Advanced statistical methods and bayesian inference in scientific research

Lecture 1

W. Dębski

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About the lecture

Title:	**Advanced statistical methods and bayesian inference in scientific research**
Lecturer:	Wojciech Dębski
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www:	private.igf.edu.pl/∼debski (Lectures)
Exam:	individual (preferable essay on selected topic)
Duration :	22-26 hours + hands-on (24 h.)
ETC points :	3

Venue + On-line access

Lecture:	CAMK, Monday 13:00 - 14:30
Hands-on:	IGF PAN Friday 14:00 - 15:30
On-line	via zoom

https://us02web.zoom.us/j/87270106568?pwd=REszR1hrbG45WXRrdUxFL1NlVFFRQT09

Meeting ID: 872 7010 6568
Passcode: 242474

General plan of the lecture

Intro.	*mathematics of probability calculus*
I	basics of the probability calculus
II	stochastic process - introduction
III	Advanced: entropy, Fisher information, Cramer-Rao bound
III	Bayesian inference
IV	Monte Carlo sampling techniques
V	Statistics of extreme values
Hands-on ...	*** **Programing in Julia** ***
Hands-on ...	Simple Monte Carlo sampling
Hands-on ...	Elementary statistical algorithms

Literature...

- N.G van Kampen **Stochastic Processes in Physics and Chemistry**, 2007
- A. Tarantola, **Inverse Problem Theory and Methods for Model Parameter Estimation**, SIAM,2005 (on-line available)
- Arfken, G. Mathematical Methods for Physicists, 1989, San Diego: Academic Press.
- Bartlett, A. A An Introduction to Stochastic Processes, 1966, Cambridge: Cambridge Univ. Press.
- Berger, L. O. (1985). Statistical Decision Theory and Bayesian Analysis. Springer-Verlag.
- Box, G. E. P. and G. C. Tiao (1973). Bayesian Inference in Statistical Analysis. Wiley.
- Brandt, S. (1999). Data Analysis. Statistical and Computational Methods for Scientists and Engineers (3rd ed.).

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- Gelman, A., J. B. Carlin, H. S. Stern, and D. B. Rubin (1997). Bayesian Data Analysis. Chapmann & Hall.
- Gamerman, D. (1997). Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference. Chapman and Hall.
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- Jeffreys, H. (1983). Theory of Probability. Oxford: Clarendon Press.
- Jaynes, E. T., 2003, Probability Theory: the Logic of Science, Cambridge University Press
- Kolmogorff, A. (1956). Foundations of the Theory of Probability. New York: Chelsea.

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- E. J. Gumbel, Statistical Theory of Extreme Values and Some Practical Applications (National Bureau of Standards Applied Mathematics Series, 33; Washington D.C., 1954)
- W. Debski Probabilistic Inverse Theory Advances in Geophys. (2010) Vol. 52, pp. 1-102 doi:10.1016/S0065-2687(10)52001-6
- W. Navidi, Statistics for Engineers and Scientists (5th Edition) , McGraw Hill Higher Education
- B. Massimiliano, Statistics and Analysis of Scientific Data
- L. Hatcher, Advanced Statistics in Research, Reading, Understanding, and Writing Up Data Analysis Results

Probability - what does it mean

The word probability has been used in a variety of ways since it was first applied to the mathematical study of games of chance. Does probability measure the real, physical, tendency of something to occur, or is it a measure of how strongly one believes it will occur, or does it draw on both these elements? In answering such questions, mathematicians interpret the probability values of ***probability theory****.*

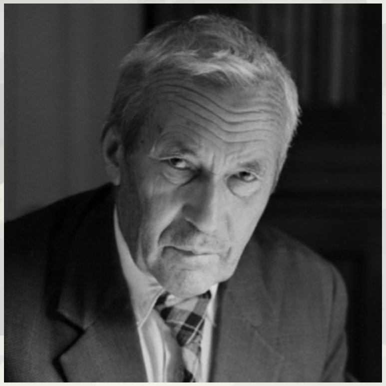
[Wikipedia]

real world ⇔ abstract mathematics

Probability theory - Axioms

The structure $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{p})$ called the **probability space**:

Ω	-	space of all elementary events
$\mathcal{F}$	-	set of all subsets of Ω
$\mathcal{A} \in \mathcal{F}$	-	are called **events**,
$\mathbf{p}()$	-	function $\Omega \to R^+$.



Probability theory - Axioms

Let Ω be a set (sample space)
Let $\mathcal{F}$ be a σ-algebra of all subsets of Ω . (event space)

σ-algebra of all subsets of Ω is a family of Ω's subsets such that

- ✦ $\Omega \in \mathcal{F}$
- ✦ $A \in \mathcal{F} \Rightarrow \bar{A} \in \mathcal{F}$ where $\bar{A}$ denotes compactness of A
- ✦ for all $A_i \in \mathcal{F}$ $\quad \bigcup_i A_i \in \mathcal{F}$

Let $\mathbf{p}(\cdot)$ be a function: $\Omega \rightarrow R^+$ (probability measure)

a) $\mathbf{p}(\varnothing) = 0$

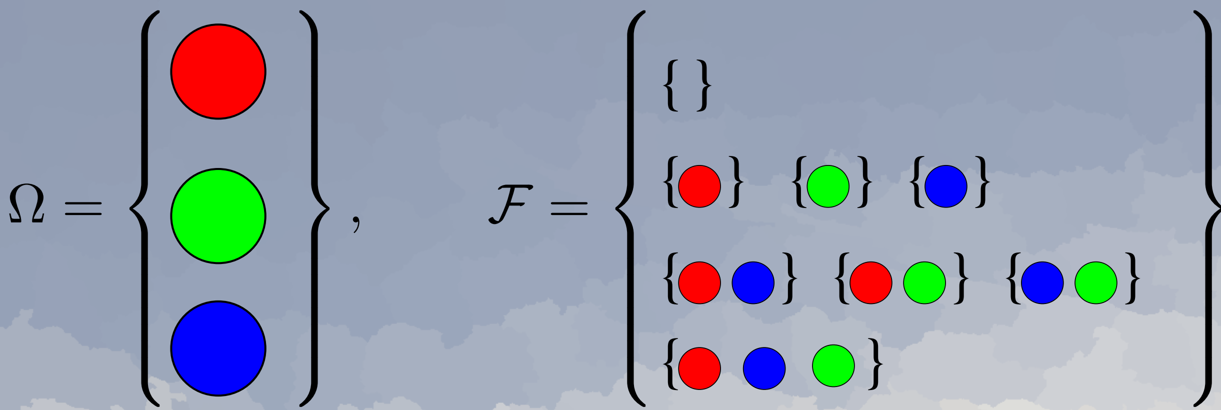
b) $\mathbf{p}(\mathcal{A}) \geq 0$ for all $\mathcal{A} \in \mathcal{F}$

c) $\mathbf{p}(\bigcup_i A_i) = \sum_i \mathbf{p}(A_i)$ for any $A_i, A_j \in \mathcal{F}$

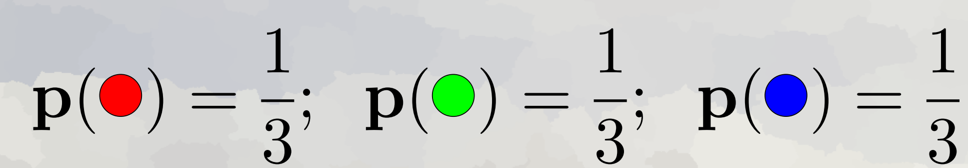
$$A_i \cap A_j = \varnothing \quad \text{if} \;\; i \neq j$$

If $\mathbf{p}(\Omega) = 1$ than $\mathbf{p}()$ is called probability

Example



Defining $\mathbf{p}()$. Experiment: drawing a single bal



Remarks:

1. $\mathbf{p}()$ does not need to be defined over the whole $\mathcal{F}$

2. attaching $\mathbf{p}()$ to elements of $\mathcal{F}$ is `arbitrary`

Subjective (classical) approach (after Jaynes 2003)

> The probability of an event is the ratio of the number of cases favorable to it, to the number of all cases possible when nothing leads us to expect that any one of these cases should occur more than any other, which renders them, for us, equally possible.
>
> J. Bernouli, P. Laplace

Comment:

This definition is essentially subjective and based on the *principle of insufficient reason*. which states that in the absence of any relevant evidence all the possible outcomes under consideration should not be distinguished. In Bayesian approach, this is the simplest non-informative or *a priori* distribution.

Frequentists approach

Probabilities are discussed only when dealing with well-defined random experiments. The set of all possible outcomes of a random experiment is called the sample space of the experiment. An event is defined as a particular subset of the sample space to be considered. For any given event, only one of two possibilities may hold: it occurs or it does not. The relative frequency of occurrence of an event, observed in a number of repetitions of the experiment, is a measure of the probability of that event.

after Wikipedia

Comment:

This definition applies only to situation of repeatable processes. It provides guidance for how to apply mathematical probability theory to real-world situations. It offers distinct guidance in the construction and design of practical experiments. Whether this guidance is useful, or leads to mis-interpretation, has been a source of controversy.

Illustration

$$\Omega = \{{\color{red}\bullet}, {\color{green}\bullet}, {\color{blue}\bullet}\}$$

- **Frequentists (physical) approach**
  Repeat drawing-and-return of a ball N times

$$\mathbf{p}({\color{red}\bullet}) = \frac{Nr}{N}; \;\; \mathbf{p}({\color{green}\bullet}) = \frac{Ng}{N}; \;\; \mathbf{p}({\color{blue}\bullet}) = \frac{Nb}{N};$$

- **Bayesian (evidential/subjective) approach**
  Since bals are indistinguishable a chance of drawing any should be *a priori* the same

$$\mathbf{p}({\color{red}\bullet}) = \frac{1}{3}; \;\; \mathbf{p}({\color{green}\bullet}) = \frac{1}{3}; \;\; \mathbf{p}({\color{blue}\bullet}) = \frac{1}{3};$$

Interpretations (philosophy)

- Physical approach

  Physical (objective, frequency) probabilities, are associated with random physical systems like rolling dice and radioactive atoms. In such systems, a given type of event tends to occur at a given relative frequency, in a long run of trials. Physical probabilities are suppose to explain these stable frequencies.

- Bayesian approach

  Bayesian (evidential) probability can be assigned to any statement even when no random process is involved, as a way to represent its subjective plausibility, or the degree to which the statement is supported by the available evidence. On most accounts, evidential probabilities are considered to be degrees of belief.

See you next week ...