Inverse Theory - a modern method of data analysis

Lecture 3

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Linear Invers Problem

$$\mathbf{m} = (m_1, m_2, \cdots m_M) \in \mathcal{M}$$

$$\mathbf{d} = (d_1, d_2, \cdots d_N) \in \mathcal{D}$$

$$\mathbf{d}(\mathbf{m}) = \mathbf{G} \cdot \mathbf{m}$$

$$d_i = \sum_j g_{ij} m_j$$

Problem: How to find **m** provided **d** is known ?

Linear problem

$$\mathbf{d} = \mathbf{G} \cdot \mathbf{m} \quad / \times \mathbf{G}^T$$

$$\mathbf{G}^T \cdot \mathbf{d} = \mathbf{G}^T \mathbf{G} \cdot \mathbf{m}$$

$$\mathbf{G}^T \mathbf{G} \implies \mathbf{G}^T \mathbf{G} + \lambda \mathbf{I}$$

$$\mathbf{m}^{est} = (\mathbf{G}^T \mathbf{G} + \lambda \mathbf{I})^{-1} \mathbf{G}^T \cdot \mathbf{d}^{obs}$$

Linear problem

$$\mathbf{m} \sim \mathbf{m}^{apr}; \qquad \bar{\mathbf{d}} = \mathbf{G} \cdot \mathbf{m}^{apr}$$

$$\mathbf{d} = \mathbf{G} \cdot \mathbf{m} \quad / - \bar{\mathbf{d}}$$

$$\mathbf{d} - \bar{\mathbf{d}} = \mathbf{G} \cdot (\mathbf{m} - \mathbf{m}^{apr})$$

$$\mathbf{G}^T \cdot (\mathbf{d} - \bar{\mathbf{d}}) = \mathbf{G}^T\mathbf{G} \cdot (\mathbf{m} - \mathbf{m}^{apr})$$

$$\mathbf{G}^T\mathbf{G} \implies \mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}$$

$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot (\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr})$$

Linear problem

λ - arbitrary parameter $\Longrightarrow$ subjective solution

$$\mathbf{m}^{est} \; = \; \mathbf{m}^{est}(\lambda)$$

1) $\lambda \rightarrow \infty$

$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda}\,\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr}\right)$$

2) $\lambda \rightarrow 0$ (if exists)

$$\mathbf{m}^{est} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs}\right)$$

Linear problem - resolution analysis

$$\mathbf{m} = (m_1, m_2, \ldots \mathbf{m}_M)$$

- how accurately one can estimate m_i ?
- are m_i, m_j correlated ?

$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{G} \cdot \mathbf{m}^{apr}\right)$$

$$\mathbf{d}^{obs} = \mathbf{d}^{true}; \qquad \mathbf{d}^{true} = \mathbf{G} \cdot \mathbf{m}^{true}$$

$$\mathbf{m}^{est} - \mathbf{m}^{apr} = \mathbf{R}(\lambda) \cdot (\mathbf{m}^{true} - \mathbf{m}^{apr})$$

Linear problem - resolution analysis

$$\text{definition}: \quad \frac{\mathbf{A}}{\mathbf{B}} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{R} = \frac{\mathbf{G}^T\mathbf{G}}{\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}}$$

$$\mathbf{m}^{est} = \mathbf{R}(\lambda) \cdot \mathbf{m}^{true}$$

$\mathbf{m}^{est}$ are filtered estimators of $\mathbf{m}^{true}$

Linear problem - well defined problem

$$\det(\mathbf{G}^T \cdot \mathbf{G}) \neq 0 \longrightarrow \lambda = 0$$

$$\mathbf{R} = \frac{\mathbf{G}^T\mathbf{G}}{\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}}$$

$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{m}^{est} = \mathbf{m}^{true}$$

Linear problem - under-determined problem

$$\det(\mathbf{G}^T \cdot \mathbf{G}) = 0 \longrightarrow \lambda \neq 0$$

$$\mathbf{R} = \frac{\mathbf{G}^T\mathbf{G}}{\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}}$$

$$\mathbf{R} \neq \mathbf{I}$$

$$\mathbf{m}^{est} \neq \mathbf{m}^{true}$$

Linear problem - regularization

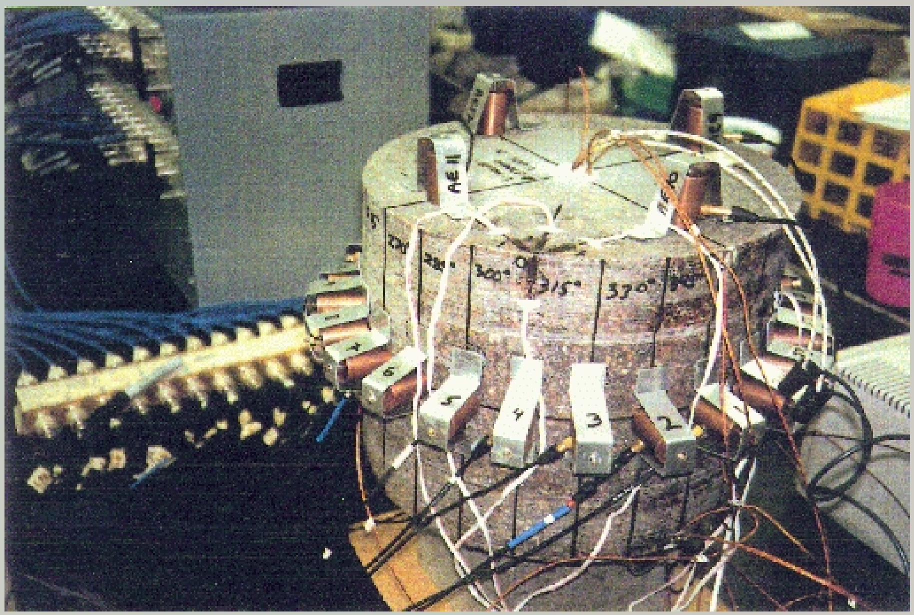
1) $\lambda \to \infty$

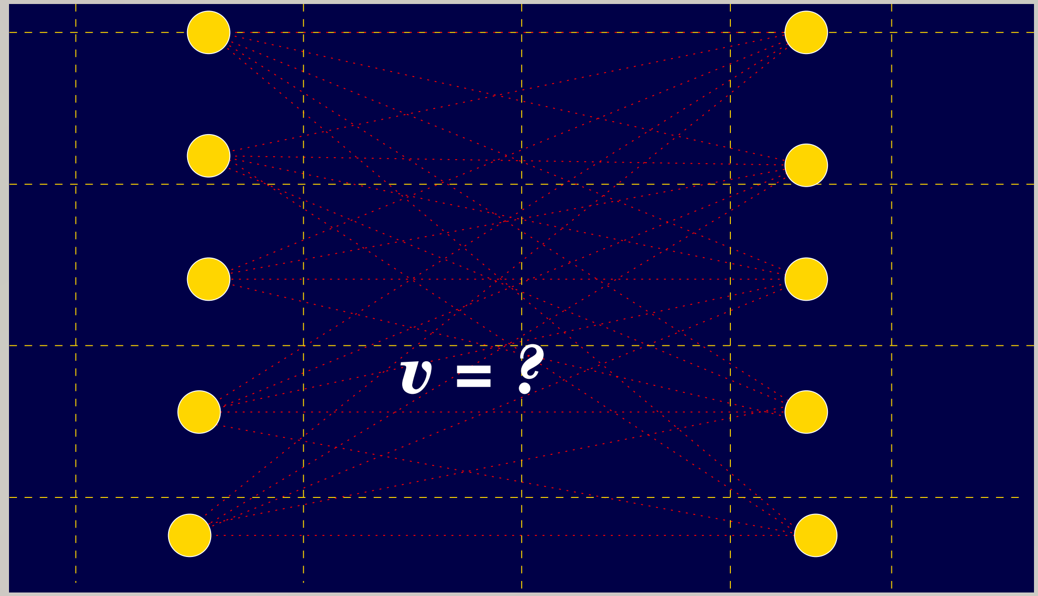
$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda}\,\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{G}\cdot\mathbf{m}^{apr}\right)$$

2) $\lambda \to 0$ (if exists)

$$\mathbf{m}^{est} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs}\right)$$

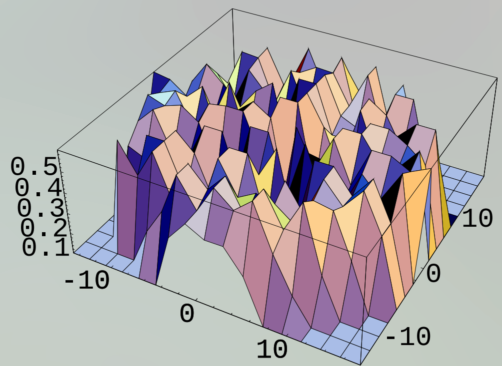
Linear problem - illustration

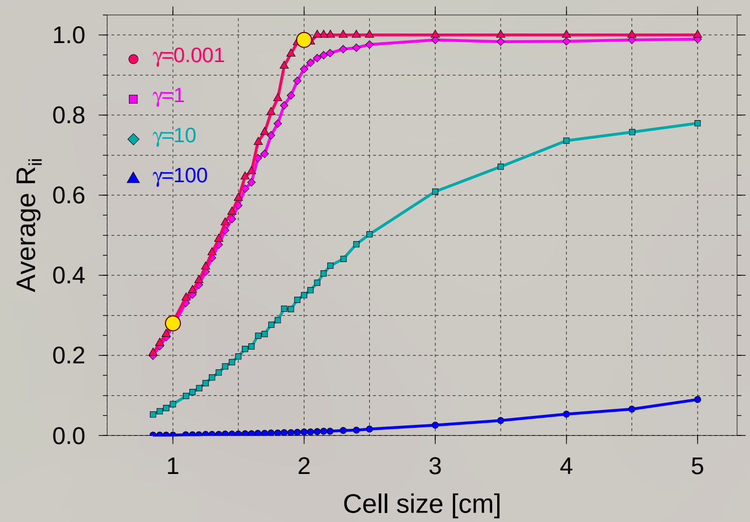


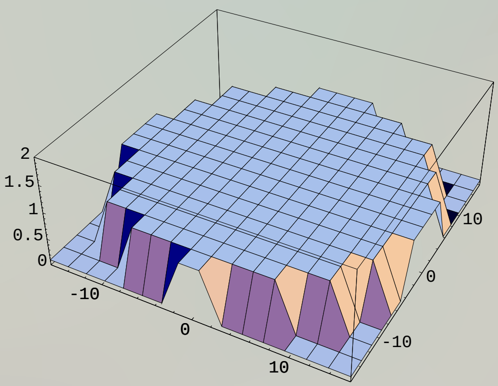


$$\Delta \mathbf{t} = \mathbf{G} \cdot \mathbf{s}$$

Linear problem - resolution illustration







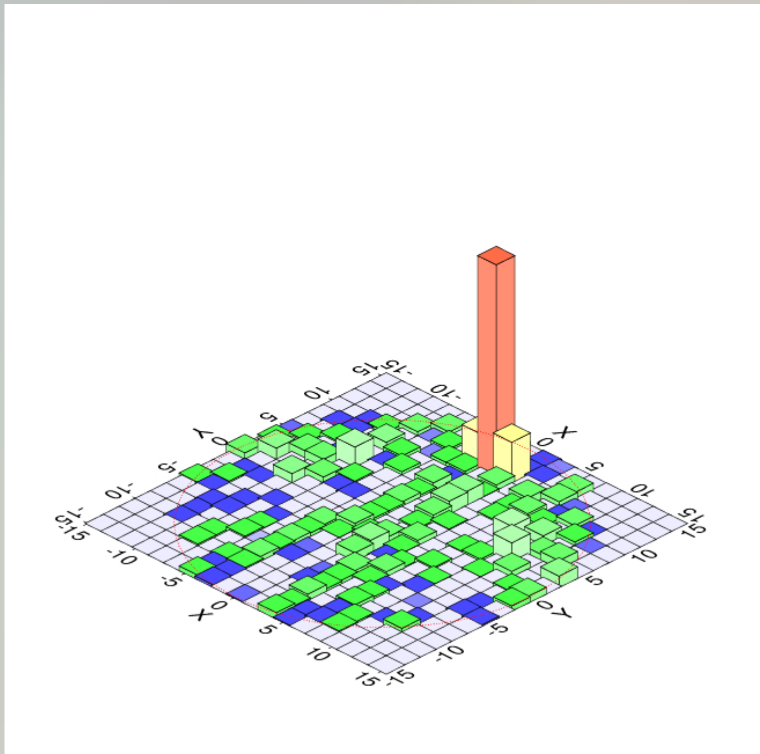
$$\mathbf{R} = \frac{\mathbf{G}^T\mathbf{G}}{\mathbf{G}^T\mathbf{G}+\lambda\mathbf{I}} \qquad \lambda \to \infty \quad \mathbf{R} \sim \frac{1}{\lambda}\cdots$$

$$\mathbf{m}^{est} \sim \mathbf{m}^{apr} + \frac{1}{\lambda}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{G}\cdot\mathbf{m}^{apr}\right)$$

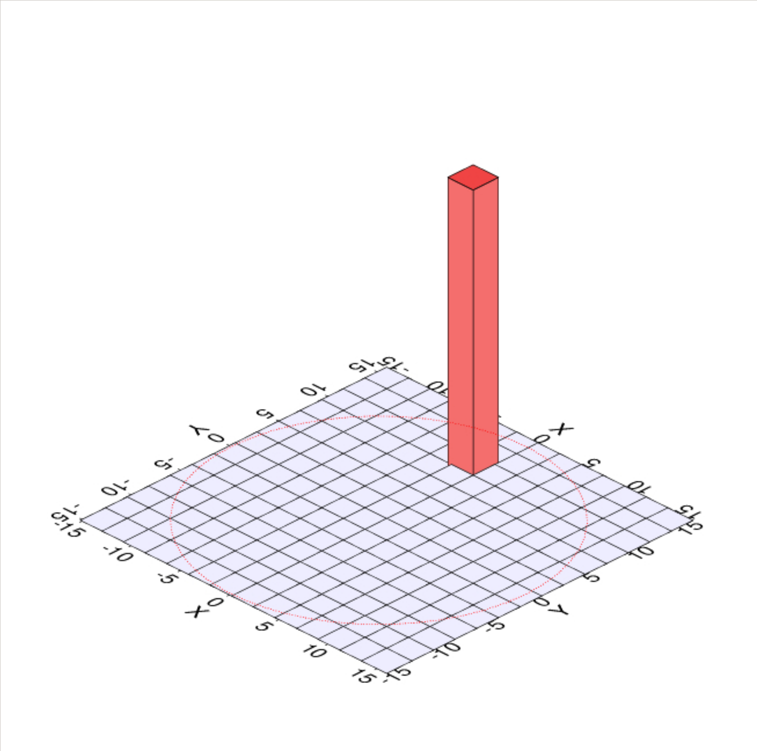
$$\mathbf{m}^{est} = \mathbf{m}^{apr} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot (\ldots)$$

Linear problem - correlations illustration

$\lambda = 1$



$\lambda = 100$



Linear problem - data error sensitivity

$$\mathbf{m}^{apr} = \mathbf{m}^{true}; \qquad \mathbf{d}^{obs} = \mathbf{d}^{true} + \delta$$

$$\mathbf{m}^{est}(\lambda) = \mathbf{m}^{true} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \left(\mathbf{d}^{obs} - \mathbf{d}^{true}\right)$$

$$\mathbf{m}^{est} = \mathbf{m}^{true} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \delta$$

$$\mathbf{\Delta m}^{\mathbf{est}} = (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \delta$$

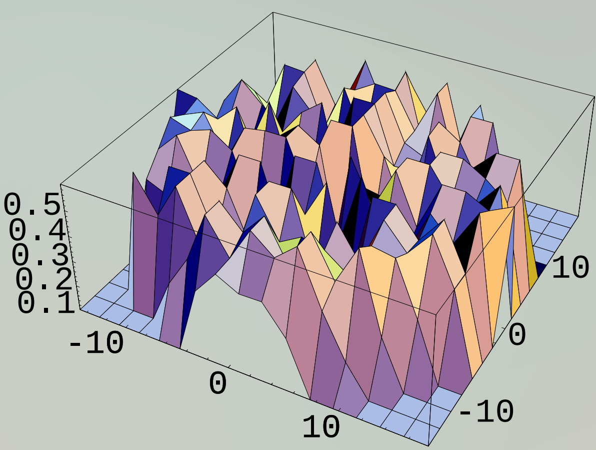
Linear problem - error propagation

$$\mathbf{m}^{est}(\lambda) = \mathbf{m}^{apr} + (\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T \cdot \begin{cases} (\mathbf{d}^{true} - \mathbf{G}\cdot\mathbf{m}^{apr}) \\ +\delta \end{cases}$$

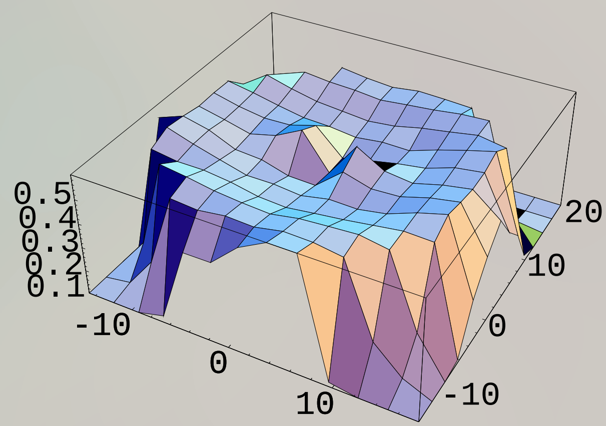
Przykład 1D: $d = gm$

$$\mathbf{m}^{est}(\lambda) = \mathbf{m}^{apr} + \frac{g}{g^2+\lambda} \cdot \begin{cases} (d^{true} - g\,\mathbf{m}^{apr}) \\ +\delta \end{cases}$$

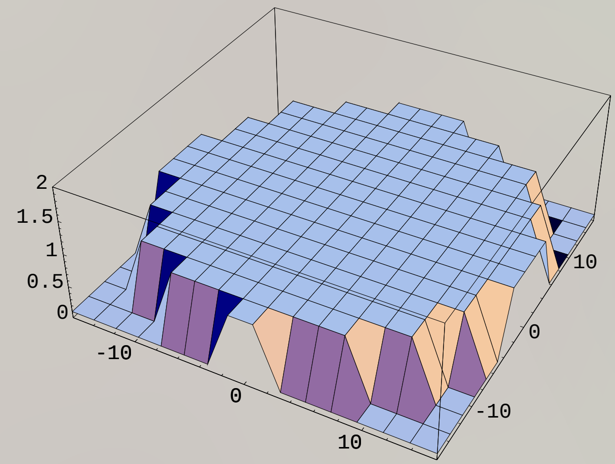
Linear problem - error propagation



$\lambda \sim 0.01$



$\lambda \sim 1$



$\lambda \sim 100$

Linear problem- Residua

$$\mathbf{d}^{est} = \begin{cases} +\left[\mathbf{I} - \mathbf{G}(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T\right]\mathbf{G}\mathbf{m}^{apr} \\ +\mathbf{G}(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1})\mathbf{G}^T\mathbf{d}^{true} \\ +\mathbf{G}(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1})\mathbf{G}^T\delta \end{cases}$$

$$\mathbf{d}^{est} - \mathbf{d}^{true} = \begin{cases} +\lambda\,\mathbf{I}\left(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}\right)^{-1}\mathbf{G}^T(\mathbf{G}\mathbf{m}^{apr} - \mathbf{d}^t) \\ +\mathbf{G}\left(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}\right)^{-1}\mathbf{G}^T\,\delta \end{cases}$$

Linear problem - inversion efficiency

Final verification of inversion results: how accurately $\mathbf{m}^{est}$ "reproduces" $\mathbf{d}^{obs}$?

RMS value of data residua

$$r_{rms} = \frac{1}{N} \sum_i \sqrt{\left(\mathbf{d}_i^{obs} - \mathbf{d}_i(\mathbf{m}^{est})\right)^2}$$

Linear problem - inversion efficiency (practical approach)

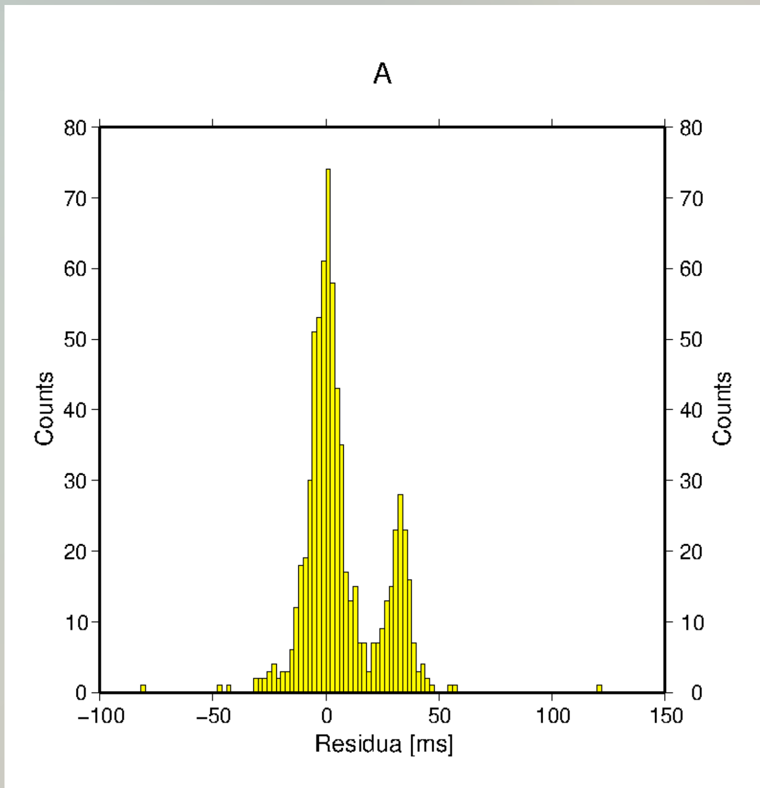
$$\bar{r} = \frac{1}{N}\sum_{1}^{N} r_i$$

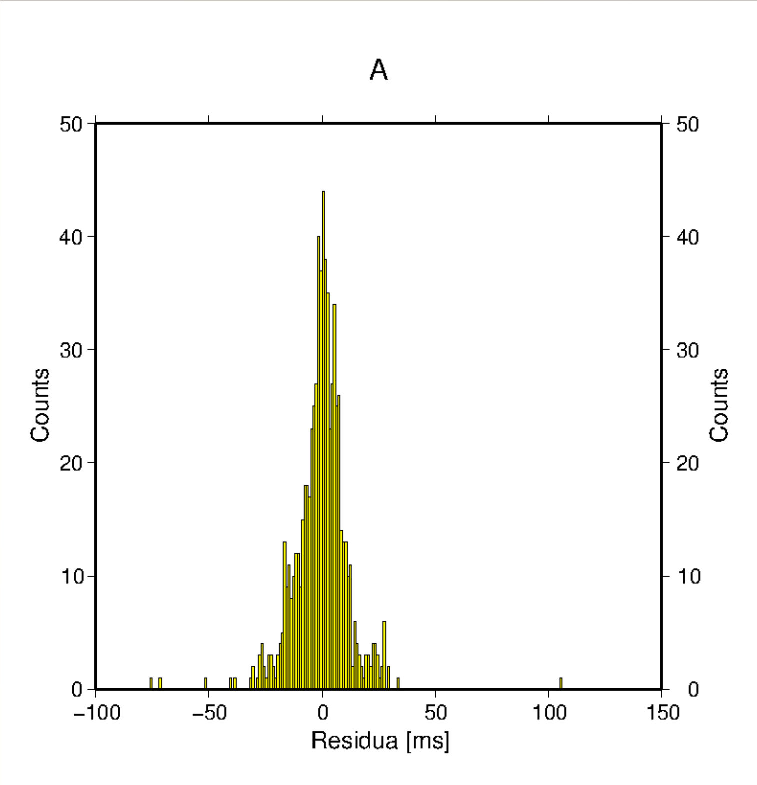
$$\delta r_i = \sqrt{\frac{1}{N}(r_i - \bar{r})^2}$$

Provides a qualitative measuer of inversion “efficiency” and simultaneously data quality

Linear problem - inversion efficiency (better approach)

compare residua histogram for $\mathbf{m}^{apr}$ and $\mathbf{m}^{est}$ models





Linear problem - 1D example

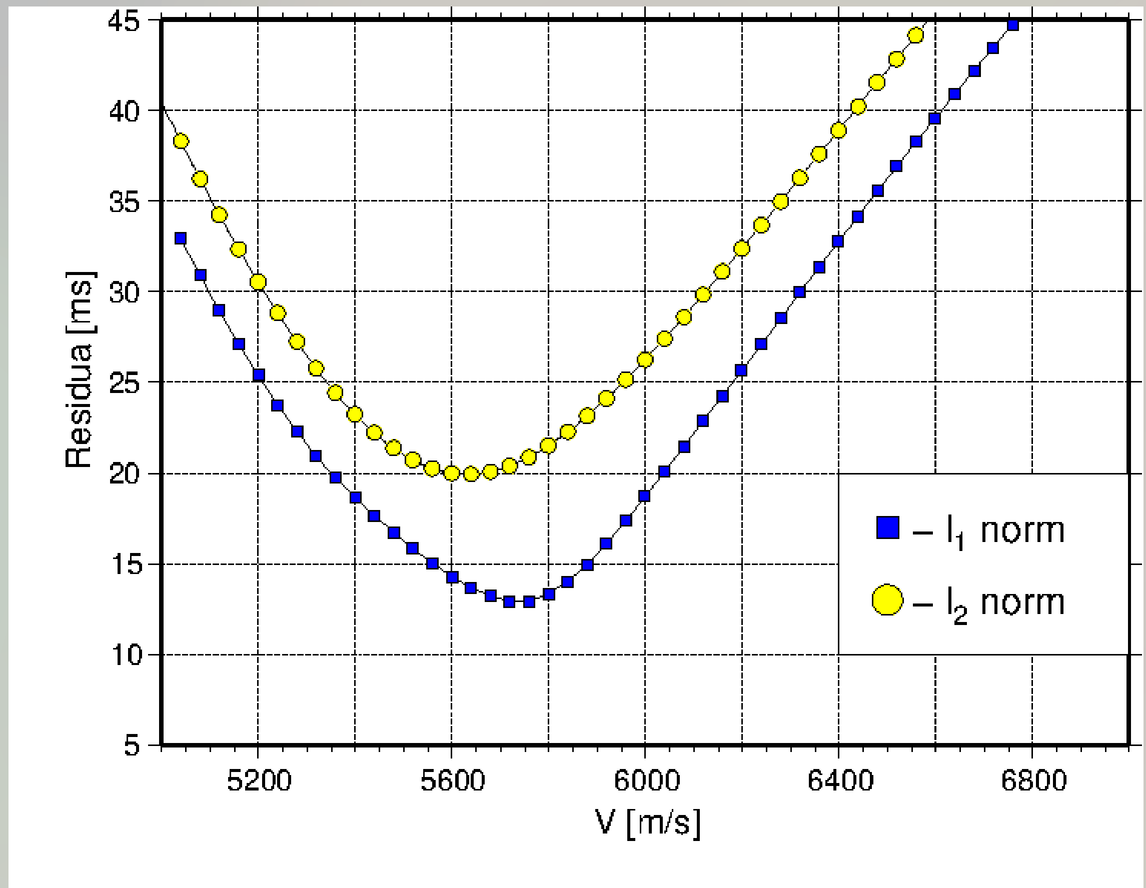
$$d = g \cdot m \qquad d^{obs} = d^{true} + \delta$$

$$m^{est}(\lambda) = \frac{\lambda}{g^2 + \lambda} m^{apr} + \frac{g}{g^2 + \lambda} (\bar{d}^{true} + \delta)$$

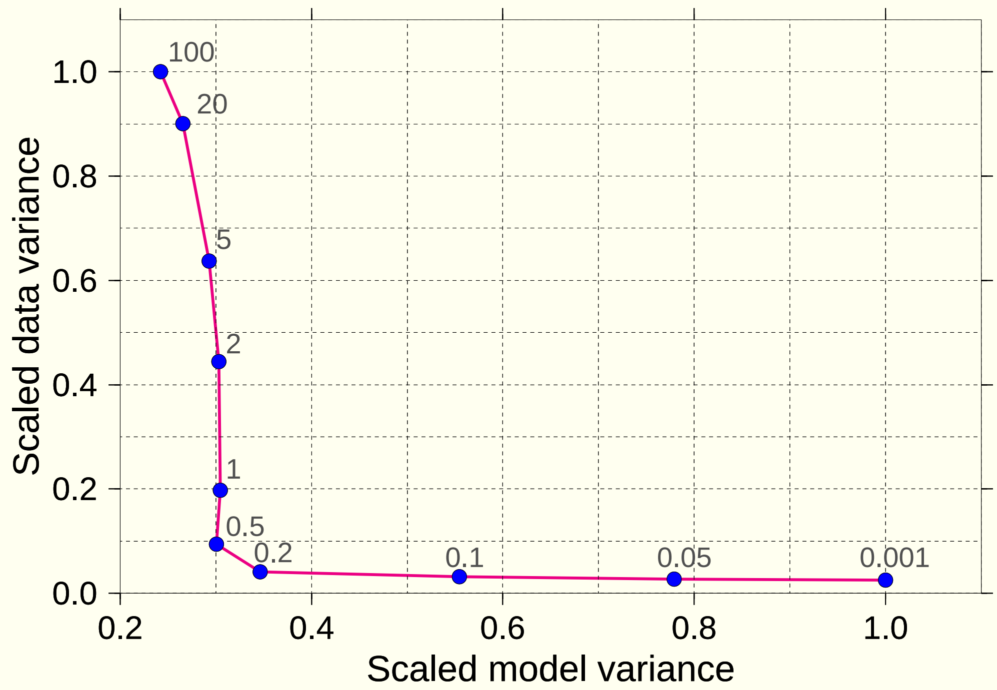
$$r(\lambda) = \frac{\lambda}{g^2 + \lambda} \left(g \, m^{apr} - d^{true} \right) + \frac{g^2}{g^2 + \lambda} \delta$$

$$\Delta \mathbf{m}^{est} = \frac{g}{g^2 + \lambda} \delta$$

Linear problem - selecting $\mathbf{m}^{apr}$



Linear problem - selecting λ



Linear problem - single λ ?

$$\mathbf{G}^T\mathbf{G} \implies \mathbf{G}^T\mathbf{G} + \lambda\mathbf{I}$$

should we use single “global value of λ ?

diagonalize $\mathbf{G}^T \cdot \mathbf{G}$

$$\begin{pmatrix} \cdot & & & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

End