Inverse Theory - a modern method of data analysis

Lecture 1

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Today

- Introduction
  - Forward problems
  - Inverse problems
- Inverse problem - different point of views
  - parameter estimation
  - direct and indirect measurements
  - inference

Forward and Invers problems - a general point of view

Actually, all problems encountered in our scientific activity can be divided into few categories:

- forward problems
- inverse problems
- administrations
- ...

Forward problems

This is a class of problems when one can try to understand qualitatively some observed phenomena or to predict new ones. The ultimate goal of solving such problems is an ability to predict (calculate) behavior of a system in hand.
Question: why...

Two examples:

- building new theories
- modelling physical processes

1. Building a new theory ED $\rightarrow$ QED

Classical electrodynamics:

$$\nabla \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$$
$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \cdot \vec{D} = 4\pi\rho + \nabla \cdot \vec{j}$$
$$\nabla \cdot \vec{B} = 0$$

Classical ED: Lorentz force

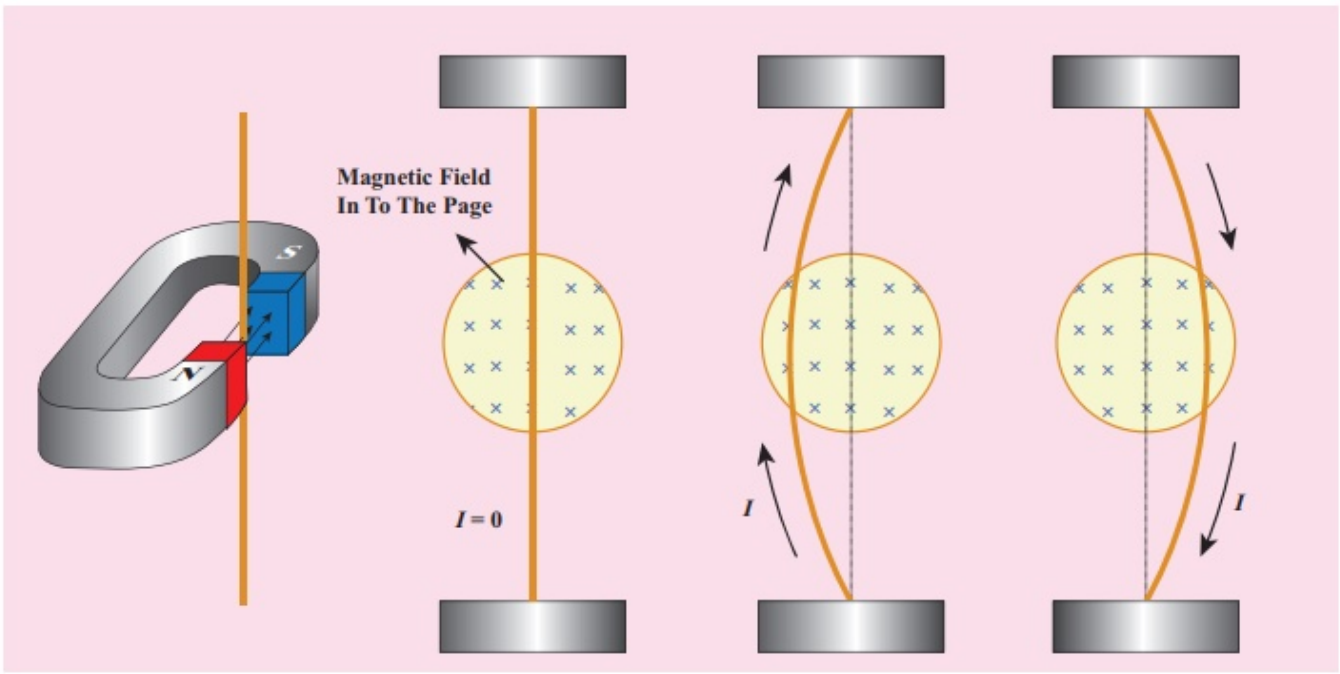
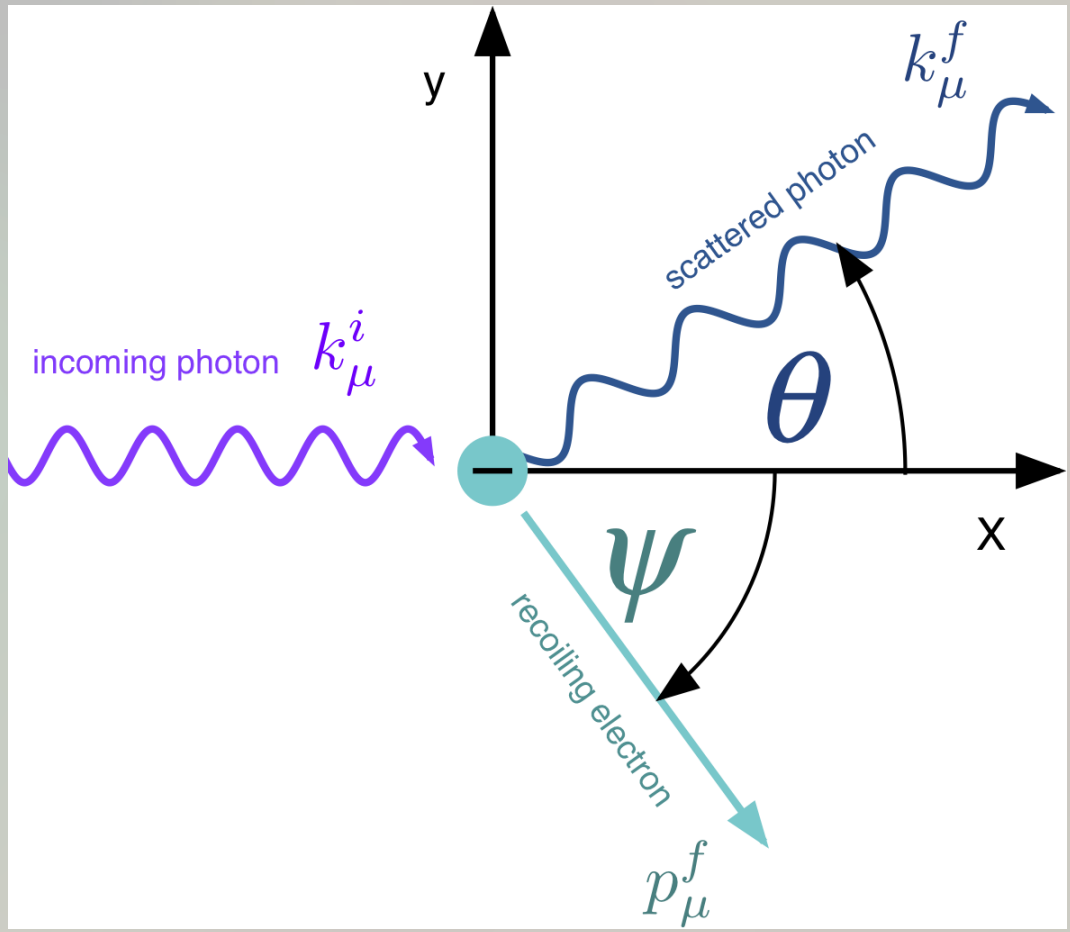
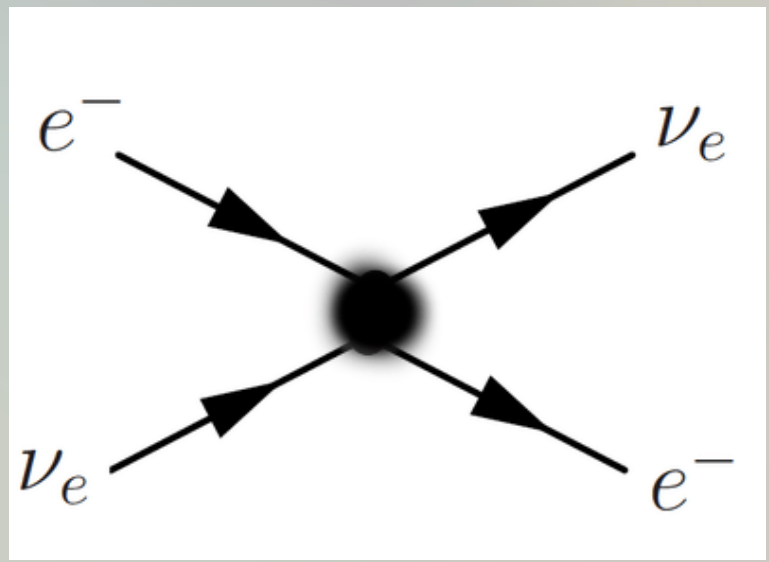


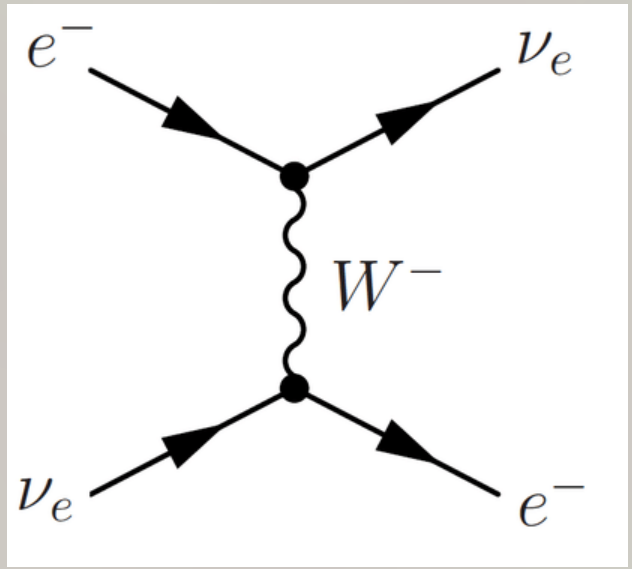
Figure 3.11 Deflection of current carrying wire in magnetic field

Compton scattering



1. Building a new theory : QED





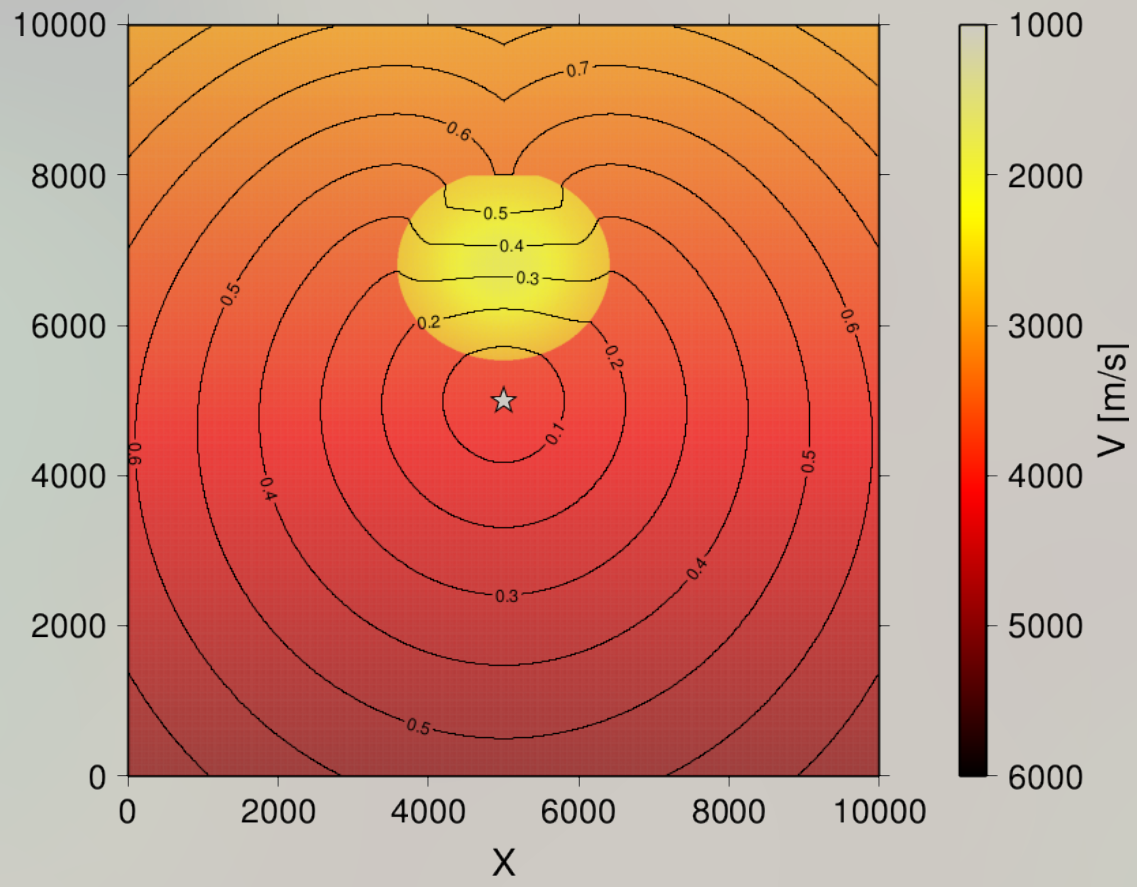
2. Forward modelling (seismic waves)

$$\rho \frac{\partial^2 u^i}{\partial t^2} - \partial_j \sigma^{ij} = S^i(\mathbf{r}, t)$$

$$\epsilon_{kl} = 1/2(\partial_k u_l + \partial_l u_k)$$

$$\sigma^{ij} = c^{ijkl} \epsilon_{kl}$$

2. Solution: wave propagation fora given velocity model

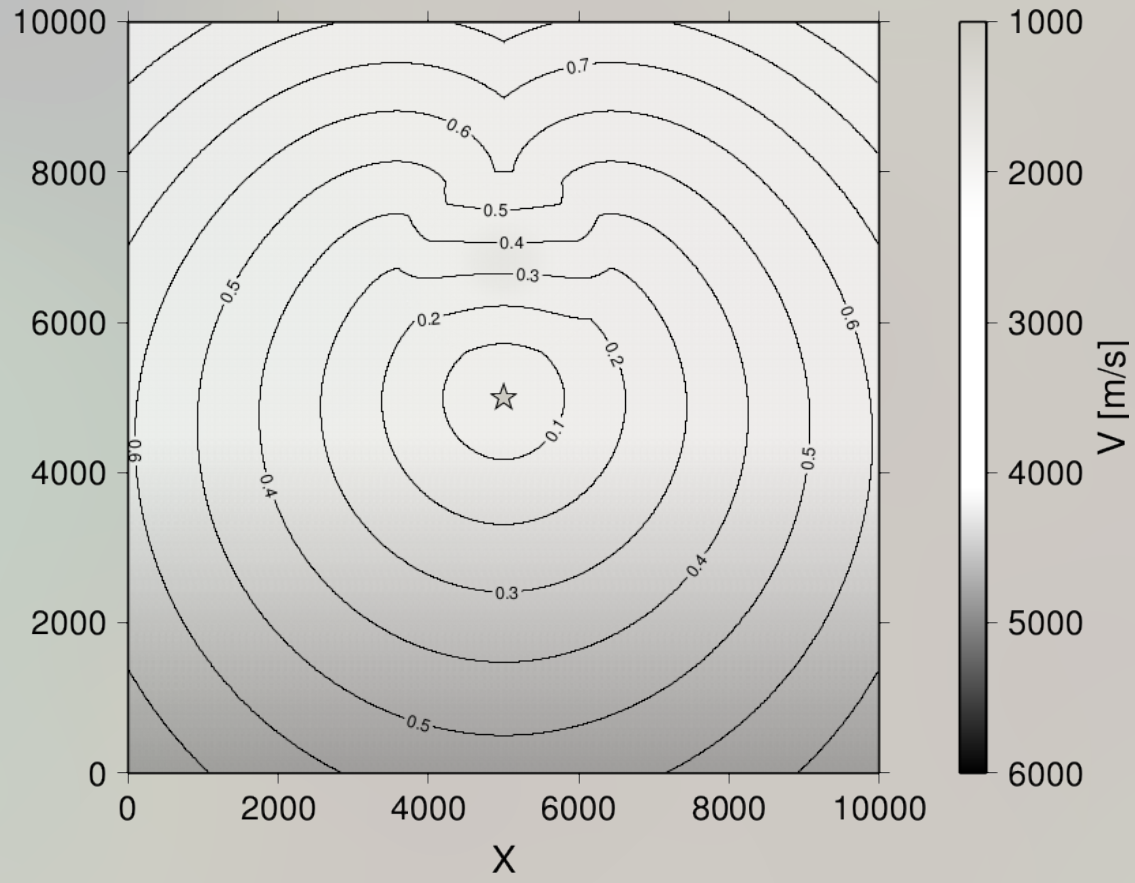


Inverse problems

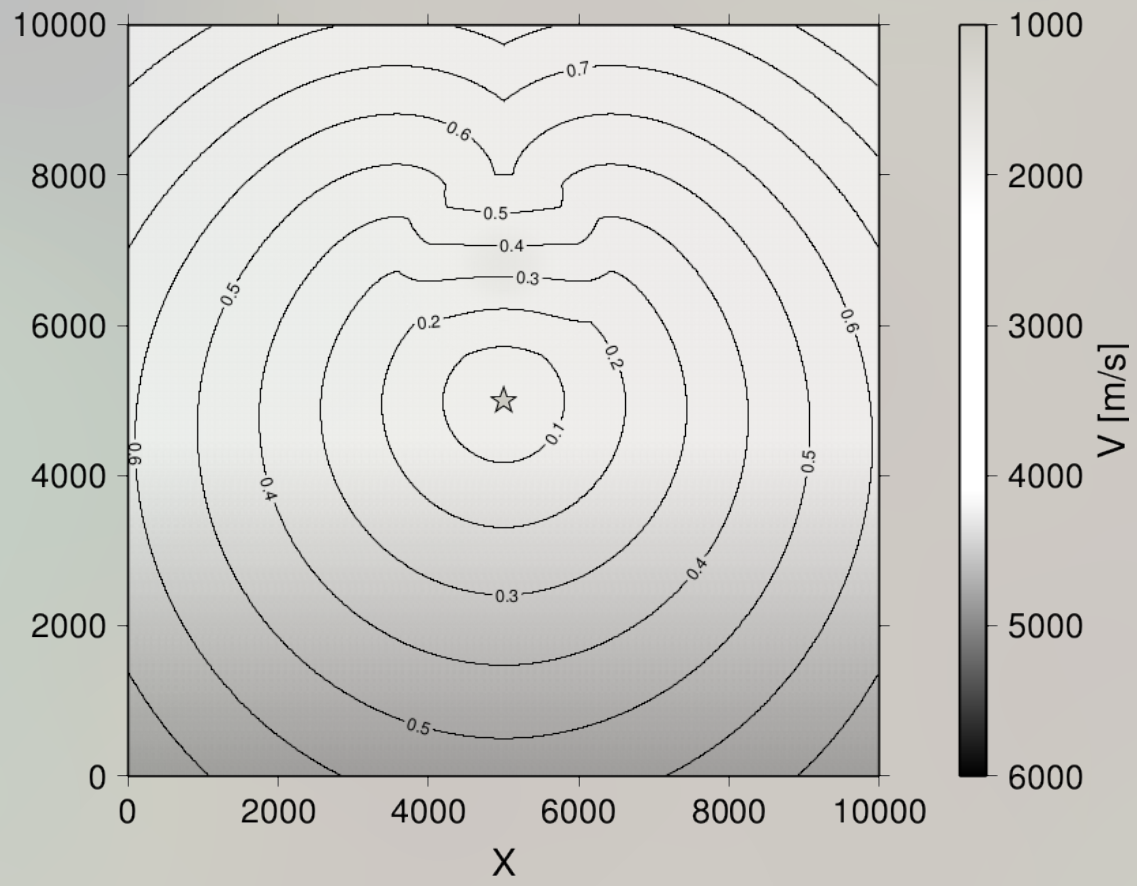
Inverse problems are tasks when one can try to grasp quantitative description of given system in hand or observed processes. The goal is not to provide a general description how the system behaves but to infer information on it allowing its realistic description.

Question: What is ....

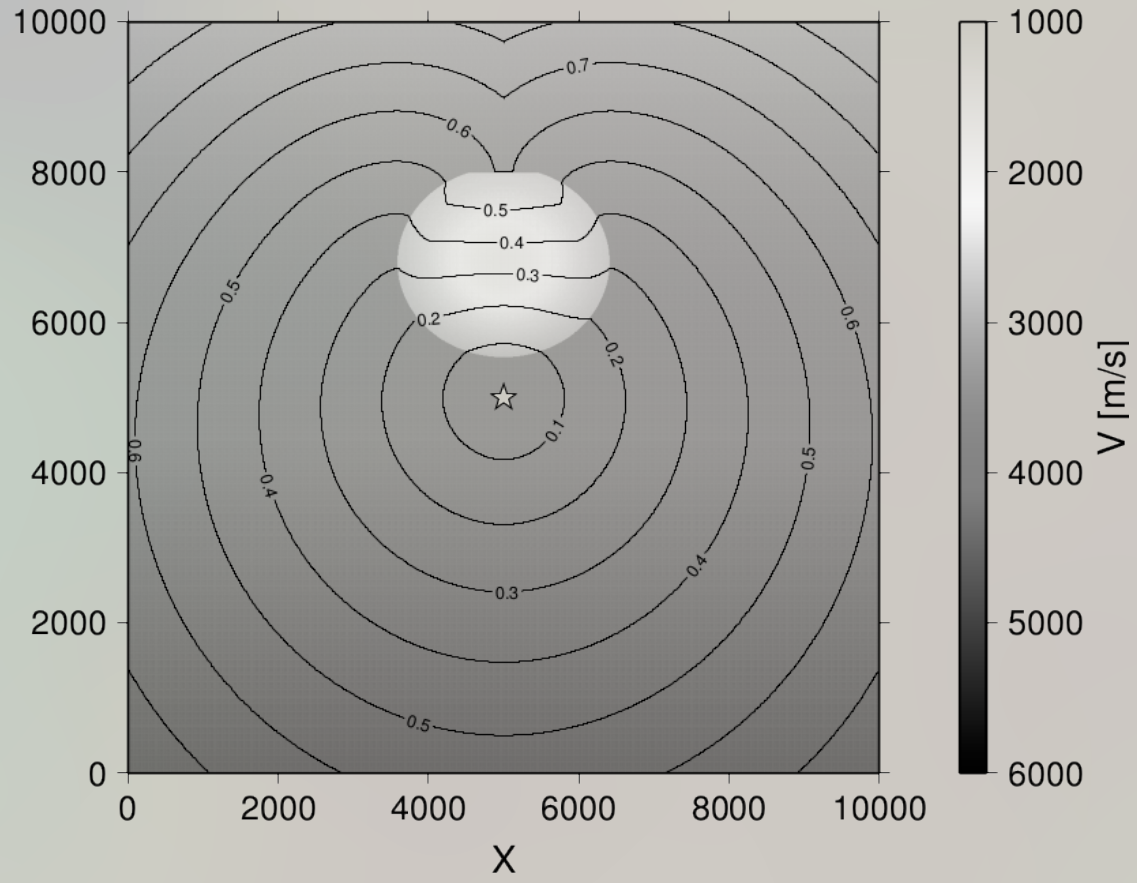
Solution (1) : velocity distribution for which we observe ...



Solution (2) : location of waves source



Solution (3) : size of an opening



Inverse problems - comments

- to solve inverse problems we HAVE TO to be able to solve a corresponding forward problem
- for given forward problem there can be many different inverse problems - we can pose different questions
- solution of inverse problems - some characteristics of studied object/process

Inverse problems - cd

- forward modelling problem is always unique - for given process/object predictions are always unique
- uniquness of inverse problems may be problematic
- solving inverse problems usually requires some observational information ....
- solution of inverse problem - interpratation of available data

Solving inverse problems

BASIC QUESTION:

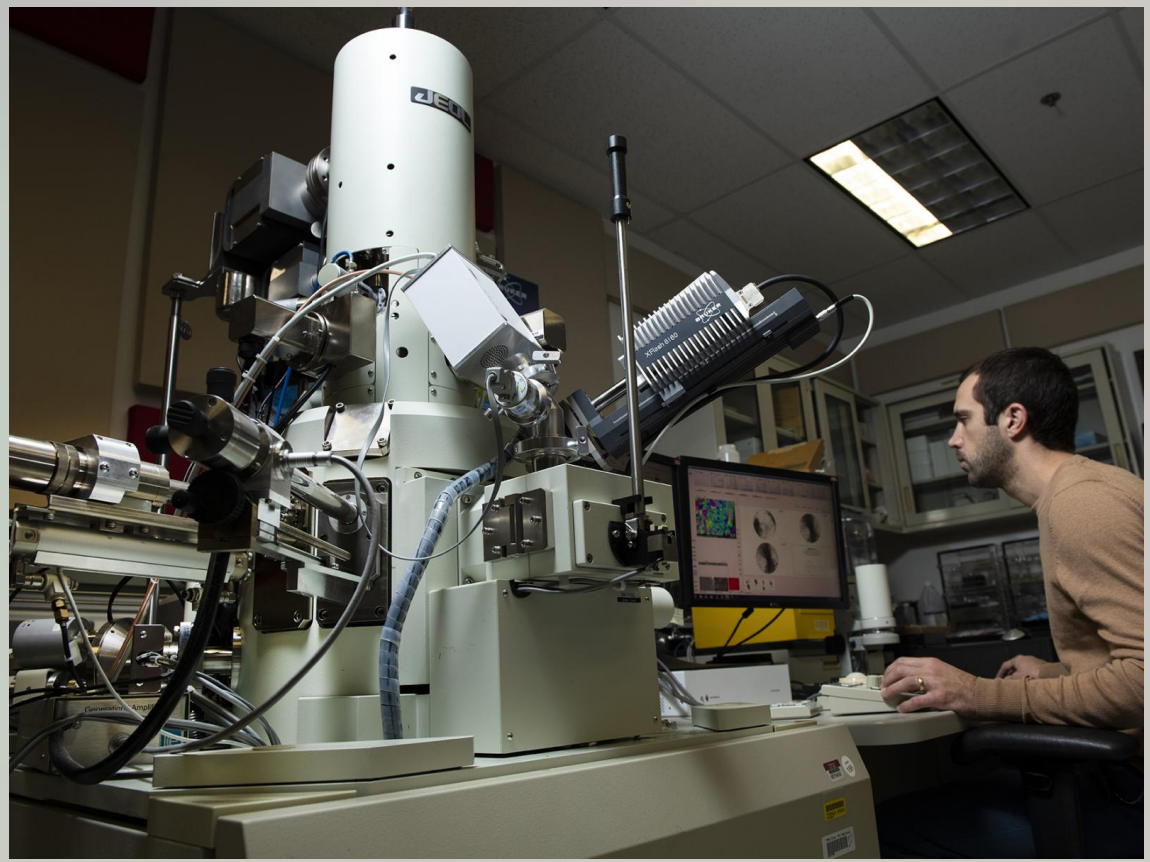
How to solve given inverse problem in hand

ANSWER:

Inverse Theory ...

Inverse theory (inverse methods)- what is it? (A)

Is a complex device, aparatus, etc?



NO

Inverse theory (inverse methods) - what is it? (A)

Is it a complex physical/chemical etc. process?

NO

Inverse theory (inverse methods) - what is it? (B)

Is it a horrible piece of mathematics?

$$
\begin{aligned}
&\mathbf{x}_1(12) - \mathbf{x}_1(21) \\
&= \left[\frac{d\Gamma^1_{12}}{du^1} - \frac{d\Gamma^1_{11}}{du^2} + \Gamma^1_{12}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{12} + \frac{d\Gamma^1_{12}}{du^1}\Gamma^1_{11} - \frac{d\Gamma^1_{11}}{du^2}\Gamma^1_{12} \right. \\
&\quad \left. + \Gamma^2_{12}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{22} + \frac{d\Gamma^2_{12}}{du^1}\Gamma^1_{21} - \frac{d\Gamma^2_{11}}{du^2}\Gamma^1_{22} \right] \mathbf{x}_1 \\
&\quad + \left[\Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \frac{d\Gamma^1_{12}}{du^1}\Gamma^2_{11} - \frac{d\Gamma^1_{11}}{du^2}\Gamma^2_{12} + \frac{d\Gamma^2_{12}}{du^1} - \frac{d\Gamma^2_{11}}{du^2} \right. \\
&\quad \left. + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} + \frac{d\Gamma^2_{12}}{du^1}\Gamma^2_{21} - \frac{d\Gamma^2_{11}}{du^2}\Gamma^2_{22} \right] \mathbf{x}_2 \\
&= \left[\frac{d\Gamma^1_{12}}{du^1} - \frac{d\Gamma^1_{11}}{du^2} + \Gamma^p_{12}\Gamma^1_{p1} - \Gamma^p_{11}\Gamma^1_{p2} + \frac{d\Gamma^p_{12}}{du^1}\Gamma^1_{p1} - \frac{d\Gamma^p_{11}}{du^2}\Gamma^1_{p2} \right] \mathbf{x}_1 \\
&\quad + \left[\Gamma^p_{12}\Gamma^2_{p1} - \Gamma^p_{11}\Gamma^2_{p2} + \frac{d\Gamma^p_{12}}{du^1}\Gamma^2_{p1} - \frac{d\Gamma^p_{11}}{du^2}\Gamma^2_{p2} + \frac{d\Gamma^2_{12}}{du^1} - \frac{d\Gamma^2_{11}}{du^2} \right] \mathbf{x}_2 .
\end{aligned}
$$

partially YES

A philosophical answer

In the most general way: by Inverse Method (Inverse Theory) we shall understand a way of reasoning and set of mathematical methods and computational tools needed to solve some specific scientific/technical problems encountered:

- (mathematics) parameter estimation
- (physics) indirect measurements
- (general) inference

Inverse theory - basic mathematical notions

Physical system:

$$p_1, p_2, \cdots p_K$$

parameters:

$$\mathbf{m} = (m_1, m_2, \cdots m_M)$$

predicted (measureable) quantities:

$$\mathbf{d} = (d_1, d_2, \cdots d_N)$$

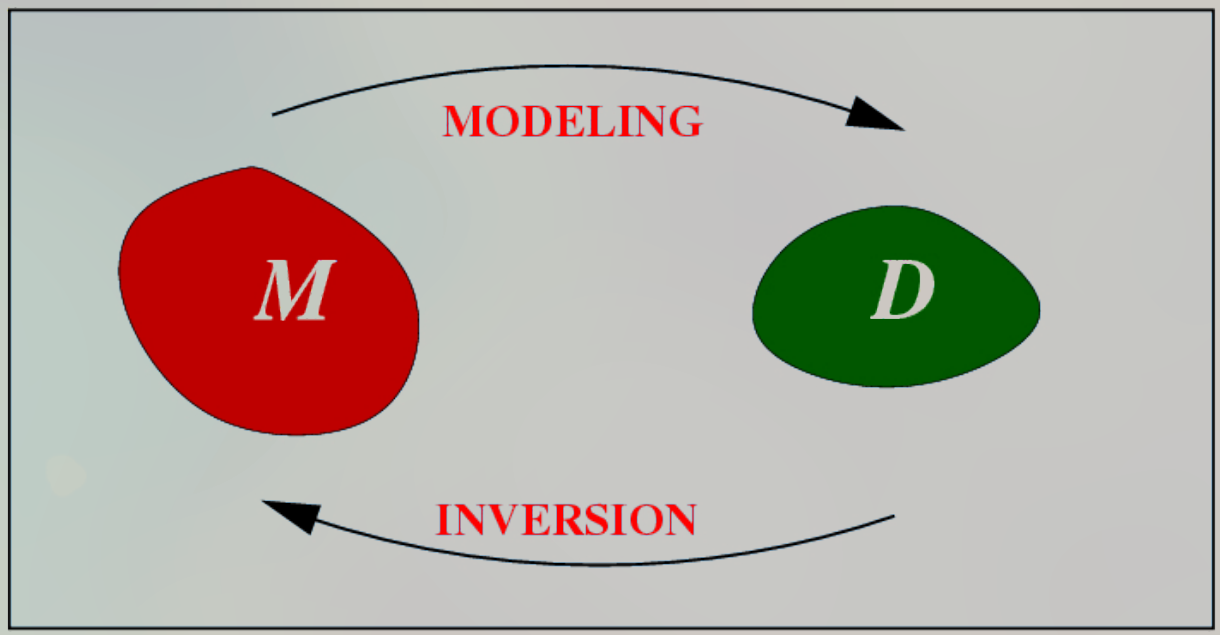
fixed parameters (*a priori* fixed):

$$\mathbf{m}^{fix} = (u_1, u_2, \cdots)$$

Forward modelling: $\mathbf{d}^{th} = f(\mathbf{m}, \mathbf{m}^{fix})$

Inversion : $\mathbf{m} = \ldots \pm \ldots$

Inverse theory - cartoon



Inverse problems - various point of view

1. (mathematics) parameter estimation
2. (physics) indirect measurements
3. (general) inference

Parameter estimation tasks

Modelling:

$$\mathbf{m} \longrightarrow \mathbf{d}^{th} = \mathbf{G}(\mathbf{m})$$

Inversion:

$$\mathbf{d}^{obs} \longrightarrow \mathbf{d}^{th}$$

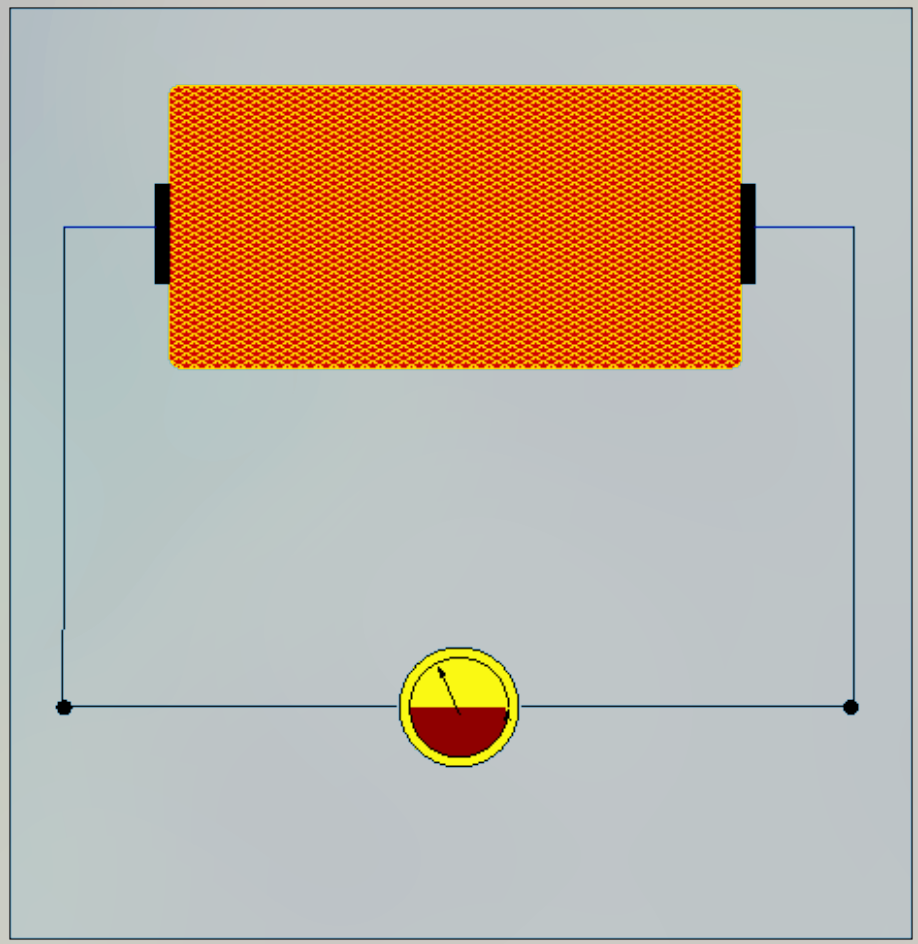
$$\mathbf{d}^{obs} = \mathbf{G}(\mathbf{m}^{est})$$

$$\mathbf{m}^{est} \longleftarrow \mathbf{d}^{obs}$$

Direct and "indirect" measurements

Our knowledge of serounding world comes entirely from observations. These observational information (data) are absolutly necessary for understanding what is happening, how the world is built, etc. The required information is obtained by proper measurements.

Direct measurement

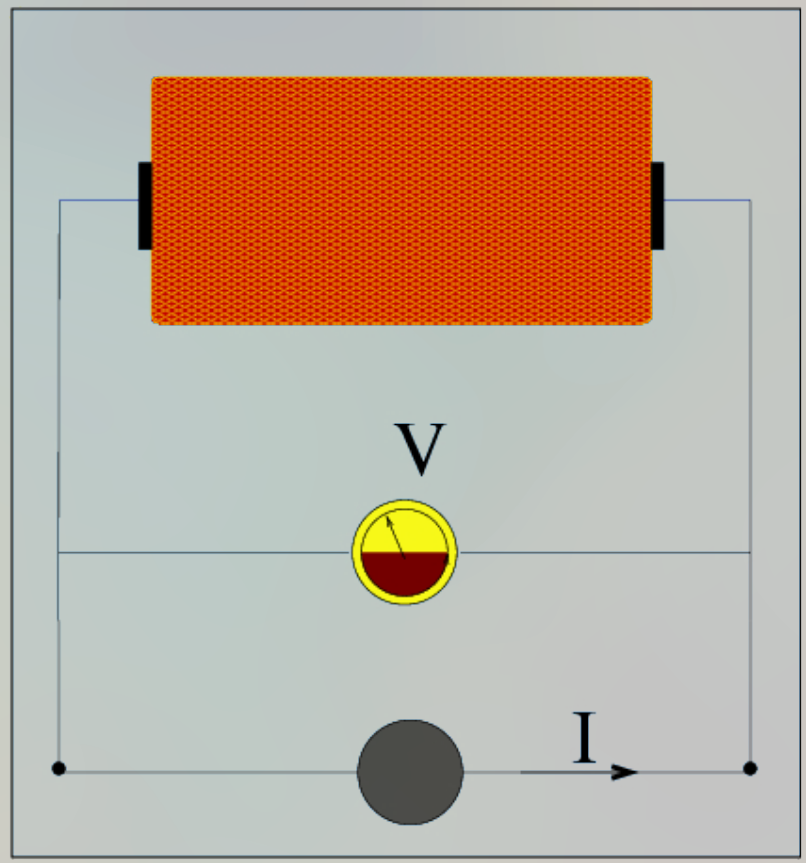


Direct measurement impossible

What to do if we cannot for some reason to measure what we want to know ???

- change job ....
- sit down and cry ...
- try to avoid obstacles and invent how we can get required information

"Indirect" measurement



Data: V

Theory (Ohm law + ...)

$$V = I\,\rho + n(\rho, \dot{\rho})$$

A priori

$$1 < \rho < 10$$

End