Large-scale convective dynamos in a stratified rotating plane layer

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We study the effect of stratification on large-scale dynamo action in convecting fluids in the presence of background rotation. The fluid is confined between two horizontal planes and both boundaries are impermeable, stress-free and perfectly conducting. An asymptotic analysis is performed in the limit of rapid rotation ($\tau \gg 1$ where τ is the Taylor number). We analyse asymptotic magnetic dynamo solutions in rapidly rotating systems generalising the results of Soward [A convection-driven dynamo I. The weak field case. *Philos. Trans. R. Soc. Lond.* A 1974, **275**, 611–651] to include the effects of compressibility. We find that in general the presence of stratification delays the efficiency of large-scale dynamo action in this regime, leading to a reduction of the onset of dynamo action and in the nonlinear regime a diminution of the large-scale magnetic energy for flows with the same kinetic energy.

1. Introduction

The generation of coherent magnetic fields in astrophysics is one of the most fundamental problems of astrophysical MHD (see, e.g. Parker 1979). Much progress has been made in this area following the pioneering work of the Potsdam group in mean field electrodynamics (see, e.g. Steenbeck *et al.* 1966, Krause and Rädler 1980). It is no exaggeration to maintain that this work has set the agenda for dynamo theory for the decades following.

In astrophysics and geophysics it is often the case that dynamo action has its origin in convective processes, and for this reason the problem of magnetic dynamo action generated by convective flows has been widely and attentively studied over past couple decades. This is mainly because the detailed understanding of the convective dynamo process would prove to be extremely useful for the study of planetary and stellar magnetic fields and possibly the inner structure of such astrophysical objects. It is a well established and accepted theory that the magnetic fields of planets and stars are often a

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§Dedicated to Professor Rädler on the occasion of his 75th Birthday.

result of convectively driven, vigorous flow of an electrically conducting fluid occupying a fairly large volume in the interior of the planet or star. The theory of the geodynamo, for which the conducting medium is simply liquid iron in the outer core, can be found e.g. in Braginsky (1991) or Soward (1991). Stellar dynamos differ from planetary dynamos by the fact that the convective medium is a compressible plasma, rather than a liquid metal. However, the standard single fluid dynamo theory (cf. Moffatt 1978, Roberts and Soward 1992) seems to work plausibly for global models of stellar dynamos. For a review on stellar and in particular solar dynamos see Rosner (2000) and Tobias and Weiss (2007a,b).

In this article we consider stratified dynamos in rapidly rotating convective flows. The analysis performed in this article is strongly related to the asymptotic analysis of Childress and Soward (1972) and Soward (1974) of a convectively driven magnetic dynamo in an incompressible medium, in a plane layer with strong background rotation. In those papers a set of nonlinear equations is derived that govern the evolution of the convective dynamo, and stable periodic dynamos are shown to exist. Furthermore Soward (1974) provides analytic solutions to these equations, involving a continuous distribution of Fourier modes. However, their model was shown to work only for magnetic fields weak enough (Elsasser number of order of inverse of Ekman number) not to influence the total kinetic energy of the flow, i.e. for solutions near to the dynamo threshold. Childress (1977) discovered that when the magnetic field is increased (i.e. for Elsasser numbers of order Ekman number to $-2/3$) it releases the rotational constraints. Consequently the finite amplitude dynamo solutions are unstable to small perturbations.

Convectively driven dynamos both in the Boussinesq and compressible regimes have been studied intensively. Of greatest relevance to the discussions in this article are the interesting discussion and observations for the plane layer dynamo model of Childress and Soward (1972) and Soward (1974), as well as of a somewhat similar annulus geodynamo model by Busse (1970), which was given in Soward (1979). Valuable information about the plane layer dynamo model can also be found in Soward (1978). Later Jones and Roberts (2000) found numerically fully three-dimensional convection-driven dynamo solutions for the planar configuration in the rapid rotation limit and proposed a simple heuristic model of the dynamo mechanism. Stellmach and Hansen (2004) also studied numerically the plane dynamo model of Childress and Soward (1972) and found a certain transition to occur from a regime in which the dynamo generated magnetic field exerts only a weak back reaction on the flow to a regime where the Lorentz force strongly influences the typical length scales of the flow. Much attention was also dedicated to dynamos driven by convection in spherical geometry. A review and numerical analysis of spherical, convection-driven dynamos was done by Busse *et al.* (1998) and Busse and Simitev (2005).

Our aim is to study the influence of compressibility, under the anelastic approximation, on convectively driven magnetic dynamo in rapidly rotating systems, in particular on the dynamo solutions of Childress and Soward (1972) and Soward (1974). Hence we consider a plane layer of compressible fluid confined between two parallel planes, and investigate the role of compressibility in modifying the nonlinear large-scale dynamo solutions found in those papers.

The structure of this article is as follows. First we formulate the problem mathematically giving the relevant equations and boundary conditions. We then study the influence of weak compressibility on the threshold for dynamo action i.e.

on the dynamo solution of Soward (1974) in sections 3.1 and 3.2. In section 3.3, we study the back reaction of the Lorentz force on the flow and in section 4 we numerically solve the nonlinear amplitude equations coupled with the mean field equations. We end with some concluding remarks in section 5.

2. Mathematical formulation

We analyse thermal convection in an infinite plane layer of fluid with gravity $g =$ constant, pointing downwards. The bottom and top boundaries are flat, impermeable perfect conductors situated at $z = 0$, d (the z-axis points vertically upwards) and stress-free conditions are imposed. The basic static state takes the form of a polytrope, for which the temperature at the bottom of the layer is T_0 and the temperature jump across the layer is $\triangle T < 0$, so that

$$\bar{T} = 1 + \theta z, \quad \bar{\rho} = (1 + \theta z)^m, \quad \bar{p} = -\frac{\mathcal{R}\sigma_m}{\sigma_\eta \theta(m+1)}(1 + \theta z)^{m+1}, \quad \text{(1a,b,c)}$$

$$\bar{s} = \frac{m+1-\gamma m}{\gamma \epsilon}\ln(1 + \theta z) + \text{const.} \quad \text{with} \quad \frac{m+1-\gamma m}{\gamma} = -\frac{\epsilon}{\theta} = \text{O}(\epsilon), \quad \text{(1d,e)}$$

where $\theta = \triangle T/T_0$. ($-1 < \theta \leq 0$). $\bar{T}$, $\bar{\rho}$, $\bar{p}$ and $\bar{s}$ are the static temperature, density, pressure and entropy, respectively, $\gamma = c_p/c_v$ is the specific heat ratio and m is the polytropic index. The basic state is assumed to be almost adiabatic, i.e. its departure from adiabaticity

$$\epsilon = -\frac{d}{T_r}\left[\left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}z}\right)_r + \frac{g}{c_p}\right] = -\frac{d}{c_p}\left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}z}\right)_r \ll 1 \quad \text{(2)}$$

is assumed small. The subscript "r" denotes a reference value, i.e. taken at $z = 0$. Furthermore, $\mathcal{R} = c_p\rho_r g d^3\epsilon/k_T\nu$ is the Rayleigh number, $\sigma_m = \nu/\eta$ is the magnetic Prandtl number and $\sigma_\eta = c_p\rho_r\eta/k_T$ is the inverse Roberts number, where ν, η and k_T are the kinematic viscosity, the magnetic diffusivity and the thermal conductivity of the system, respectively, all assumed constant.

The equations describing the evolution of the system is the set of hydro-magnetic equations under the anelastic approximation i.e.

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla\left(\frac{\mathscr{P}}{\bar{\rho}}\right) + \mathcal{R}\sigma_m\sigma_\eta^{-1}s\hat{\boldsymbol{e}}_z - \tau^{1/2}\sigma_m\hat{\boldsymbol{e}}_z \times \boldsymbol{u} + M^2\sigma_m\frac{1}{\bar{\rho}}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$+ \sigma_m\nabla^2\boldsymbol{u} + \frac{\sigma_m m\theta}{1+\theta z}\left[\frac{\partial \boldsymbol{u}}{\partial z} + \frac{2}{3}\nabla u_z + \frac{1}{3}(1+2m)\theta\frac{u_z}{1+\theta z}\hat{\boldsymbol{e}}_z\right], \quad \text{(3a)}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2\boldsymbol{B}, \quad \text{(3b)}$$

$$\nabla \cdot \boldsymbol{u} = -\frac{m\theta}{1+\theta z}u_z, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \text{(3c,d)}$$

$$\frac{\partial s}{\partial t} + \boldsymbol{u} \cdot \nabla s = \frac{u_z}{1 + \theta z} + \frac{\sigma_\eta^{-1}}{\bar{\rho}} \nabla^2 s + \frac{\sigma_\eta^{-1} \theta}{(1 + \theta z)^{m+1}} \frac{\partial s}{\partial z} - \frac{\theta M^2 \sigma_\eta}{\mathcal{R}} \frac{1}{(1 + \theta z)^{m+1}} (\nabla \times \boldsymbol{B})^2$$

$$- \frac{\sigma_\eta \theta}{\mathcal{R}} \frac{1}{(1 + \theta z)} \left[2 \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial x_i} \right)^2 + \sum_{i<j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 - \frac{2}{3} (\nabla \cdot \boldsymbol{u})^2 \right], \qquad (3e)$$

where we have chosen d^2/η, d and η/d as units of time, length and velocity, respectively (for derivation see Gough 1969, Lantz and Fan 1999, Mizerski and Tobias 2011). In the above equations, the Taylor number is denoted by $\tau = 4\Omega^2 d^4/\nu^2$ and the Hartmann number is $M = B_S d/\sqrt{\mu \rho_r \nu \eta}$, where Ω, μ and B_S are the rotation rate, magnetic permeability of the fluid and the scale of the magnetic field, respectively. We stress here again that $-1 < \theta \leq 0$ so that the final term in equation (3e) is indeed a heating term.

Furthermore, we have assumed here that the conductive heat flux due to molecular diffusion is an order of magnitude in ϵ smaller than the turbulent heat flux due to unresolved small-scale turbulence expressed in terms of entropy as $kT\nabla s$ (cf. Braginsky and Roberts 1995, Jones *et al.* 2009, Mizerski and Tobias 2011) and hence only turbulent heat flux was included in the entropy equation. Thus all the diffusivity coefficients, such as ν, η and k_T are turbulent.

The parameter θ, which measures the temperature jump across the fluid layer is also a measure of compressibility of the system. Indeed, from equations (3c,d) we get $\nabla \cdot \boldsymbol{u} \sim \theta u_z$ for small θ, so that if $|\theta| \ll 1$ the fluid is weakly compressible. In the following section, we will study the influence of compressibility of the medium, defined in the above way, on large-scale dynamo action.

3. The effect of compressibility on Soward's solutions for convection-driven dynamo

We now proceed to the magnetic dynamo analysis, particularly the influence of compressibility on the dynamo threshold in rapidly rotating systems ($\tau \gg 1$). Technically the analysis presented in this section is a generalisation of the earlier work by Childress and Soward (1972) and Soward (1974) to include the compressibility of the medium. To simplify the notation let us introduce a new pressure variable

$$p = \mathscr{P}/\bar{\rho}, \qquad (4)$$

where $\mathscr{P}$ is the pressure perturbation in (3a).

We shall assume that the system is always only slightly above the threshold for convection and hence based on the results of Mizerski and Tobias (2011) for the onset of compressible convection we assume

$$\boldsymbol{x} = (\varepsilon x, \varepsilon y, z), \qquad (5)$$

where $\varepsilon = \tau^{-1/6}$. After Childress and Soward (1972) and Soward (1974) we also separate the variables $\boldsymbol{u}$, $\boldsymbol{B}$, s and p into two parts: the horizontal average (depending only on z and t) and a fluctuation (denoted by a prime), and assume the following scalings:

$$\boldsymbol{u} = \varepsilon^{-1/2} \left[\varepsilon^3 \boldsymbol{U}_h(z,t) + \boldsymbol{u}'(\boldsymbol{x}_h, z, t) \right], \quad \boldsymbol{B} = \boldsymbol{B}_h(z,t) + \varepsilon^{1/2} \boldsymbol{B}'(\boldsymbol{x}_h, z, t), \qquad (6\text{a,b})$$

$$s = \varepsilon \left[S(z,t) + \varepsilon^{1/2} s'(\boldsymbol{x}_h, z, t) \right], \qquad p = \left[\varepsilon^{1/2} P(z,t) + \varepsilon p'(\boldsymbol{x}_h, z, t) \right], \qquad (6\text{c,d})$$

$$\mathcal{R} = \tau^{2/3} \tilde{\mathcal{R}} = \varepsilon^{-4} \tilde{\mathcal{R}}, \qquad (6\text{e})$$

where subscript h denotes the horizontal component. Introducing the above forms of the velocity, magnetic field and the entropy into (3a–e), and dropping the $o(\varepsilon^3)$ terms, we obtain

$$
\sigma_m^{-1}\left[\varepsilon^3\frac{\partial \boldsymbol{u}_h'}{\partial t} + \varepsilon^{3/2}(\boldsymbol{u}_h'\cdot\boldsymbol{\nabla})\boldsymbol{u}_h' + \varepsilon^{5/2}w\frac{\partial \boldsymbol{u}_h'}{\partial z}\right]
$$

$$
= -\boldsymbol{\nabla}_h p' - \hat{\boldsymbol{e}}_z\times\boldsymbol{u}_h' - \varepsilon^3\hat{\boldsymbol{e}}_z\times\boldsymbol{U}_h + \varepsilon^3 M^2\frac{1}{(1+\theta z)^m}(\boldsymbol{B}_h\cdot\boldsymbol{\nabla})\boldsymbol{B}_h' + \varepsilon\nabla_h^2\boldsymbol{u}_h' + \varepsilon^3\frac{\partial^2\boldsymbol{u}_h'}{\partial z^2}
$$

$$
+ \varepsilon^3\frac{m\theta}{1+\theta z}\frac{\partial\boldsymbol{u}_h'}{\partial z} + \frac{2}{3}\varepsilon^2\frac{m\theta}{1+\theta z}\boldsymbol{\nabla}_h w, \tag{7a}
$$

$$
\sigma_m^{-1}\left[\varepsilon^3\frac{\partial w}{\partial t} + \varepsilon^{3/2}(\boldsymbol{u}_h'\cdot\boldsymbol{\nabla})w + \varepsilon^{5/2}w\frac{\partial w}{\partial z}\right]
$$

$$
= -\varepsilon^{1/2}\frac{\partial P}{\partial z} - \varepsilon\frac{\partial p'}{\partial z} + \varepsilon^3 M^2\frac{1}{(1+\theta z)^m}(\boldsymbol{B}_h\cdot\boldsymbol{\nabla})B_z' + \varepsilon^{1/2}\sigma_\eta^{-1}\tilde{\mathcal{R}}S + \varepsilon\sigma_\eta^{-1}\tilde{\mathcal{R}}s' + \varepsilon\nabla_h^2 w
$$

$$
+ \varepsilon^3\frac{\partial^2 w}{\partial z^2} + \varepsilon^3\frac{m\theta}{1+\theta z}\left(\frac{5}{3}\frac{\partial w}{\partial z} + \frac{1+2m}{3}\theta\frac{w}{1+\theta z}\right), \tag{7b}
$$

$$
\varepsilon^{3/2}\frac{\partial\boldsymbol{B}_h}{\partial t} + \varepsilon^2\frac{\partial\boldsymbol{B}'}{\partial t} = (\boldsymbol{B}_h\cdot\boldsymbol{\nabla})\boldsymbol{u}' + \varepsilon^{1/2}(\boldsymbol{B}_h'\cdot\boldsymbol{\nabla})\boldsymbol{u}' + \varepsilon^{3/2}B_z'\frac{\partial\boldsymbol{u}'}{\partial z} - \varepsilon^{1/2}(\boldsymbol{u}_h'\cdot\boldsymbol{\nabla})\boldsymbol{B}' - \varepsilon w\frac{\partial\boldsymbol{B}_h}{\partial z}
$$

$$
- \varepsilon^{3/2}w\frac{\partial\boldsymbol{B}'}{\partial z} + \varepsilon\frac{m\theta}{1+\theta z}w\boldsymbol{B}_h + \varepsilon^{3/2}\frac{m\theta}{1+\theta z}w\boldsymbol{B}' + \varepsilon^{3/2}\frac{\partial^2\boldsymbol{B}_h}{\partial z^2}
$$

$$
+ \nabla_h^2\boldsymbol{B}' + \varepsilon^2\frac{\partial^2\boldsymbol{B}'}{\partial z^2}, \tag{7c}
$$

$$
\varepsilon\mathfrak{D}w = -\boldsymbol{\nabla}\cdot\boldsymbol{u}_h', \qquad \varepsilon\frac{\partial B_z'}{\partial z} = -\boldsymbol{\nabla}\cdot\boldsymbol{B}_h', \tag{7d,e}
$$

$$
\varepsilon^{3/2}\frac{\partial S}{\partial t} + \varepsilon^2\frac{\partial s'}{\partial t} = -\varepsilon^{1/2}\boldsymbol{u}_h'\cdot\boldsymbol{\nabla}s' - \varepsilon w\frac{\partial S}{\partial z} - \varepsilon^{3/2}w\frac{\partial s'}{\partial z} + \frac{w}{1+\theta z}
$$

$$
+ \frac{\sigma_\eta^{-1}}{(1+\theta z)^m}\left[\nabla_h^2 s' + \varepsilon^{3/2}\frac{\partial^2 S}{\partial z^2} + \varepsilon^2\frac{\partial^2 s'}{\partial z^2}\right] + \frac{\sigma_\eta^{-1}\theta}{(1+\theta z)^{m+1}}\left[\varepsilon^{3/2}\frac{\partial S}{\partial z} + \varepsilon^2\frac{\partial s'}{\partial z}\right]
$$

$$
- \frac{\sigma_\eta\theta\varepsilon^{3/2}}{\tilde{\mathcal{R}}(1+\theta z)}\left[\frac{1}{2}\nabla_h^2(\boldsymbol{u}')^2 - \boldsymbol{u}'\cdot\nabla_h^2\boldsymbol{u}' + \boldsymbol{\nabla}\cdot\left((\boldsymbol{u}_h'\cdot\boldsymbol{\nabla})\boldsymbol{u}_h'\right)\right]
$$

$$
- \frac{\sigma_\eta\theta M^2\varepsilon^3}{\tilde{\mathcal{R}}(1+\theta z)^{m+1}}\left[2\nabla_h^2(\boldsymbol{B}_h\cdot\boldsymbol{B}') - \boldsymbol{B}_h\cdot\nabla_h^2\boldsymbol{B}' + \boldsymbol{B}_h\cdot\boldsymbol{\nabla}\left(\frac{\partial B_z'}{\partial z}\right)\right]
$$

$$
- \frac{\sigma_\eta\theta\varepsilon^{5/2}}{\tilde{\mathcal{R}}(1+\theta z)}\left[\boldsymbol{\nabla}\cdot\left(w\frac{\partial\boldsymbol{u}_h'}{\partial z}\right) + \frac{\partial}{\partial z}(\boldsymbol{u}_h'\cdot\boldsymbol{\nabla}w) + \frac{m\theta}{1+\theta z}\boldsymbol{u}_h'\cdot\boldsymbol{\nabla}w\right], \tag{7f}
$$

where w is the vertical component of the small scale velocity $\boldsymbol{u}'$ and the operator

$$
\mathfrak{D} = \frac{\partial}{\partial z} + \frac{m\theta}{1+\theta z}. \tag{8}
$$

Note in (7b) that the leading order balance is between horizontally averaged entropy and pressure gradients.

The aim of this section is to study the dynamo effect in a compressible medium. Hence, we will need the explicit equation for the large-scale magnetic field $\boldsymbol{B}_h(z,t)$,

which is obtained by averaging (7c),

$$\frac{\partial \boldsymbol{B}_h}{\partial t} = \frac{\partial}{\partial z}\left[\hat{\boldsymbol{e}}_z \times \langle \boldsymbol{u}' \times \boldsymbol{B}'\rangle\right] + \frac{\partial^2 \boldsymbol{B}_h}{\partial z^2}. \tag{9}$$

We note the presence of the interactions of fluctuating velocity and magnetic field to yield a mean electromotive force that in turn leads to the generation of large-scale magnetic field. We impose stress-free, isentropic boundary conditions and assume additionally that both boundaries at $z = 0, 1$ are perfectly conducting. This implies

$$w|_{z=0,1} = \frac{\partial}{\partial z}\mathfrak{D}w\Big|_{z=0,1} = 0, \quad s'|_{z=0,1} = S|_{z=0,1} = 0, \tag{10a,b}$$

$$B'_z\big|_{z=0,1} = \frac{\partial^2 B'_z}{\partial z^2}\Big|_{z=0,1} = \frac{\partial \boldsymbol{B}_h}{\partial z}\Big|_{z=0,1} = 0. \tag{10c}$$

All the dependent variables are expanded in $\varepsilon^{1/2}$, so that e.g.

$$s = \varepsilon\left[S^{(0)}(z,t) + \varepsilon^{1/2}S^{(1)}(z,t) + \cdots\right] + \varepsilon^{3/2}\left[s'^{(0)}(\boldsymbol{x}_h, z, t) + \varepsilon^{1/2}s'^{(1)}(\boldsymbol{x}_h, z, t) + \cdots\right], \tag{11}$$

and the Rayleigh number is assumed to have the expansion

$$\tilde{\mathcal{R}} = \tilde{\mathcal{R}}^{(0)}(k) + \varepsilon\tilde{\mathcal{R}}^{(2)}(k) + \varepsilon^2\tilde{\mathcal{R}}^{(4)}(k) + \cdots. \tag{12}$$

We will now derive the equation for $w^{(0)}$. Equations (7a,d,f) at leading order give

$$\boldsymbol{\nabla}_h p'^{(0)} = -\hat{\boldsymbol{e}}_z \times \boldsymbol{u}_h'^{(0)}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u}_h'^{(0)} = 0, \quad -w^{(0)} = \frac{\sigma_\eta^{-1}}{(1+\theta z)^{m-1}}\nabla_h^2 s'^{(0)}. \tag{13a,b,c}$$

Further, the first two equations lead to

$$\boldsymbol{u}_h'^{(0)} = \hat{\boldsymbol{e}}_z \times \boldsymbol{\nabla}_h p'^{(0)} \quad \text{and} \quad \frac{\partial \boldsymbol{u}_h'^{(0)}}{\partial z} = 0. \tag{14a,b}$$

Equating the order ε terms in (7a,b,d) we obtain

$$\boldsymbol{\nabla}_h p'^{(2)} = -\hat{\boldsymbol{e}}_z \times \boldsymbol{u}_h'^{(2)} + \nabla_h^2 \boldsymbol{u}_h'^{(0)}, \tag{15a}$$

$$\mathfrak{D}w^{(0)} = -\boldsymbol{\nabla}\cdot\boldsymbol{u}_h'^{(2)}, \tag{15b}$$

$$\frac{\partial p'^{(0)}}{\partial z} = \sigma_\eta^{-1}\tilde{\mathcal{R}}s'^{(0)} + \nabla_h^2 w^{(0)}, \tag{15c}$$

and from (7c) at leading order and (14a) we have

$$-\nabla_h^2 \boldsymbol{B}'^{(0)} = \left(\boldsymbol{B}_h^{(0)}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}'^{(0)}, \quad \boldsymbol{u}'^{(0)} = \hat{\boldsymbol{e}}_z \times \boldsymbol{\nabla}_h p'^{(0)} + w^{(0)}\hat{\boldsymbol{e}}_z. \tag{16a,b}$$

Next we assume that all the above small-scale fields (labelled with prime) as well as $p'^{(0)}$ have a harmonic dependence on x and y, i.e. they are all of the type

$$w^{(0)}(\boldsymbol{x},t) = \bar{w}^{(0)}(\boldsymbol{x},t)W^{(0)}(z) = \sum_{|\boldsymbol{k}|=k}\hat{w}^{(0)}(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}W^{(0)}(z). \tag{17}$$

Note that since we have introduced the amplitude $\hat{w}^{(0)}(\boldsymbol{k},t)$, in general we may impose a normalisation factor on the magnitude of $W^{(0)}$. We shall choose $dW^{(0)}/dz = 1$ at $z = 0$, in

section 4. Furthermore, we assume in here, that the system is close to critical and therefore that the wave number k is close to its critical value k_c minimising $\tilde{\mathcal{R}}^{(0)}(k)$ (also $\tilde{\mathcal{R}}^{(0)}(k)$ is close to $\tilde{\mathcal{R}}^0(k_c)$). Thus by assumption all the wave vectors $\boldsymbol{k}$ in the sum in (17) lie on a circle of radius k. From (13)–(15) and (17) we infer

$$\frac{\mathrm{d}^2 W^{(0)}}{\mathrm{d}z^2} + \frac{m\theta}{1+\theta z}\frac{\mathrm{d}W^{(0)}}{\mathrm{d}z} + \left[\tilde{\mathcal{R}}^{(0)}k^2(1+\theta z)^{m-1} - k^6 - \frac{m\theta^2}{(1+\theta z)^2}\right]W^{(0)} = 0. \qquad (18)$$

3.1. *The weakly compressible ($\theta \ll 1$) stationary solutions*

To study the effect of finite θ, further analytic progress can be made if we assume that the compressibility is small, but finite. We now assume $1 \gg |\theta| \gg \varepsilon^{1/2} = \tau^{-1/12}$ and expand the dependent variables in θ. We obtain exactly the same solution up to order θ as in Mizerski and Tobias (2011), i.e.

$$W^{(0)} = A\sin(\pi z) + \theta[C\sin(\pi z) + A\mathcal{D}_1 z(z-1)\cos(\pi z) + A\mathcal{D}_2 z\sin(\pi z)], \qquad (19)$$

$$\mathcal{R} \approx \tau^{2/3}\left(\tilde{\mathcal{R}}_0^{(0)} + \theta\tilde{\mathcal{R}}_1^{(0)}\right), \qquad (20)$$

where A and C are arbitrary constants at this stage and

$$\tilde{\mathcal{R}}_0^{(0)} = \frac{1}{k^2}\left[k^6 + \pi^2\right], \quad \tilde{\mathcal{R}}_1^{(0)} = -\frac{1}{2}(m-1)\tilde{\mathcal{R}}_0^{(0)}, \qquad (21\text{a,b})$$

$$\mathcal{D}_1 = \frac{\tilde{k}^2\tilde{\mathcal{R}}_0^{(0)}(m-1)}{4\pi}, \quad \mathcal{D}_2 = -\frac{1}{2\pi}\left[\frac{\tilde{k}^2\tilde{\mathcal{R}}_0^{(0)}(m-1)}{2\pi} + m\pi\right]. \qquad (22\text{a,b})$$

At the onset of convection we have $k_c = \pi^{1/3}/2^{1/6}$ and $\tilde{\mathcal{R}}_0^{(0)}(k_c) = 3\pi^{4/3}/2^{2/3}$. The assumption (17) leads to the following form of the vertical velocity component:

$$w^{(0)} = \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{A}(t,\boldsymbol{k})[\sin(\pi z) + \theta f(z)] + \theta\sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{C}(t,\boldsymbol{k})\sin(\pi z), \qquad (23)$$

where $\hat{w}^{(0)}$ was set to unity since $\hat{A}(t,\boldsymbol{k})$ and $\hat{C}(t,\boldsymbol{k})$ were introduced, $f(z) = \mathcal{D}_1 z(z-1)\cos(\pi z) + \mathcal{D}_2 z\sin(\pi z)$, and since $w^{(0)}$ is real we have $\hat{A}(t,\boldsymbol{k}) = \hat{A}^*(t,-\boldsymbol{k})$ and $\hat{C}(t,\boldsymbol{k}) = \hat{C}^*(t,-\boldsymbol{k})$ where upper star denotes a complex conjugate. Note that since also $w^{(0)^2}$ is real $\hat{A}(t,\boldsymbol{k})\hat{C}^*(t,\boldsymbol{k})$ must be real as well. The sum is taken over all wave vectors, which, again, lie on a circle of radius k. We also define $A(t,\boldsymbol{x}) = \sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{A}(t,\boldsymbol{k})$ and $C(t,\boldsymbol{x}) = \sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{C}(t,\boldsymbol{k})$. From equations (14a,b) and (15a,b) we get

$$p'^{(0)} = -\frac{\pi}{k^4}\left\{\sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{A}(t,\boldsymbol{k})[\cos(\pi z) + \theta F(z)] + \theta\sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{C}(t,\boldsymbol{k})\cos(\pi z)\right\}, \qquad (24)$$

where $\pi F(z) = \mathrm{d}f/\mathrm{d}z + m\sin\pi z$.

At this point let us define the average kinetic energy of the flow

$$\mathcal{T}^{(0)} = \frac{1}{2}\overline{\left\langle\left(\boldsymbol{u}_h'^{(0)}\right)^2 + (w^{(0)})^2\right\rangle} = \frac{\tilde{\mathcal{R}}_0^{(0)}}{4k^4}\left[(1+\theta\mathcal{D}_2)\sum_{\boldsymbol{k}}|\hat{A}(\boldsymbol{k})|^2 + 2\theta\sum_{\boldsymbol{k}}\left(\hat{A}(\boldsymbol{k})\hat{C}^*(\boldsymbol{k})\right)\right], \qquad (25)$$

where the overbar indicates the vertical average, hence

$$\mathcal{T}^{(0)} = \mathcal{T}_0^{(0)}[1 + \theta(\mathcal{D}_2 + 2\mathcal{Q})], \qquad (26\text{a})$$

$$\mathcal{T}_0^{(0)} = \frac{\tilde{\mathcal{R}}_0^{(0)}}{4k^4}\sum_{\boldsymbol{k}}|\hat{A}(\boldsymbol{k})|^2, \quad \mathcal{T}_{ac} = \frac{\tilde{\mathcal{R}}_0^{(0)}}{4k^4}\sum_{\boldsymbol{k}}\left(\hat{A}(\boldsymbol{k})\hat{C}^*(\boldsymbol{k})\right), \qquad (26\text{b,c})$$

where

$$\mathcal{Q} = \mathcal{T}_{ac}/\mathcal{T}_0^{(0)}. \tag{27}$$

Therefore $\mathcal{T}^{(0)}$ is the average kinetic energy of the flow equal to $\mathcal{T}_0^{(0)}$ if the fluid is incompressible (i.e. if $\theta = 0$) (cf. Soward 1974).

Using (16a,b) we obtain

$$\langle \boldsymbol{u}'^{(0)} \times \boldsymbol{B}'^{(0)} \rangle = -\frac{\pi}{k^6} \left\{ [\sin(2\pi z) + \theta h(z)] \boldsymbol{B}_h^{(0)} \cdot \langle \nabla A \otimes \nabla A \rangle + 2\theta \sin(2\pi z) \boldsymbol{B}_h^{(0)} \cdot \langle (\nabla A \otimes \nabla C)_S \rangle \right\}, \tag{28}$$

where

$$h(z) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}z} [\mathcal{D}_1 z(z-1) \sin(2\pi z) + \mathcal{D}_2 z(1 - \cos(2\pi z))] + \frac{m}{\pi}(1 - \cos(2\pi z)). \tag{29}$$

Above, $\otimes$ is the tensor multiplication and subscript $(\cdot)_S$ denotes the symmetric part of a tensor. Thus equation (9) for the evolution of the large-scale magnetic field takes the following form:

$$\frac{\partial B_i^{(0)}}{\partial t} + 2\pi \Lambda \frac{\partial}{\partial z} \left\{ \sin(2\pi z) \mathbb{M}_{ij} B_j^{(0)} + \theta \left[h(z) \mathbb{M}_{ij} B_j^{(0)} + 2\frac{\Xi}{\Lambda} \sin(2\pi z) \mathbb{N}_{ij} B_j^{(0)} \right] \right\}$$
$$- \frac{\partial^2 B_i^{(0)}}{\partial z^2} = 0 + \mathrm{O}(\theta^2), \tag{30}$$

where the magnetic field has yet to be expanded in powers of θ and

$$\Lambda = \mathcal{T}_0^{(0)} / \tilde{\mathcal{R}}_0^{(0)}, \quad \Xi = \mathcal{T}_{ac} / \tilde{\mathcal{R}}_0^{(0)}, \tag{31a,b}$$

$$\mathbb{M} = \begin{bmatrix} -\alpha_{21} & -\alpha_{22} \\ \alpha_{11} & \alpha_{12} \end{bmatrix}, \quad \mathbb{N} = \begin{bmatrix} -\beta_{21} & -\beta_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix}, \tag{31c,d}$$

$$\alpha_{ij} = \frac{1}{2k^6 \Lambda} \langle \nabla A \otimes \nabla A \rangle_{ij} = \sum_{\boldsymbol{k}} \frac{k_i k_j}{k^2} q(\boldsymbol{k}), \tag{31e}$$

$$\beta_{ij} = \frac{1}{2k^6 \Xi} \langle (\nabla A \otimes \nabla C)_S \rangle_{ij} = \sum_{\boldsymbol{k}} \frac{k_i k_j}{k^2} p(\boldsymbol{k}), \tag{31f}$$

$$q(\boldsymbol{k}) = \frac{1}{2k^4 \Lambda} |\hat{A}(\boldsymbol{k})|^2, \quad p(\boldsymbol{k}) = \frac{1}{2k^4 \Xi} \left(\hat{A}(\boldsymbol{k}) \hat{C}^*(\boldsymbol{k}) \right). \tag{31g,h}$$

It follows that

$$\alpha_{11} + \alpha_{22} = 2, \quad \beta_{11} + \beta_{22} = 2, \quad \mathcal{Q} = \Xi/\Lambda. \tag{32a,b,c}$$

We will now seek a stationary solution of the large-scale field equation (30). Expanding the large-scale magnetic field in powers of θ, i.e. $\boldsymbol{B}_h^{(0)} = \boldsymbol{B}_{0,h}^{(0)} + \theta \boldsymbol{B}_{1,h}^{(0)} + \mathrm{O}(\theta^2)$ we obtain

$$2\pi \Lambda \frac{\mathrm{d}}{\mathrm{d}z} \left[\sin(2\pi z) \mathbb{M}_{ij} B_{0,j}^{(0)} \right] - \frac{\mathrm{d}^2 B_{0,i}^{(0)}}{\mathrm{d}z^2} = 0, \tag{33}$$

$$2\pi \Lambda \frac{\mathrm{d}}{\mathrm{d}z} \left[\sin(2\pi z) \mathbb{M}_{ij} B_{1,j}^{(0)} \right] - \frac{\mathrm{d}^2 B_{1,i}^{(0)}}{\mathrm{d}z^2} = 2\pi \Lambda \frac{\mathrm{d}}{\mathrm{d}z} \left[h(z) \mathbb{M}_{ij} B_{0,j}^{(0)} + 2\mathcal{Q} \sin(2\pi z) \mathbb{N}_{ij} B_{0,j}^{(0)} \right], \tag{34}$$

with the boundary conditions $(\mathrm{d}B_{i,j}^{(0)}/\mathrm{d}z)|_{z=0,1} = 0$ for $i = 1, 2$ and $j = x, y$. If we define

$$\alpha_{11}^{1/2}\phi_0 = \alpha_{11}B_{0,x}^{(0)} + (\alpha_{12} + \mathrm{i}\alpha)B_{0,y}^{(0)}, \tag{35}$$

$$\alpha_{11}^{1/2}\phi_1 = \alpha_{11}B_{1,x}^{(0)} + (\alpha_{12} + \mathrm{i}\alpha)B_{1,y}^{(0)}, \tag{36}$$

$$\alpha^2 = \det \alpha_{ij}, \quad \beta^2 = \det \beta_{ij}, \tag{37a,b}$$

we may reduce the problem to only two equations for ϕ_0 and ϕ_1

$$2\pi\mathrm{i}\alpha\Lambda\frac{\mathrm{d}}{\mathrm{d}z}\{\phi_0 \sin(2\pi z)\} - \frac{\mathrm{d}^2\phi_0}{\mathrm{d}z^2} = 0, \tag{38}$$

$$2\pi\mathrm{i}\alpha\Lambda\frac{\mathrm{d}}{\mathrm{d}z}\{\phi_1 \sin(2\pi z)\} - \frac{\mathrm{d}^2\phi_1}{\mathrm{d}z^2}$$
$$= 2\pi\mathrm{i}\alpha\Lambda\frac{\mathrm{d}}{\mathrm{d}z}\left\{h(z)\phi^0 + 2\mathcal{Q}\sin(2\pi z)\left[\left(\frac{\mathrm{i}\alpha + \alpha_{12}}{\mathrm{i}\alpha}\beta_{11} - \frac{\alpha_{11}}{\mathrm{i}\alpha}\beta_{21}\right)B_{0,x}^{(0)}\right.\right.$$
$$\left.\left. + \left(\frac{\mathrm{i}\alpha + \alpha_{12}}{\mathrm{i}\alpha}\beta_{12} - \frac{\alpha_{11}}{\mathrm{i}\alpha}\beta_{22}\right)B_{0,y}^{(0)}\right]\right\}. \tag{39}$$

The first of these equations, expressing the leading order in θ balance, is the same as in Soward (1974). As noted by Soward, there exists a solution of (38) of the form

$$\phi_0 = \exp[-\mathrm{i}\alpha\Lambda\cos(2\pi z)]. \tag{40}$$

The solution of the homogeneous part of (39) has exactly the same form, i.e.

$$\phi_1^H = \exp[-\mathrm{i}\alpha\Lambda\cos(2\pi z)], \tag{41}$$

where superscript H denotes the solution of a homogeneous equation. A specific solution of a non-homogeneous equation (39) with ϕ_0 defined in (40) may be sought in the following form:

$$\phi_1 = f_1(z)\cos[\alpha\Lambda\cos(2\pi z)] + f_2(z)\sin[\alpha\Lambda\cos(2\pi z)]. \tag{42}$$

Substituting this form into (39) we get

$$2\pi\alpha\Lambda\sin(2\pi z)(\mathrm{i}f_1 + f_2) - \frac{\mathrm{d}f_1}{\mathrm{d}z} = 2\pi\mathrm{i}\alpha\Lambda\left\{h(z) + 2\mathcal{Q}\sin(2\pi z)\left(\frac{\beta_{11}}{\alpha_{11}} + \mathrm{i}\frac{\alpha_{11}\beta_{12} - \alpha_{12}\beta_{11}}{\alpha\alpha_{11}}\right)\right\}, \tag{43a}$$

$$2\pi\alpha\Lambda\sin(2\pi z)(\mathrm{i}f_2 - f_1) - \frac{\mathrm{d}f_2}{\mathrm{d}z}$$
$$= 2\pi\mathrm{i}\alpha\Lambda\left\{-\mathrm{i}h(z) + 2\mathcal{Q}\sin(2\pi z)\left[-\frac{\alpha_{11}\beta_{12} - \alpha_{12}\beta_{11}}{\alpha\alpha_{11}} + \frac{\mathrm{i}}{\alpha^2}\left(2\alpha_{12}\beta_{12} - \alpha_{11}\beta_{22} - \frac{\alpha_{12}^2\beta_{11}}{\alpha_{11}}\right)\right]\right\}, \tag{43b}$$

which, after introducing

$$F_1 = f_1 + \mathrm{i}f_2, \quad F_2 = f_1 - \mathrm{i}f_2, \tag{44a,b}$$

leads to

$$\frac{\mathrm{d}F_1}{\mathrm{d}z} = -4\pi\mathrm{i}\alpha\Lambda\left\{h(z) + 2\mathcal{Q}\sin(2\pi z)\frac{\chi^2}{\alpha^2}\right\}, \tag{45a}$$

$$4\pi i \alpha \Lambda \sin(2\pi z) F_2 - \frac{dF_2}{dz}$$

$$= 4\pi i \alpha \Lambda \mathcal{Q} \sin(2\pi z) \left\{ \frac{1}{\alpha^2} \left(2\alpha_{12}\beta_{12} - \alpha_{11}\beta_{22} + \frac{\alpha^2 - \alpha_{12}^2}{\alpha_{11}} \beta_{11} \right) + i\frac{2}{\alpha} \left(\beta_{12} - \alpha_{12}\frac{\beta_{11}}{\alpha_{11}} \right) \right\},$$

$$(45b)$$

where $2\chi^2 = (\alpha_{11}\beta_{22} + \alpha_{22}\beta_{11}) - 2\alpha_{12}\beta_{12}$. From the above we obtain a specific solution

$$f_1 = -2i\alpha\Lambda \left[\mathcal{D}_1 z(z-1)\sin(2\pi z) + \mathcal{D}_2 z(1 - \cos(2\pi z)) \right.$$

$$\left. + mz - \frac{m}{2\pi}\sin(2\pi z) - \mathcal{Q}\frac{\chi^2}{\alpha^2}\cos(2\pi z) \right] + C', \qquad (46a)$$

$$f_2 = -2\alpha\Lambda \left[\mathcal{D}_1 z(z-1)\sin(2\pi z) + \mathcal{D}_2 z(1 - \cos(2\pi z)) \right.$$

$$\left. + mz - \frac{m}{2\pi}\sin(2\pi z) - \mathcal{Q}\frac{\chi^2}{\alpha^2}\cos(2\pi z) \right] + iC', \qquad (46b)$$

where

$$C' = \mathcal{Q}\left[\frac{1}{\alpha^2}\left(2\alpha_{12}\beta_{12} - \alpha_{11}\beta_{22} + \frac{\alpha^2 - \alpha_{12}^2}{\alpha_{11}}\beta_{11} \right) + i\frac{2}{\alpha}\left(\beta_{12} - \alpha_{12}\frac{\beta_{11}}{\alpha_{11}} \right) \right] = \text{const.} \quad (47)$$

Hence

$$\phi_1 = \text{const.} \times \exp[-i\alpha\Lambda\cos(2\pi z)] + (f_1(z) - C')\exp[-i\alpha\Lambda\cos(2\pi z)] + C'\exp[i\alpha\Lambda\cos(2\pi z)],$$

$$(48)$$

with $f_1(z)$ defined in (46a) and the boundary conditions $d(\phi_0 + \theta\phi_1)/dz|_{z=0,1} = 0$ are satisfied.

As in Soward (1974), to exclude a uniform magnetic field we assume the following condition

$$\int_0^1 \phi dz \approx \int_0^1 (\phi_0 + \theta\phi_1)dz = 0, \qquad (49)$$

which holds true for all time. We now use the integral representation $J_0(z) = (1/\pi)\int_0^\pi \exp(-iz\cos\vartheta)d\vartheta$ (cf. Abramowitz and Stegun 1972) to determine that $\alpha\Lambda$ must be a zero of a Bessel function $J_0(\alpha\Lambda)$, since at leading order $\int_0^1 \phi_0 dz = 0$. Furthermore the condition $\int_0^1 \phi_1 dz = 0$ leads to

$$\mathcal{D}_2 J_1(\alpha\Lambda) = -2J_1(\alpha\Lambda)\mathcal{Q}\frac{\chi^2}{\alpha^2} \qquad \Longrightarrow \qquad \mathcal{Q} = -\frac{1}{2}\mathcal{D}_2\frac{\alpha^2}{\chi^2}, \qquad (50a,b)$$

and so

$$\mathcal{T}^{(0)} = \mathcal{T}_0^{(0)}\left[1 + \theta\mathcal{D}_2\left(1 - \frac{\alpha^2}{\chi^2}\right) \right], \qquad (51)$$

where the constant $\mathcal{D}_2$ was defined in (22b). Since the polytropic index m is usually greater than unity, the constant $\mathcal{D}_2$ is likely to be negative. However, the sign of the correction $\theta\mathcal{D}_2(1 - \alpha^2/\chi^2)$ to the average kinetic energy of the flow in (51), resulting

from the compressibility of the medium depends also on the horizontal structure of the flow (the planform of modes), i.e. the ratio α^2/χ^2. For different values of this ratio one may obtain stationary dynamo solutions for either smaller or greater average kinetic energy of the flow than in the incompressible case. However, it is important to realise, that a higher order analysis in θ and ε would put some constraints on the solution and restrictions for the choice of $\hat{A}(\boldsymbol{k})$ and $\hat{C}(\boldsymbol{k})$ and what follows for the ratio α^2/χ^2. In section 3.3, we show that stationary solutions of the mean field equation (9) exist only in the kinematic regime, when the field is very weak, i.e. $M^2 \ll 1$. Furthermore not all the choices of planform lead necessarily to stable stationary solutions. In fact the compressibility breaks the Boussinesq up-down symmetry introducing smaller scales to the flow (cf. Glatzmaier and Gilman 1981, Jones *et al.* 2009, Jones and Kuzanyan 2009, Mizerski and Tobias 2011), and thus we would expect dissipation to play a more significant role in compressible, convective flows. This suggests that in compressible systems larger kinetic energies should be necessary to maintain a stationary large-scale magnetic field than in Boussinesq systems.

As will become clear later in section 3.3, the use of the arbitrarily prescribed planform such as squares or regular hexagons in this theory is only allowed for very weak magnetic field, i.e. for $M^2 \ll 1$; then the deviations from the symmetric planform are weak and occur at higher order (see the discussion below equation (73) in section 3.3). In other words such solutions, which we consider in the following section, exist only in kinematic regime.

3.2. *Dynamo threshold for regular planforms in the weak compressibility limit*

We present the kinematic dynamo solutions in the limit of rapid rotation, $\varepsilon \ll 1$, and weak compressibility, $\theta \ll 1$, with convective velocity field given in (23) and regular planform: squares, hexagons, etc.. In other words we provide the explicit form of the solution of equations (33) and (34) given in (40) and (48) (cf. also (35)–(37)).

For any regular planform according to definitions (25)–(27) and (31a–h) we obtain

$$\alpha_{11} = \alpha_{22} = 1, \quad \alpha_{12} = \alpha_{21} = 0, \tag{52a,b}$$

$$\beta_{11} = \beta_{22} = 1, \quad \beta_{12} = \beta_{21} = 0, \tag{52c,d}$$

$$\mathcal{T}_0^{(0)} = \frac{N\tilde{\mathcal{R}}_0^{(0)}}{4k^4}|A|^2, \quad \mathcal{T}_{ac} = \frac{N\tilde{\mathcal{R}}_0^{(0)}}{4k^4}AC^*, \tag{52e,f}$$

where N is the number of different wavevectors in the planform (four for squares, six for hexagons, etc.). Hence we have

$$\alpha^2/\chi^2 = 1, \tag{53}$$

which means that, for this particular form of rolls, the average kinetic energy $\mathcal{T}^{(0)}$ of the flow necessary to sustain a stationary magnetic field of the form found in section 3.1, in the compressible case, up to order θ, is the same as the average kinetic energy $\mathcal{T}_0^{(0)}$ of an analogous incompressible flow necessary to sustain a similar stationary magnetic field (cf. (51)),

$$\mathcal{T}^{(0)} = \mathcal{T}_0^{(0)} + \mathrm{O}(\theta^2). \tag{54}$$

Furthermore, concentrating on modes with square planform the stationary dynamo solution is possible if the following holds:

$$\frac{\mathcal{T}_0^{(0)}}{\tilde{\mathcal{R}}_0^{(0)}} = \frac{|\hat{A}|^2}{k^4} = \mathcal{Z}_{J_0}, \quad \frac{\mathcal{T}_{ac}}{\mathcal{T}_0^{(0)}} = \frac{\hat{C}^*}{\hat{A}^*} = -\frac{1}{2}\mathcal{D}_2, \quad (55a,b)$$

where $\mathcal{Z}_{J_0}$ is a zero of the Bessel function J_0 (e.g. the first stationary solution to appear is at the first zero which is $\mathcal{Z}_{J_0} = 2.4048$) and, as explained in section 3.3 if $M^2 \ll 1$.† In the case of modes with square planform the form of the large-scale magnetic field is given by

$$B_x^{(0)} = \mathcal{C}_1 \cos\!\big(\mathcal{Z}_{J_0} \cos 2\pi z\big) + \theta\big[\mathcal{C}_2 \cos\!\big(\mathcal{Z}_{J_0} \cos 2\pi z\big) - 2\mathcal{C}_1 \mathcal{Z}_{J_0}\mathcal{F}(z)\sin\!\big(\mathcal{Z}_{J_0}\cos 2\pi z\big)\big] + \mathrm{O}(\theta^2),$$
$$(56a)$$

$$B_y^{(0)} = -\mathcal{C}_1 \sin\!\big(\mathcal{Z}_{J_0}\cos 2\pi z\big) + \theta\big[-\mathcal{C}_2 \sin\!\big(\mathcal{Z}_{J_0}\cos 2\pi z\big)$$
$$- 2\mathcal{C}_1\mathcal{Z}_{J_0}\mathcal{F}(z)\cos\!\big(\mathcal{Z}_{J_0}\cos 2\pi z\big)\big] + \mathrm{O}(\theta^2), \quad (56b)$$

where

$$\mathcal{F}(z) = \left[\mathcal{D}_1 z(z-1) - \frac{m}{2\pi}\right]\sin(2\pi z) - \left(z - \frac{1}{2}\right)\mathcal{D}_2\cos(2\pi z) + (\mathcal{D}_2 + m)z, \quad (57)$$

and the constants $\mathcal{C}_1$ and $\mathcal{C}_2$ are at this stage arbitrary, i.e. are not set by our linear, kinematic dynamo analysis. Note that equation (30) is linear and the boundary conditions plus the initial condition (49) are homogeneous. This means that the amplitude of the magnetic field is constant up to order $\mathrm{O}(\theta)$ and is equal to $\mathcal{C}_1^2 + 2\mathcal{C}_1\mathcal{C}_2\theta$. Furthermore, equations (56a,b) define a spiral which in the Boussinesq case is symmetric with respect to the $z = 1/2$ plane. In other words, the direction of the magnetic vector changes with height i.e. as z increases the magnetic vector is going around a circle of radius $1 + 2\theta$, but instead of completing the circle turns back at $z = 1/2$, and in the Boussinesq case for $z > 1/2$ forms a perfect reflection of the lower half. Compressibility breaks the Boussinesq symmetry rotating the field vector towards the y-axis at the point $z = 1/2 + \theta(4\mathcal{D}_2 + \pi\mathcal{D}_1 + 4m)/4\pi^2$ where the spiral turns back (which is shifted by the presence of compressibility) and also changing the field's direction at the ends of the spiral. Figure 1 presents the solution given in (56a,b).

Furthermore, the leading order current density is given by the formulae (A.1a–c) in Appendix A, and in the case of square-planform modes it reduces to

$$j_x^{(0)} = -B_y^{(0)}(z)W^{(0)}(z)\cos ky, \quad (58a)$$

$$j_y^{(0)} = B_x^{(0)}(z)W^{(0)}(z)\cos kx, \quad (58b)$$

$$j_z^{(0)} = -\frac{1}{k^3}\Big[B_x^{(0)}(z)\sin kx + B_y^{(0)}(z)\sin ky\Big]\mathfrak{D}W^{(0)}, \quad (58c)$$

where $\hat{w}^{(0)}$ has in this case been set to unity. This means that the vertical component of the Lorentz force at $(x, y) = (0, 0)$ is equal to $-\big[B_x^2(z) + B_x^2(z)\big]W^{(0)}(z)$ and hence,

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†Using our results of weakly nonlinear theory of rapidly rotating, weakly compressible, convective systems from the previous paper Mizerski and Tobias (2011) we infer that such solution holds for $\tilde{\mathcal{R}}_{2,0}^{(0)} = k^2\tilde{\mathcal{R}}_{c,0}^{(0)}\mathcal{Z}_{J_0}/2$ and $\tilde{\mathcal{R}}_{2,1}^{(0)} = (m+1)\tilde{\mathcal{R}}_{2,0}^{(0)}/2 = (m+1)k^2\tilde{\mathcal{R}}_{c,0}^{(0)}\mathcal{Z}_{J_0}/4$, where the notation of Mizerski and Tobias (2011) was used, i.e. the Rayleigh number has the following expansion $\tilde{\mathcal{R}}^{(0)} = \tilde{\mathcal{R}}_{c,0}^{(0)} + \sum_{i,j=0} a^i\theta^j\tilde{\mathcal{R}}_{i,j}^{(0)}$ and $a \gg \varepsilon^{1/2}$ is the amplitude of the velocity field, $\tilde{\mathcal{R}}_{0,0}^{(0)} = 0$.
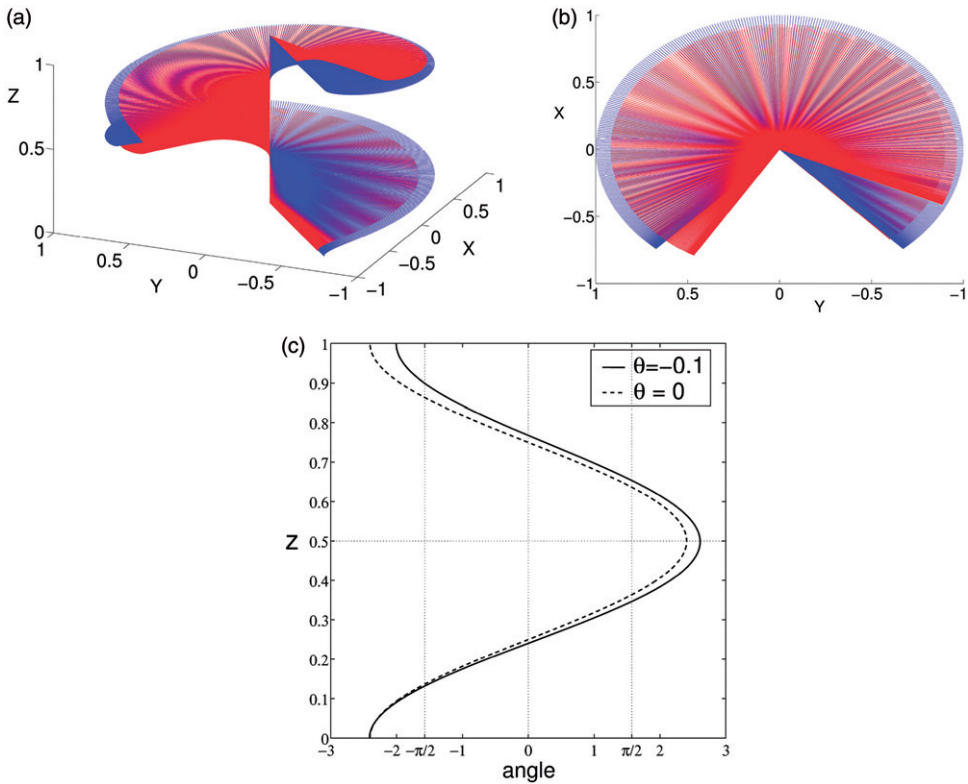
Figure 1.   The large-scale magnetic field averaged over horizontal directions at the dynamo threshold in the Boussinesq and weakly compressible ($\theta = -0.1$) cases for modes with square planform. The magnetic spiral is shown in (a), i.e. the magnetic field vector along the z-axis with the view from above shown in (b). The Boussinesq spiral is drawn with blue and weakly compressible spiral with red. The dependence of the angle between the magnetic field vector and the x axis on the vertical coordinate z for both, Boussinesq and weakly compressible fluid is shown in (c).

in accordance with the Lenz rule, opposes the upwelling of convective current, i.e. the buoyancy force.

We now proceed to higher orders in the rotation parameter ε to study the back reaction of the Lorentz force on the flow, which under current assumptions (cf. equations (5) and (6)) is the first nonlinearity to appear in the asymptotic expansions and will be used to establish the amplitude of the perturbations. This analysis will yield a system that can only be solved numerically for arbitrary θ, but this is not surprising as this is the case for the analogous Boussinesq system (Soward 1974).

3.3. *The back reaction of the Lorentz force for arbitrary compressibility*

To obtain the amplitude equations, which will determine the planform of convective modes and the magnetic field, we now proceed to higher orders in parameter ε assuming arbitrary compressibility θ. Following exactly the steps of Soward (1974) we write down the evolution equations (7a–f) for $w(\boldsymbol{x}, t)$ in the form (11) and (17) up to the order ε^3

together with their solvability conditions which leads to the following system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(0)} + \left[\tilde{\mathcal{R}}^{(0)}k^2(1+\theta z)^{m-1} - k^6\right]W^{(0)} = 0, \tag{59a}$$

$$\frac{\partial S^{(0)}}{\partial t} = -\frac{\sigma_\eta}{k^2}\mathcal{A}^{(0)}\mathfrak{D}\left[(1+\theta z)^{m-1}W^{(0)2}\right] + \frac{\sigma_\eta^{-1}}{(1+\theta z)^m}\left[\frac{\partial^2 S^{(0)}}{\partial z^2} + \frac{\theta}{1+\theta z}\frac{\partial S^{(0)}}{\partial z}\right]$$

$$-\frac{\sigma_\eta\theta\mathcal{A}^{(0)}}{k^4\tilde{\mathcal{R}}^{(0)}(1+\theta z)}\left[k^6 W^{(0)2} + \left(\mathfrak{D}W^{(0)}\right)^2\right], \tag{59b}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(2)} + \left[\tilde{\mathcal{R}}^{(0)}k^2(1+\theta z)^{m-1} - k^6\right]W^{(2)}$$

$$= k^2\tilde{\mathcal{R}}^{(0)}(1+\theta z)^m W^{(0)}\frac{\partial S^{(0)}}{\partial z} - k^2\tilde{\mathcal{R}}^{(2)}(1+\theta z)^{m-1}W^{(0)}, \tag{60a}$$

$$\frac{\partial S^{(2)}}{\partial t} = -\frac{\sigma_\eta}{k^2}\mathfrak{D}\left\{(1+\theta z)^{m-1}\left[2\mathcal{A}^{(2)}W^{(0)2} + 2\mathcal{A}^{(0)}W^{(0)}W^{(2)} - \mathcal{A}^{(0)}(1+\theta z)\frac{\partial S^{(0)}}{\partial z}W^{(0)2}\right]\right\}$$

$$+ \frac{\sigma_\eta^{-1}}{(1+\theta z)^m}\left[\frac{\partial^2 S^{(2)}}{\partial z^2} + \frac{\theta}{(1+\theta z)}\frac{\partial S^{(2)}}{\partial z}\right]$$

$$-\frac{\sigma_\eta\theta\mathcal{A}^{(0)}}{k^4\tilde{\mathcal{R}}^{(0)}(1+\theta z)}\left(\frac{2\mathcal{A}^{(2)}}{\mathcal{A}^{(0)}} - \frac{\tilde{\mathcal{R}}^{(2)}}{\tilde{\mathcal{R}}^{(0)}}\right)\left[k^6 W^{(0)2} + \left(\mathfrak{D}W^{(0)}\right)^2\right]$$

$$-\frac{2\sigma_\eta\theta\mathcal{A}^{(0)}}{k^4\tilde{\mathcal{R}}^{(0)}(1+\theta z)}\left[k^6 W^{(0)}W^{(2)} + \mathfrak{D}W^{(0)}\mathfrak{D}W^{(2)}\right], \tag{60b}$$

$$4k^8\sigma_m\left\{\frac{\partial}{\partial z}\mathfrak{D}w^{(3)} - \tilde{\mathcal{R}}^{(0)}(1+\theta z)^{m-1}\nabla_h^2 w^{(3)} + \nabla_h^6 w^{(3)}\right\}$$

$$= \left[2k^6 W^{(0)}\mathfrak{D}W^{(0)} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\mathfrak{D}W^{(0)}\right)^2 - 2\sigma\theta(1+\theta z)^{m-1}\left(\mathfrak{D}W^{(0)}\right)^2\right]\nabla_h^6\bar{w}^{(0)2}$$

$$+ \left[4k^8\mathfrak{D}W^{(0)2} - 4k^2\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathfrak{D}W^{(0)}\right)^2 - 2k^4\sigma\tilde{\mathcal{R}}^{(0)}(1+\theta z)^{2m-1}W^{(0)}\mathfrak{D}W^{(0)}\right.$$

$$\left. - 8k^2\sigma\theta(1+\theta z)^{m-1}\left(\frac{1}{4}k^6 W^{(0)2} + \left(\mathfrak{D}W^{(0)}\right)^2\right)\right]\nabla_h^4\bar{w}^{(0)2}$$

$$- \left\{2k^4\frac{\mathrm{d}}{\mathrm{d}z}\left(W^{(0)}\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(0)} - \left(\mathfrak{D}W^{(0)}\right)^2\right) + 4k^6\sigma\tilde{\mathcal{R}}^{(0)}(1+\theta z)^m\mathfrak{D}\left[(1+\theta z)^{m-1}W^{(0)2}\right]\right.$$

$$\left. + 4k^4\sigma\theta(1+\theta z)^{m-1}\left(k^6 W^{(0)2} + \left(\mathfrak{D}W^{(0)}\right)^2\right)\right\}\nabla_h^2\bar{w}^{(0)2}, \tag{61}$$

where

$$w^{(0)}(\boldsymbol{x}, t) = \bar{w}^{(0)}(\boldsymbol{x}, t)W^{(0)}(z) = \sum_{\boldsymbol{k}}\hat{w}^{(0)}(\boldsymbol{k}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}W^{(0)}(z), \quad \text{with} \quad \mathrm{d}W^{(0)}/\mathrm{d}z|_{z=0} = 1, \tag{62a}$$

$$w^{(2)}(\boldsymbol{x}, t) = \bar{w}^{(2)}(\boldsymbol{x}, t)W^{(0)}(z) + \bar{w}^{(0)}(\boldsymbol{x}, t)W^{(2)}(z)$$

$$= \sum_{\boldsymbol{k}}\hat{w}^{(2)}(\boldsymbol{k}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}W^{(0)}(z) + \sum_{\boldsymbol{k}}\hat{w}^{(0)}(\boldsymbol{k}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}W^{(2)}(z), \tag{62b}$$

$$w^{(3)}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{w}^{(3)}(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} W^{(0)}(z)$$

$$+ \frac{1}{4k^8\sigma_m} \sum_{\boldsymbol{k},\boldsymbol{k}'} |\boldsymbol{k} + \boldsymbol{k}'|^2 \hat{w}^{(0)}(\boldsymbol{k}, t)\hat{w}^{(0)}(\boldsymbol{k}', t) W^{(3)}_{|\boldsymbol{k}+\boldsymbol{k}'|}(z) e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}, \qquad (62c)$$

$$\mathcal{A}^{(0)} = \sum_{\boldsymbol{k}} \left|\hat{w}^{(0)}(\boldsymbol{k}, t)\right|^2, \quad \mathcal{A}^{(2)} = \sum_{\boldsymbol{k}} \left\{\hat{w}^{(0)}(\boldsymbol{k}, t)\left[\hat{w}^{(2)}(\boldsymbol{k}, t)\right]^*\right\}, \qquad (63a,b)$$

the operator $\mathfrak{D}$ is defined in (8) and the first-order terms are zero. As noted earlier, because we have introduced the Fourier amplitude $\hat{w}^{(0)}$ in (62a), some condition establishing the magnitude of $W^{(0)}$ is needed – we have chosen to set $\mathrm{d}W^{(0)}/\mathrm{d}z = 1$ at $z = 0$.

The solvability condition for (60a) yields

$$\int_0^1 (1 + \theta z)^{m-1} W^{(0)2} \left[\tilde{\mathcal{R}}^{(0)}(1 + \theta z)\frac{\partial S^{(0)}}{\partial z} - \tilde{\mathcal{R}}^{(2)}\right]\mathrm{d}z = 0, \qquad (64)$$

and since (59b) admits solutions with separated variables of the type $S^{(0)} = T(t)Z(z)$,† this implies that $S^{(0)}$ is independent of time. The above condition (64) together with (59b) is used to obtain $\mathcal{A}^{(0)}$ which, in turn, is also independent of time. Furthermore, since the average total kinetic energy at leading order is

$$\mathcal{T}^{(0)} = \frac{1}{2}\overline{\left\langle\left(\boldsymbol{u}_h'^0\right)^2 + \left(w'^0\right)^2\right\rangle} = \frac{1}{2}\left[\frac{1}{k^6}\overline{\left(\mathfrak{D}W^{(0)}\right)^2} + \overline{W^{(0)2}}\right]\mathcal{A}^{(0)}, \qquad (65)$$

where the overbar indicates a vertical average, $\mathcal{A}^{(0)}$ is a measure of the kinetic energy of the flow, which also must be constant. From (59b) and (64) we can also clearly see that the value of $\mathcal{A}^{(0)}$ and hence the average kinetic energy is established by the value of the correction to the Rayleigh number $\tilde{\mathcal{R}}^{(2)}$ and hence the magnetic field does not influence the kinetic energy of the flow at leading order. This is why these dynamo solutions with the Hartmann number of order unity, $M^2 \sim 1$, were called the weak field solutions by Childress and Soward (1972).

Finally, the solvability condition for the equations obtained at order ε^3 provides the equation governing the evolution of the amplitudes $\hat{w}^{(0)}(\boldsymbol{k}, t)$, which, for general planform, is given in Appendix B. The amplitude equation (B.8) is coupled with the evolution equation for the mean magnetic field $\boldsymbol{B}_h^{(0)}$, (B.9–B.10), also provided in Appendix B. Those equations are correct for a general planform of convection close to onset. However, more progress can be made by assuming the planform for convection and simplifying the equations. Here we again follow Soward by selecting two planforms to analyse. In the first one, the rectangular planform, the rolls are aligned at right angles to each other. For this case we derive the relevant equations and give numerical solutions in a subsequent section. For the more complicated case of a

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†Since the function $S^{(0)}$ must vanish on the boundaries, it can be expanded in Fourier series $S^{(0)}(z, t) = \sum_n T_n(t)\sin(n\pi z)$ with $n = 1, 2, \ldots$. Next we multiply equation (59b) by $\sigma_n\theta^{-1}(1 + \theta z)^{m+1}$ and note that in the resulting equation the zero boundary condition is satisfied by all the terms except for $\partial_z S^{(0)} = \sum_n n\pi T_n(t)\cos(n\pi z)$ and the term appearing in the viscous heating proportional to $\sim \mathcal{A}^{(0)}(1 + \theta z)^m(\mathrm{d}_z W^{(0)})^2$, which, therefore, are the only terms in the equation possessing the functions $\cos(n\pi z)$ in the Fourier expansion. This implies that each of the functions $T_n(t)$ must be proportional to $\mathcal{A}^{(0)}$ and hence we get separation of variables, i.e. the function $S^{(0)}(z, t)$ can be represented as $S^{(0)}(z, t) = \mathcal{A}^{(0)}(t)Z(z)$.

hexagonal planform, where three sets of rolls at $\pi/3$ are considered we give the equations without the numerical solution.

3.3.1. Equations for a rectangular planform.

We now consider the simplest case, i.e. solutions in the form of superposition of two perpendicular rolls. For this case the vertical velocity is

$$w^{(0)}(\boldsymbol{x}, t) = \left(\hat{w}_1^{(0)}(\boldsymbol{k}_1, t) e^{i\boldsymbol{k}_1 \cdot \boldsymbol{x}} + \hat{w}_2^{(0)}(\boldsymbol{k}_2, t) e^{i\boldsymbol{k}_2 \cdot \boldsymbol{x}} + \text{c.c.} \right) W^{(0)}(z), \tag{66}$$

where

$$\boldsymbol{k}_1 = (k, 0), \quad \boldsymbol{k}_2 = (0, k). \tag{67a,b}$$

Taking advantage of the fact that $\mathcal{A}^{(0)} = 2(|\hat{w}_1^{(0)}|^2 + |\hat{w}_2^{(0)}|^2)$ is constant in time, we may reduce the number of variables by eliminating $\hat{w}_2^{(0)}$ from the equations. Defining

$$q_1 = |\hat{w}_1^{(0)}|^2 / \mathcal{A}^{(0)}, \quad q_2 = |\hat{w}_2^{(0)}|^2 / \mathcal{A}^{(0)}, \tag{68a,b}$$

we get

$$q_1 + q_2 = \frac{1}{2} \Longrightarrow q_2 = \frac{1}{2} - q_1. \tag{69}$$

Equation (B.8) can now be rewritten and takes a simplified form

$$\frac{dq_1}{dt} = \frac{4\sigma_m}{\mathcal{F}_1} q_1 \left(\frac{1}{2} - q_1 \right) \left[\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_1, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_2, m, \theta) \right], \tag{70}$$

with

$$\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_1, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_2, m, \theta)$$
$$= \frac{M^2}{k^2} \int_0^1 \left(\mathfrak{D} W^{(0)} \frac{dW^{(0)}}{dz} - k^6 W^{(0)2} \right) \frac{B_x^{(0)2} - B_y^{(0)2}}{(1 + \theta z)^m} \, dz. \tag{71}$$

This equation is coupled with the equations governing the evolution of the mean magnetic field, which in this case reduce to

$$\frac{\partial B_x^{(0)}}{\partial t} = \frac{4}{k^4} \mathcal{A}^{(0)} \left(\frac{1}{2} - q_1 \right) \frac{\partial}{\partial z} \left[W^{(0)} \mathfrak{D} W^{(0)} B_y^{(0)} \right] + \frac{\partial^2 B_x^{(0)}}{\partial z^2} = 0, \tag{72a}$$

$$\frac{\partial B_y^{(0)}}{\partial t} = -\frac{4}{k^4} \mathcal{A}^{(0)} q_1 \frac{\partial}{\partial z} \left[W^{(0)} \mathfrak{D} W^{(0)} B_x^{(0)} \right] + \frac{\partial^2 B_y^{(0)}}{\partial z^2} = 0, \tag{72b}$$

and we assume that the boundaries are perfectly conducting, thus $(\partial \boldsymbol{B}_h / \partial z)|_{z=0,1} = 0$. As noted above, stationary solutions of the above system of equations (70)–(72) are only possible in the limit of very weak magnetic field, $M^2 \ll 1$, at leading order in the Hartmann number, i.e. when the problem is kinematic. For such solutions the values of the amplitudes q_1 and q_2 are set by the condition

$$\int_0^1 B_x^{(0)} \, dz = \int_0^1 B_y^{(0)} \, dz = 0, \tag{73}$$

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which is simply imposed to exclude a uniform magnetic field from the solution and by the value of the correction to the Rayleigh number $\tilde{\mathcal{R}}_2$. For $M^2 \sim 1$ stationary solutions of (72a,b) would not satisfy the independent condition $\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_1, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_2, m, \theta) = 0$ resulting from (70) and hence are not allowed.

The numerical procedure developed to solve the above set of equations and the numerical results are presented in section 4.

3.3.2. Equations for a hexagonal planform.

Here we provide the explicit form of equations for the case when the velocity field is composed of three rolls whose axes form an angle of $\pi/3$ with each other. In this case, the vertical velocity is

$$w^{(0)}(\boldsymbol{x}, t) = \left(\hat{w}_1^{(0)}(\boldsymbol{k}_1, t) e^{i\boldsymbol{k}_1 \cdot \boldsymbol{x}} + \hat{w}_2^{(0)}(\boldsymbol{k}_2, t) e^{i\boldsymbol{k}_2 \cdot \boldsymbol{x}} + \hat{w}_3^{(0)}(\boldsymbol{k}_2, t) e^{i\boldsymbol{k}_3 \cdot \boldsymbol{x}} + \text{c.c.} \right) W^{(0)}(z), \quad (74)$$

where

$$\boldsymbol{k}_1 = (k, 0), \quad \boldsymbol{k}_2 = \left(-k/2, \sqrt{3}k/2\right), \quad \boldsymbol{k}_3 = \left(-k/2, -\sqrt{3}k/2\right). \quad (75\text{a,b,c})$$

Here the constant $\mathcal{A}^{(0)} = 2\left(|\hat{w}_1^{(0)}|^2 + |\hat{w}_2^{(0)}|^2 + |\hat{w}_3^{(0)}|^2\right)$. Thus defining,

$$q_1 = |\hat{w}_1^{(0)}|^2 / \mathcal{A}^{(0)}, \quad q_2 = |\hat{w}_2^{(0)}|^2 / \mathcal{A}^{(0)}, \quad q_3 = |\hat{w}_3^{(0)}|^2 / \mathcal{A}^{(0)}, \quad (76\text{a, b,c})$$

we get

$$q_1 + q_2 + q_3 = \frac{1}{2} \Longrightarrow q_3 = \frac{1}{2} - q_1 - q_2. \quad (77)$$

The amplitude equations take the form

$$\begin{aligned}
\frac{dq_1}{dt} = \frac{4\sigma_m}{\mathcal{F}_1} q_1 &\left\{ \left(\frac{1}{2} - q_1\right) \left[\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_1, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_3, m, \theta) \right] \right. \\
&\left. - q_2 \left[\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_2, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_3, m, \theta) \right] \right\} \\
&+ q_1 \frac{\sqrt{3}}{2} \frac{\mathcal{A}^{(0)}}{k^2 \sigma_m \mathcal{F}_1} \left[\Gamma\left(\sqrt{3}k, k, \sigma, m, \theta\right) - \Gamma(k, k, \sigma, m, \theta) \right] \left(q_1 + 2q_2 - \frac{1}{2} \right), \quad (78\text{a})
\end{aligned}$$

$$\begin{aligned}
\frac{dq_2}{dt} = \frac{4\sigma_m}{\mathcal{F}_1} q_2 &\left\{ \left(\frac{1}{2} - q_2\right) \left[\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_2, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_3, m, \theta) \right] \right. \\
&\left. - q_1 \left[\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_1, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_3, m, \theta) \right] \right\} \\
&- q_2 \frac{\sqrt{3}}{2} \frac{\mathcal{A}^{(0)}}{k^2 \sigma_m \mathcal{F}_1} \left[\Gamma\left(\sqrt{3}k, k, \sigma, m, \theta\right) - \Gamma(k, k, \sigma, m, \theta) \right] \left(2q_1 + q_2 - \frac{1}{2} \right), \quad (78\text{b})
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(\boldsymbol{B}_h^{(0)}, &\boldsymbol{k}_1, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_3, m, \theta) \\
&= \frac{M^2}{k^2} \int_0^1 \left(\mathfrak{D}W^{(0)} \frac{dW^{(0)}}{dz} - k^6 W^{(0)2} \right) \frac{(3/4)\left(B_x^{(0)2} - B_y^{(0)2}\right) - (\sqrt{3}/2)B_x^{(0)}B_y^{(0)}}{(1 + \theta z)^m} dz, \quad (79\text{a})
\end{aligned}$$

$$\mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_2, m, \theta) - \mathcal{M}(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}_3, m, \theta) = -\frac{M^2}{k^2} \int_0^1 \left(\mathfrak{D}W^{(0)} \frac{dW^{(0)}}{dz} - k^6 W^{(0)2} \right) \frac{\sqrt{3}B_x^{(0)}B_y^{(0)}}{(1 + \theta z)^m} dz. \quad (79\text{b})$$

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This equation is coupled with the equations governing the evolution of the mean magnetic field, which in this case reduce to

$$\frac{\partial B_x^{(0)}}{\partial t} = \frac{\sqrt{3}}{k^4} \mathcal{A}^{(0)} \frac{\partial}{\partial z} \left\{ W^{(0)} \mathfrak{D} W^{(0)} \left[\left(\frac{1}{2} - q_1 - 2q_2 \right) B_x^{(0)} + \left(\frac{1}{2} - q_1 \right) \sqrt{3} B_y^{(0)} \right] \right\} + \frac{\partial^2 B_x^{(0)}}{\partial z^2} = 0,$$

(80a)

$$\frac{\partial B_y^{(0)}}{\partial t} = -\frac{\sqrt{3}}{k^4} \mathcal{A}^{(0)} \frac{\partial}{\partial z} \left\{ W^{(0)} \mathfrak{D} W^{(0)} \left[\left(q_1 + \frac{1}{6} \right) \sqrt{3} B_x^{(0)} + \left(\frac{1}{2} - q_1 - 2q_2 \right) B_y^{(0)} \right] \right\} + \frac{\partial^2 B_y^{(0)}}{\partial z^2} = 0.$$

(80b)

To evaluate $\Gamma\left(\sqrt{3}k, k, \sigma, m, \theta\right)$ and $\Gamma(k, k, \sigma, m, \theta)$ it is necessary first to solve for $W_{\sqrt{3}k}^{(3)}$ and $W_k^{(3)}$ from (61) (cf. (62c)), respectively. The equation for $W_K^{(3)}$, where $K = \sqrt{3}k$ or $K = k$ can be readily obtained from (61) and (62c),

$$\frac{\partial}{\partial z} \mathfrak{D} W_K^{(3)} + K^2 \tilde{\mathcal{R}}^{(0)} (1 + \theta z)^{m-1} W_K^{(3)} - K^6 W_K^{(3)}$$

$$= -K^4 \left[2k^6 W^{(0)} \mathfrak{D} W^{(0)} - \frac{\mathrm{d}}{\mathrm{d}z} \left(\mathfrak{D} W^{(0)} \right)^2 - 2\sigma\theta(1+\theta z)^{m-1} \left(\mathfrak{D} W^{(0)} \right)^2 \right]$$

$$+ K^2 \left[4k^8 \mathfrak{D} W^{(0)^2} - 4k^2 \frac{\mathrm{d}}{\mathrm{d}z} \left(\mathfrak{D} W^{(0)} \right)^2 - 2k^4 \sigma \tilde{\mathcal{R}}^{(0)} (1+\theta z)^{2m-1} W^{(0)} \mathfrak{D} W^{(0)} \right.$$

$$\left. - 8k^2 \sigma\theta(1+\theta z)^{m-1} \left(\frac{1}{4} k^6 W^{(0)^2} + \left(\mathfrak{D} W^{(0)} \right)^2 \right) \right]$$

$$+ \left\{ 2k^4 \frac{\mathrm{d}}{\mathrm{d}z} \left(W^{(0)} \frac{\mathrm{d}}{\mathrm{d}z} \mathfrak{D} W^{(0)} - \left(\mathfrak{D} W^{(0)} \right)^2 \right) + 4k^6 \sigma \tilde{\mathcal{R}}^{(0)} (1+\theta z)^m \mathfrak{D} \left[(1+\theta z)^{m-1} W^{(0)^2} \right] \right.$$

$$\left. + 4k^4 \sigma\theta(1+\theta z)^{m-1} \left(k^6 W^{(0)^2} + \left(\mathfrak{D} W^{(0)} \right)^2 \right) \right\}.$$

(81)

4. Nonlinear solutions for a rectangular planform

In this section, we describe a numerical procedure to determine the nonlinear evolution of the large-scale field and the amplitude of the convective modes for a range of stratification parameters θ.

4.1. *Numerical algorithm*

The numerical procedure adopted is a two-stage process involving both the solution of a two-point boundary eigenvalue system and the subsequent evolution of a one-dimensional initial value problem. We describe below each of these procedures in turn.

The first stage involves the solution of the eigenvalue problem defined by (59a) together with the condition $W^{(0)} = 0$ at $z = 0$, 1 and the normalisation condition $\mathrm{d}W^{(0)}/\mathrm{d}z = 1$ at $z = 0$. This enables the calculation of $\tilde{R}^{(0)}$ and $W^{(0)}$ as a function of k for each θ chosen. This procedure is repeated in order to calculate the critical $\tilde{R}^{(0)}$ and corresponding wavenumber k_c at which it occurs (as well as the corresponding $W^{(0)}$) for each value of θ. Note at this point we also can calculate $\mathcal{F}_1$ given by (B.3). The solutions for $W^{(0)}$ together with the corresponding values of $\tilde{R}^{(0)}$, k_c^2 and $\mathcal{F}_1$ are given in figure 2. Here the weak breaking of symmetry in the velocity can be seen as θ is varied from zero.
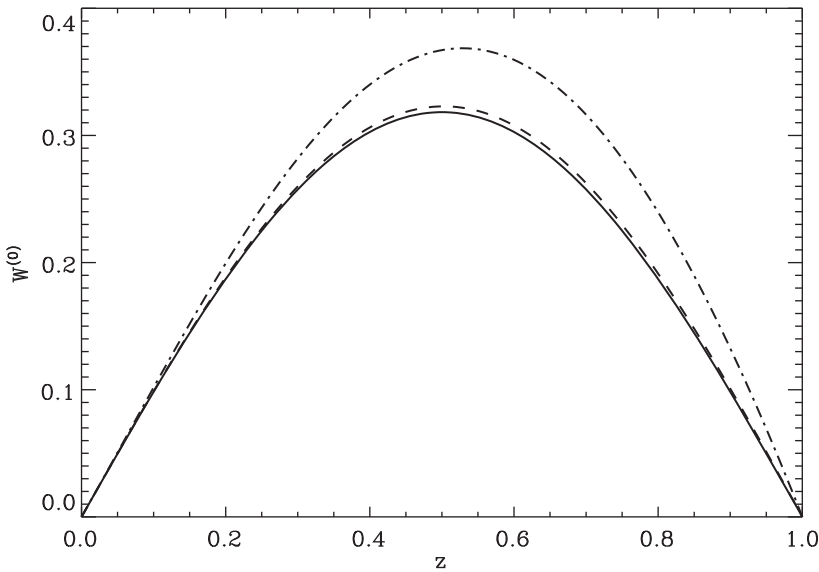
Figure 2. Eigenfunction $W^{(0)}$ as a function of z for $\theta = 0.0$ (—) $\theta = -0.05$ (- - -) and $\theta = -0.5$ ($\cdots$). All solutions are normalised so that $dW^{(0)}/dz = 1$ at $z = 0$. Here and throughout this article $m = 1.495$ and $\sigma = \sigma_\eta = 1$. The corresponding calculated eigenvalues are: $\tilde{R}^{(0)} = 8.695$, $k_c^2 = 1.70$, $\mathcal{F}_1 = 0.294$ for $\theta = 0$; $\tilde{R}^{(0)} = 8.81$, $k_c^2 = 1.7$, $\mathcal{F}_1 = 0.284$ for $\theta = -0.05$ and $\tilde{R}^{(0)} = 10.43$, $k_c^2 = 1.74$, $\mathcal{F}_1 = 0.1985$ for $\theta = -0.5$.

The second stage of the procedure involves the solution of the dynamo system defined by equations (70)–(72a,b). We do this for a range of stratifications θ. In order to make meaningful comparisons for systems of differing stratifications, we choose to compare dynamo solutions with the same kinetic energies in the flow. In order to achieve this we

- Fix the kinetic energy, $\mathcal{T}^{(0)}$, and calculate $\mathcal{A}^{(0)}$ via equation (65) for the $W^{(0)}$ calculated in the first stage.
- Once $\mathcal{A}^{(0)}$ is known it can be substituted into the dynamo equations (70)–(72a,b) together with the selected values of the other input parameters and the calculated value of $\mathcal{F}_1$.
- The dynamo equations are timestepped forward using a fourth-order finite difference spatial discretisation and an Adams–Bashforth timestepping scheme, until an attractor of the system is reached.
- We also note that once $\mathcal{A}^{(0)}$ is known the correction to the Rayleigh number $\tilde{\mathcal{R}}^{(2)}$ defined by condition (64) can also be calculated – though this does not enter into the dynamo equations directly.

The results of this procedure are presented in the following section.

4.2. Numerical results

We begin by reproducing the Boussinesq results of Soward. We therefore choose $\theta = 0$, $\mathcal{T}^{(0)} = 13.46$ and $M^2 = 18.8$ and $\sigma = \sigma_\eta = 1$ and integrate the equations
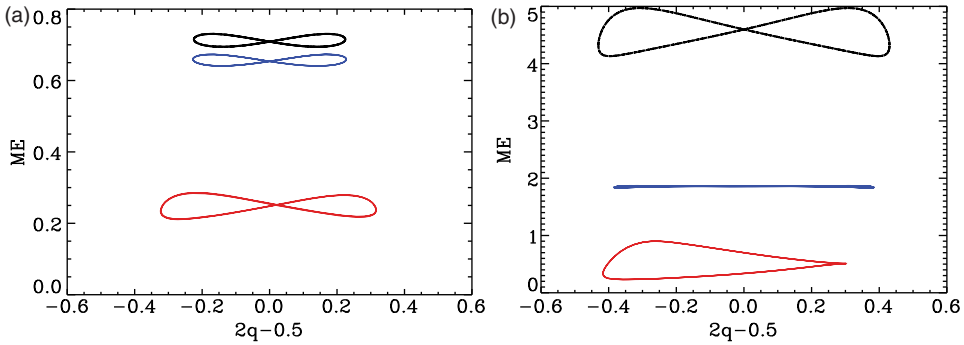
Figure 3. Attractors for nonlinear dynamo solutions. Periodic solutions in the magnetic energy, $2q - 0.5$ plane for $\theta = 0$ (black) $\theta = -0.05$ (blue) and $\theta = -0.5$ (red), with $M^2 = 18.8$ and (a) $\mathcal{T}^{(0)} = 13.46$ and (b) $\mathcal{T}^{(0)} = 15.19$.

forward in time until a simple attractor is reached. This attractor is shown in figure 3(a), where the magnetic energy of the large-scale field is plotted against $2q - 0.5$. The upmost curve corresponds to figure 2a of Soward (1974). As noted by Soward, this solution is subcritical in that the dynamo is operating below the critical magnetic Reynolds number for dynamo action. The components of the magnetic field together with the root mean square field strength, at a fixed time, for these parameters are plotted as a function of z in figure 4(a). This figure clearly shows that the Boussinesq symmetry of the system leads to the generation of a field that is antisymmetric about the mid-plane $z = 0.5$ (with the r.m.s. magnetic field being symmetric).

As θ is decreased, with all other parameters except for k_c and $\mathcal{F}_1$ fixed, the subcritical solution loses magnetic energy in favour of kinetic energy. By $\theta = -0.5$ the magnetic energy is approximately one-third of that for the Boussinesq case, as shown in figure 3. Further decrease of θ leads to the eventual loss of the subcritical solution. The change in stratification also leads to a significant modification of the magnetic field as shown in figures 4(b) and (c). The magnetic field is no longer antisymmetric about the mid-plane with field being preferentially generated near the top of the domain.

Figures 3(b) and 5 give the same results for an increased kinetic energy $\mathcal{T}^{(0)} = 15.19$ (supercritical kinetic energy), and the behaviour is largely similar. Note for the most stratified case, figure 3(b) shows that there has been a symmetry breaking bifurcation and the orbit is not symmetric. This behaviour has been seen in the Boussinesq calculations of Soward (1974) – see figure 3(a) of that paper. We note that the branch with the other symmetry is also stable and can be found by a change in the initial conditions. This symmetry breaking will be investigated in a future paper. Again at fixed kinetic energy, increasing the stratification decreases the dynamo efficiency (as measured by the ratio of the magnetic energy to kinetic energy). Moreover increasing the stratification again focuses the magnetic field in the upper half plane, so that field generation is preferred where the density is lower.
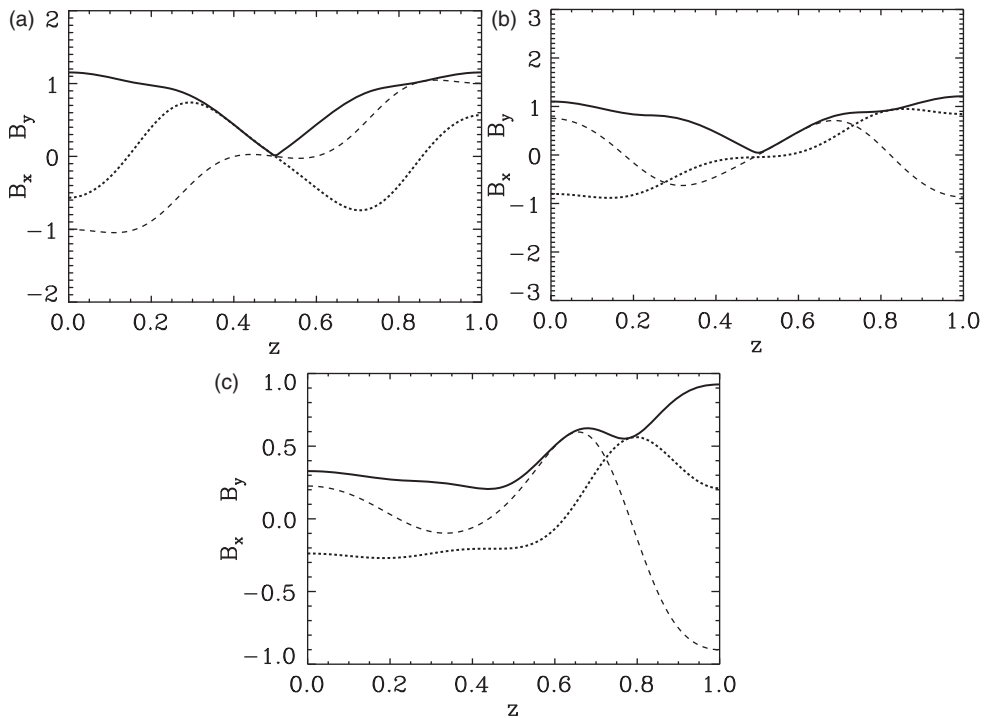
Figure 4. B_x (- - -) B_y ($\cdots$) and $\left(B_x^2 + B_y^2\right)^{1/2}$ (—) as a function of z for (a) $\theta = 0.0$ (b) $\theta = -0.05$ and (c) $\theta = -0.5$, with $\mathcal{T}^{(0)} = 13.46$.

5. Concluding remarks

In this article, we have studied the influence of compressibility on dynamo action in a rapidly rotating plane layer. We have utilised the anelastic formulation with the thermal energy flux expressed in terms of an entropy gradient (cf. Braginsky and Roberts 1995, Jones *et al.* 2009, Mizerski and Tobias 2011) to perform an asymptotic analysis in the limit of rapid background rotation; we use a judicious combination of analytic and numerical techniques to make the problem tractable.

The main assumptions in the paper (in addition to rapid rotation) is that the fluid is in a parameter regime near to the onset of convection. Utilising this, we were able to study the effect of compressibility on the Boussinesq dynamo solution found by Childress and Soward (1972) and Soward (1974).

Our main result is that in general the presence of stratification is detrimental to large-scale dynamo action. This is the case for a number of calculations of differing complexity. Specifically we find that for compressible convective flows typically a larger kinetic energy is necessary at onset to maintain a large-scale magnetic field than that which would be necessary to maintain an analogous magnetic field by an incompressible flow. Furthermore, an asymptotic analysis for small stratifications ($\tau^{-1/12} \ll |\theta| \ll 1$) determined that, for regular planforms in the kinematic regime, a
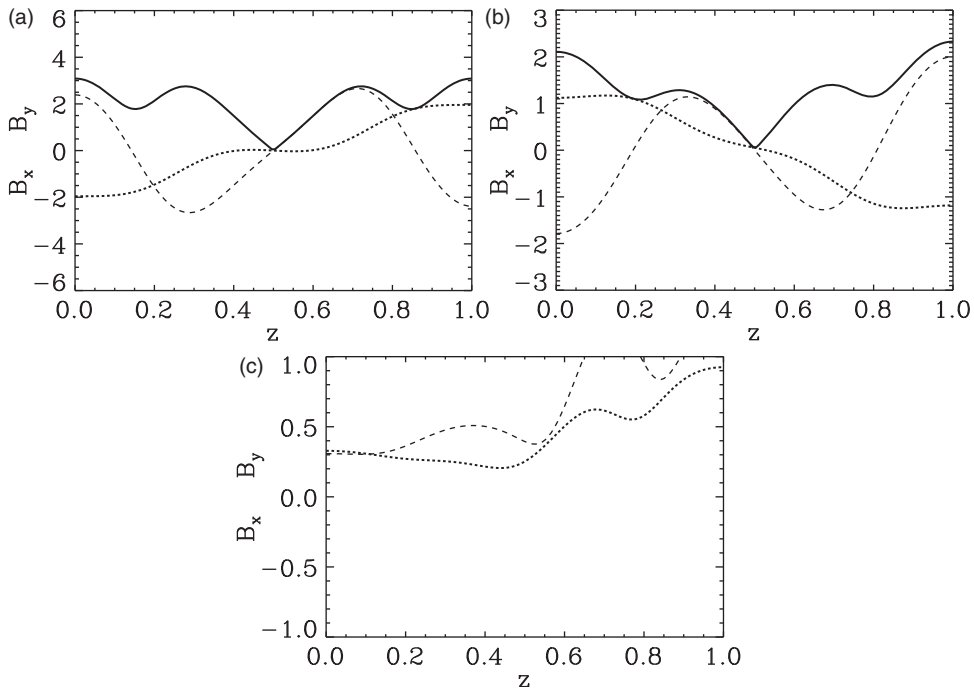
Figure 5. As in figure 4, but $\mathcal{T}^{(0)} = 15.19$.

stationary dynamo solution turned out to have the same average kinetic energy as an analogous Boussinesq solution up to order θ^2. This means that weak compressibility does not influence the critical magnetic Reynolds number at the first order and significant change can only be seen for stronger compressibilities. Finally, when the back-reaction of the magnetic field is included, the dynamo solutions were calculated numerically for arbitrary stratifications. We found that stratification inhibited large-scale dynamo action in the sense that a weaker magnetic field was generated by compressible flows with the same kinetic energy as their Boussinesq counterparts.

We note that several other authors have argued (cf. Glatzmaier and Gilman 1981, Jones *et al.* 2009, Jones and Kuzanyan 2009, Mizerski and Tobias 2011) that the presence of compressibility reduces the length scales of convective cells, by confining convection to the upper region of small density. They argue that the introduction of small-scale variations also affects the mean magnetic field (cf. figure 2(a)) and tends to enhance the dissipative effects and thus it is expected that more kinetic energy is needed to sustain a large-scale magnetic field if the flow is compressible. It follows that for a compressible medium larger magnetic Reynolds number then for an incompressible one should be necessary for the system to maintain a large-scale magnetic field. Our results are consistent with this picture though it is not clear whether the mechanism leading to the relative inefficiency of dynamo action is the same in our highly specific limit of rapid rotation and near-marginality.

Acknowledgements

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Appendix A: The formulae for electric current

The density of the electric current for section 3 at leading order is given by the following formulae:

$$j_x^{(0)} = \varepsilon^{-1/2}\partial_y B_z'^{(0)} = -\sum_{\boldsymbol{k}} \frac{k_y}{k^2}\left(k_x B_x^{(0)} + k_y B_y^{(0)}\right)\hat{w}^{(0)}(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}W^{(0)}(z), \tag{A.1a}$$

$$j_y^{(0)} = -\varepsilon^{-1/2}\partial_x B_z'^{(0)} = \sum_{\boldsymbol{k}} \frac{k_x}{k^2}\left(k_x B_x^{(0)} + k_y B_y^{(0)}\right)\hat{w}^{(0)}(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}W^{(0)}(z), \tag{A.1b}$$

$$j_z^{(0)} = \varepsilon^{-1/2}\left(\partial_x B_y'^{(0)} - \partial_y B_x'^{(0)}\right) = \sum_{\boldsymbol{k}} \frac{i}{k^4}\left(k_x B_x^{(0)} + k_y B_y^{(0)}\right)\hat{w}^{(0)}(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathfrak{D}W^{(0)}. \tag{A.1c}$$

Appendix B: The evolution equations for general planform

From the solvability condition for the equations obtained at order ε^3 one obtains the following equation governing the evolution of the amplitudes $\hat{w}^{(0)}(\boldsymbol{k},t)$:

$$\frac{\mathcal{F}_1\left(k,\tilde{\mathcal{R}}^{(0)},\sigma,m,\theta\right)}{\sigma_m}\frac{d\hat{w}^{(0)}}{dt} = \mathcal{M}\left(\boldsymbol{B}_h^{(0)},\boldsymbol{k},m,\theta\right)\hat{w}^{(0)} - \hat{w}^{(0)}k^2\tilde{\mathcal{R}}^{(0)}\int_0^1 (1+\theta z)^m W^{(0)2}\frac{\partial S^{(2)}}{\partial z}dz$$
$$+ \mathcal{F}_2\left(k,\tilde{\mathcal{R}}^{(0)},\tilde{\mathcal{R}}^{(2)},\tilde{\mathcal{R}}^{(4)},\sigma_\eta,m,\theta\right)\hat{w}^{(0)}$$
$$+ \frac{\hat{w}^{(0)}}{2k^4\sigma_m^2}\sum_{\boldsymbol{k}'}(\boldsymbol{k}\times\boldsymbol{k}')\cdot\hat{\boldsymbol{e}}_z\left|\hat{w}^{(0)}(\boldsymbol{k}')\right|^2\Gamma(K,k,\sigma,m,\theta), \tag{B.1}$$

where

$$\mathcal{M}\left(\boldsymbol{B}_h^{(0)},\boldsymbol{k},m,\theta\right) = \frac{M^2}{k^4}\int_0^1\left(\mathfrak{D}W^{(0)}\frac{dW^{(0)}}{dz} - k^6 W^{(0)2}\right)\frac{B_x^{(0)2}k_x^2 + B_y^{(0)2}k_y^2 + 2B_x^{(0)}B_y^{(0)}k_x k_y}{(1+\theta z)^m}dz, \tag{B.2}$$

$$\mathcal{F}_1\left(k,\tilde{\mathcal{R}}^{(0)},\sigma,m,\theta\right) = \int_0^1\left\{2k^4 + \tilde{\mathcal{R}}^0(1+\theta z)^{m-1}[\sigma(1+\theta z)^m - 1]\right\}W^{(0)2}dz, \tag{B.3}$$

$$\mathcal{F}_2\big(k, \tilde{\mathcal{R}}^{(0)}, \tilde{\mathcal{R}}^{(2)}, \tilde{\mathcal{R}}^{(4)}, \sigma_\eta, m, \theta\big)$$

$$= \int_0^1 \bigg\{ k^2(1+\theta z)^{m-1} W^{(0)}\big(\tilde{\mathcal{R}}^{(2)} W^{(2)} + \tilde{\mathcal{R}}^{(4)} W^{(0)}\big)$$

$$- k^2(1+\theta z)^m W^{(0)} \frac{\partial S^{(0)}}{\partial z} \big(\tilde{\mathcal{R}}^{(2)} W^{(0)} + \tilde{\mathcal{R}}^{(0)} W^{(2)}\big)$$

$$- \tilde{\mathcal{R}}^{(0)}\bigg(\frac{\mathrm{d}W^{(0)}}{\mathrm{d}z} - \frac{\theta}{1+\theta z} W^{(0)}\bigg) \frac{\mathrm{d}}{\mathrm{d}z}\big[(1+\theta z)^{m-1} W^{(0)}\big] + \frac{1}{k^2} W^{(0)}\bigg(\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}\bigg)^2 W^{(0)}$$

$$+ k^4 W^{(0)}\bigg[2\frac{\mathrm{d}^2 W^{(0)}}{\mathrm{d}z^2} + \frac{m\theta}{1+\theta z}\bigg(\frac{4}{3}\frac{\mathrm{d}W^{(0)}}{\mathrm{d}z} + \frac{1+2m}{3}\theta\frac{W^{(0)}}{1+\theta z}\bigg)\bigg]\bigg\}\mathrm{d}z, \tag{B.4}$$

$$\Gamma(K, k, \sigma, m, \theta) = \frac{2k^4}{K^2}\int_0^1 W^{(0)}\bigg(\frac{K^2-k^2}{k^8}\mathfrak{D}W^{(0)} - W^{(0)}\bigg)\bigg[(\mathfrak{D}W^{(0)})^2 - W^{(0)}\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(0)}\bigg]\mathrm{d}z$$

$$+ \int_0^1 W^{(0)}\bigg[\mathfrak{D}W_K^{(3)}\frac{1}{K^2}\bigg(\frac{K^2-k^2}{k^8}\mathfrak{D}W^{(0)} - W^{(0)}\bigg) + \frac{K^2}{k^4} W_K^{(3)}\mathfrak{D}W^{(0)}\bigg]\mathrm{d}z$$

$$+ 2\sigma\int_0^1 (1+\theta z)^m W^{(0)}\mathfrak{D}W^{(0)}\bigg[2k^2\mathfrak{D}W^{(0)^2} - K^2 W^{(0)}\mathfrak{D}W^{(0)}$$

$$- \frac{4k^2-K^2}{2k^6}\frac{\mathrm{d}}{\mathrm{d}z}(\mathfrak{D}W^{(0)})^2\bigg]\mathrm{d}z$$

$$- \frac{2\sigma}{k^2 K^2}\int_0^1 (1+\theta z)^m\bigg[(\mathfrak{D}W^{(0)})^2 - W^{(0)}\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(0)}\bigg]$$

$$\times\bigg[k^6 W^{(0)2} - (\mathfrak{D}W^{(0)})^2 - 2W^{(0)}\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(0)}\bigg]\mathrm{d}z$$

$$- \frac{\sigma}{k^6 K^2}\int_0^1 (1+\theta z)^m W^{(0)}\bigg[\mathfrak{D}W^{(0)}\bigg(\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W_K^{(3)} - K^6 W_K^{(3)}\bigg)$$

$$- \mathfrak{D}W_K^{(3)}\bigg(\frac{\mathrm{d}}{\mathrm{d}z}\mathfrak{D}W^{(0)} - k^6 W^{(0)}\bigg)\bigg]\mathrm{d}z \tag{B.5}$$

and

$$\boldsymbol{K} = \boldsymbol{k} - \boldsymbol{k}', \tag{B.6}$$

Multiplying (B.1) by $(\hat{w}^{(0)})^*$ and summing over $\boldsymbol{k}$ and taking advantage of the fact that $\mathcal{A}^{(0)}$ is constant in time we obtain

$$k^2\tilde{\mathcal{R}}^{(0)}\int_0^1 (1+\theta z)^m W^{(0)2}\frac{\partial S^{(2)}}{\partial z}\mathrm{d}z = \sum_{\boldsymbol{k}}\frac{\mathcal{M}\big(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}, m, \theta\big) + \mathcal{F}_2}{\mathcal{A}^{(0)}}\big|\hat{w}^{(0)}(\boldsymbol{k})\big|^2, \tag{B.7}$$

which leads to the following amplitude equation:

$$\frac{\mathcal{F}_1\big(k, \tilde{\mathcal{R}}^{(0)}, \sigma, m, \theta\big)}{\sigma_m}\frac{\mathrm{d}\hat{w}^{(0)}}{\mathrm{d}t} = \mathcal{M}\big(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}, m, \theta\big)\hat{w}^{(0)} - \hat{w}^{(0)}\sum_{\boldsymbol{k}'}\frac{\mathcal{M}\big(\boldsymbol{B}_h^{(0)}, \boldsymbol{k}', m, \theta\big)}{\mathcal{A}^{(0)}}\big|\hat{w}^{(0)}(\boldsymbol{k}')\big|^2$$

$$+ \frac{\hat{w}^{(0)}}{2k^4\sigma_m^2}\sum_{\boldsymbol{k}'}(\boldsymbol{k}\times\boldsymbol{k}')\cdot\hat{\boldsymbol{e}}_z\big|\hat{w}^{(0)}(\boldsymbol{k}')\big|^2\Gamma(K, k, \sigma, m, \theta), \tag{B.8}$$

coupled with the mean magnetic field equation

$$\frac{\partial B_i}{\partial t} = -2\frac{\partial}{\partial z}\left[W^{(0)}\mathfrak{D}\,W^{(0)}\mathbb{A}_{ij}B_j\right] + \frac{\partial^2 B_i}{\partial z^2} = 0, \tag{B.9}$$

where

$$\mathbb{A} = \begin{bmatrix} -a_{21} & -a_{22} \\ a_{11} & a_{12} \end{bmatrix}, \quad a_{ij} = \sum_{|k|=k}\frac{k_i k_j}{k^6}\left|\hat{w}^{(0)}(t,\boldsymbol{k})\right|^2. \tag{B.10a,b}$$