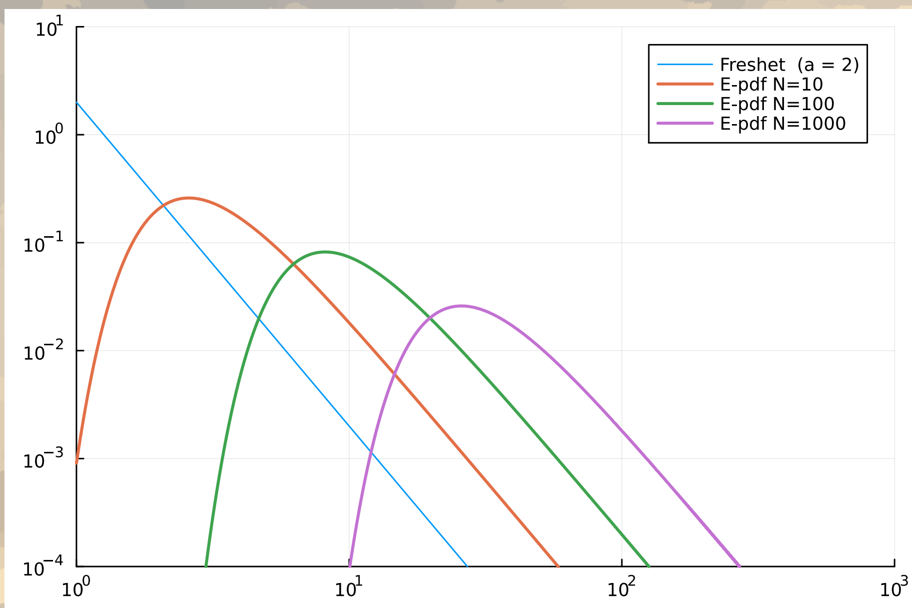
$$p(x) \;=\; \alpha x^{-\alpha-1} \qquad \alpha = 3$$

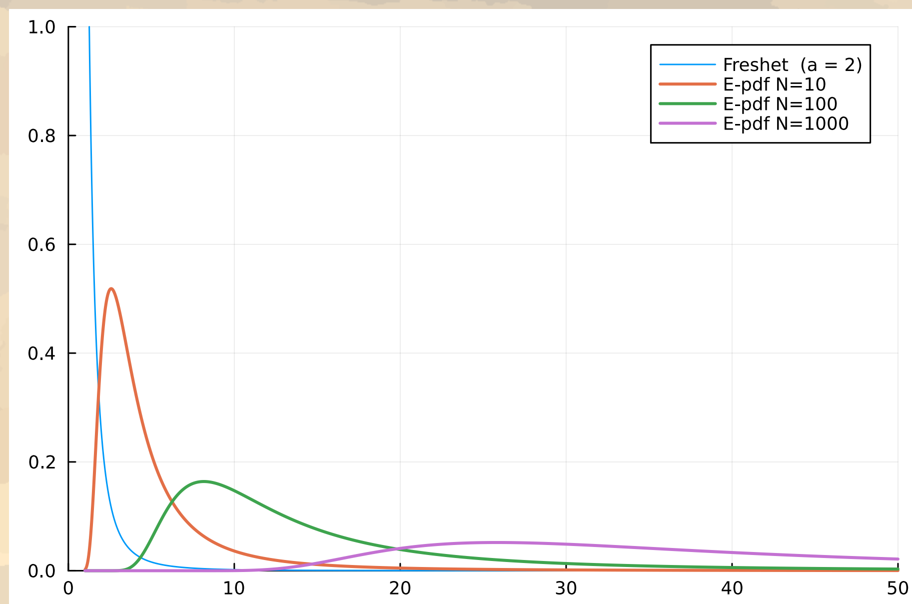




Fréshet pdf - example

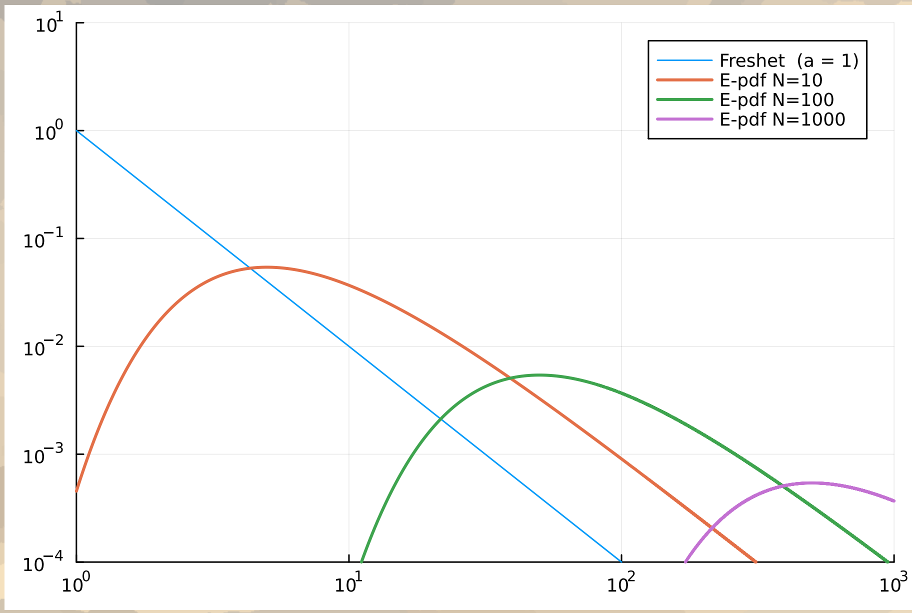
$$p(x) \ = \ \alpha x^{-\alpha-1} \qquad \alpha = 2$$

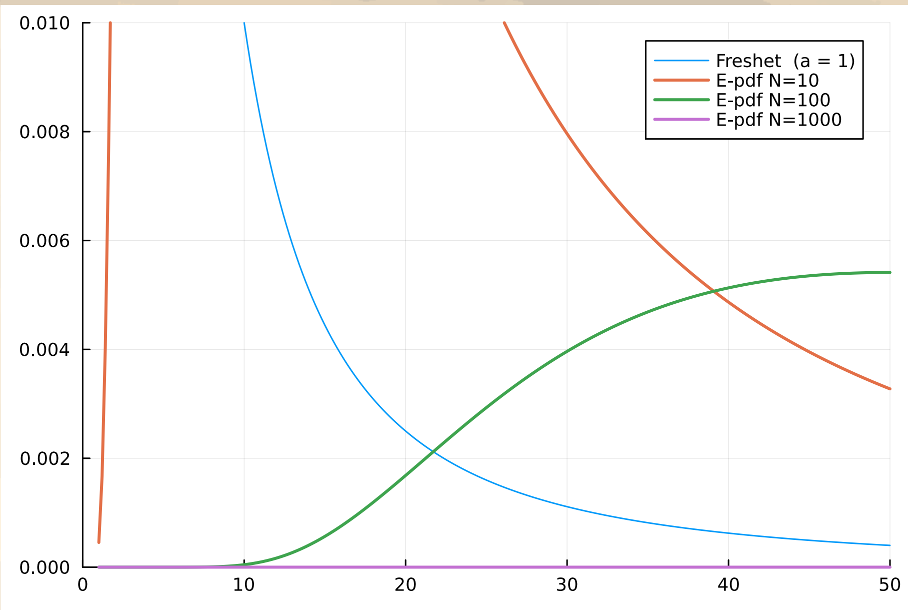




Fréshet pdf - example

$$p(x) = \alpha x^{-\alpha-1} \qquad \alpha = 1$$





Extreme value statistics - summary

Extreme values of random variable x i.e. values essentially larger then expected or median values described by probability distribution $p(x)$ depends only on asymptotic behavior of $p(x)$ for large x Actually, their distributions falls into one of three classes:

- ✦ Weibull's distribution for $p(x)$ with strong upper bound
- ✦ Fréchet distributions for power-like decreasing $p(x)$
- ✦ Gumbel class in most general cases of $p(x)$

Lecture - summary

Just a time for questions ...

Exam: List of exam subjects available at
https://private.igf.edu.pl/ debski/HTML/teaching.html

Although there is no deadline, please do not thing toooo long.

END