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The effect of stratification and compressibility on anelastic convection in a rotating plane layer

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We study the effect of stratification and compressibility on the threshold of convection and the heat transfer by developed convection in the nonlinear regime in the presence of strong background rotation. We consider fluids both with constant thermal conductivity and constant thermal diffusivity. The fluid is confined between two horizontal planes with both boundaries being impermeable and stress-free. An asymptotic analysis is performed in the limits of weak compressibility of the medium and rapid rotation ($\tau^{-1/12} \ll |\theta| \ll 1$, where τ is the Taylor number and θ is the dimensionless temperature jump across the fluid layer). We find that the properties of compressible convection differ significantly in the two cases considered. Analytically, the correction to the characteristic Rayleigh number resulting from small compressibility of the medium is positive in the case of constant thermal conductivity of the fluid and negative for constant thermal diffusivity. These results are compared with numerical solutions for arbitrary stratification. Furthermore, by generalizing the nonlinear theory of Julien and Knobloch [Fully nonlinear three-dimensional convection in a rapidly rotating layer. *Phys. Fluids* 1999, **11**, 1469–1483] to include the effects of compressibility, we study the Nusselt number in both cases. In the weakly nonlinear regime we report an increase of efficiency of the heat transfer with the compressibility for fluids with constant thermal diffusivity, whereas if the conductivity is constant, the heat transfer by a compressible medium is more efficient than in the Boussinesq case only if the specific heat ratio γ is larger than two.

Keywords: Anelastic convection; Heat transfer; Compressibility

1. Introduction

An understanding of convection, is the key to determining the dynamics of the fluid planetary cores and stellar interiors. It is believed that flow in the interiors of many astrophysical bodies is convectively driven and this flow is important not only for transporting energy, but also angular momentum. Moreover, if the medium is electrically conducting the convectively driven flow may also generate planetary and stellar magnetic fields (see, e.g. Soward 1991, Tobias and Weiss 2007). This is the reason why the problem of convection has received so much attention in the astrophysical and geophysical community over the last few decades. The Boussinesq approximation, for thin layers of fluid in which variations in density are only included in the

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buoyancy force, has been extensively studied (e.g. Roberts 1968, Busse 1970, Matthews 1999, Jones *et al*. 2000, Dormy *et al*. 2004; see also Busse 1994, Busse 2002). In most astrophysical applications, however, the thin layer approximation is not enough, since the depth of the convective layer is comparable with (or larger than) a typical scale height of the system. This led to the formulation of the *anelastic approximation* by Ogura and Phillips (1962) and Gough (1969). The properties of convection under this approximation have been investigated in many subsequent papers, which we describe below.

In a series of papers Gilman and Glatzmaier (1981) and Glatzmaier and Gilman (1981a, b) studied the linear problem and the effects of dissipation, boundary conditions and zone depth in fully nonlinear compressible convection in spherical shell under the anelastic approximation. One of their important discoveries was that the depth-dependence of the thermal conductivity strongly influences the localization of the region where the convective instability first sets in and in the case when thermal diffusivity is assumed constant, anelastic convection develops small structures close to the outer boundary. The onset of compressible convection in spherical geometry with constant kinematic or dynamic viscosity was also investigated by Drew *et al*. (1995) and an asymptotic linear theory under the anelastic approximation was developed for rapidly rotating spherical shells by Jones *et al*. (2009). A very detailed derivation and comprehensive discussion of the compressible convection equations in the geophysical context was presented by Braginsky and Roberts (1995). Later Bannon (1996) investigated the applicability of the anelastic approximation to the dynamics of a compressible atmosphere and provided a survey of different approaches to stratified convection. A comparison of the Boussinesq and anelastic approximations can be found in Lilly (1996) and in the context of Earth's core dynamics in Anufriev *et al*. (2005). The anelastic formulation was also used by Julien *et al*. (1999) who derived a reduced set of equations governing the evolution of stratified convection. Finally, Lantz and Fan (1999) generalized this approximation to the magnetohydrodynamic case and (Julien *et al*. 2000, 2003) exploited the MHD formulation of anelastic equations to study the nonlinear effects in compressible convection in the presence of strong external field oblique to the boundaries.

Our aim is to study the influence of compressibility, under the anelastic approximation, on the convection threshold and the total heat transfer by fully developed rotating convection. We consider the simplest formalism of the anelastic approximation; for details of other formalisms and how they relate see Berkoff *et al*. (2010). We consider both the linear and nonlinear problems analytically and numerically. Bassom and Zhang (1994) by analytical methods determined the strongly nonlinear properties of convection cells in rapidly rotating systems. Julien and Knobloch (1999) with the use of similar techniques studied the nonlinear development of rapidly rotating Boussinesq convection and derived an equation governing the nonlinear evolution of modes with square planform. We generalize this equation to include the effects of stratification and compressibility in the anelastic formulation. Two cases are considered for the thermal properties of the system, i.e. first with the thermal conductivity assumed constant and second with constant thermal diffusivity.

The structure of this article is as follows. First, we give the mathematical formulation of the mathematical basis of the model including the equations (under the anelastic approximation) and the boundary conditions. Then, we study the influence of compressibility on the threshold of convection in section 3. In section 4, we generalize the Julien and Knobloch's (1999) nonlinear equations for rapidly rotating Boussinesq convection to a compressible case and investigate the influence of compressibility on the heat transfer in a convective system. We end this article with some concluding remarks in section 5.

2. Mathematical formulation

We analyse thermal convection in an infinite plane layer of fluid with gravity **g** pointing downwards. The top and bottom boundaries are flat, and are situated at $z=0,\ d$. The equations describing the evolution of the system is the set of hydrodynamic equations under the anelastic approximation (cf. Gough 1969, Lantz and Fan 1999).

The formalism is such that the basic state is assumed to be almost adiabatic, i.e. its departure from adiabaticity

$$\epsilon = -\frac{d}{T_r}\left[\left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}z}\right)_r + \frac{g}{c_p}\right] = -\frac{d}{c_p}\left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}z}\right)_r \ll 1 \tag{1}$$

is assumed to be small. Here the subscript "r" denotes a reference value (i.e. taken at the bottom wall). The equations (2)–(5) are derived by expanding the pressure, density, temperature and the entropy in the parameter ϵ, i.e $\bar{p}+\epsilon p$, $\bar{\rho}+\epsilon\rho$, $\bar{T}+\epsilon T$, $\epsilon c_p(\bar{s}+s)+\text{const.}$, assuming additionally that velocity scales like $\epsilon^{1/2}$ and time as $\epsilon^{-1/2}$ (the origin of this scaling comes from an estimate obtained by assuming a balance between the work done by the buoyancy force on a rising bubble over a characteristic length and its kinetic energy gain. In addition $Re \gtrsim 1$ and $Rm \gtrsim 1$ must hold, where Re and Rm are the Reynolds and magnetic Reynolds numbers, respectively). Then, taking the leading order form of the Navier-Stokes, mass conservation and energy equations yields

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\left(\frac{p}{\bar{\rho}}\right) + \mathcal{R}\sigma s\hat{\mathbf{e}}_z - \tau^{1/2}\sigma\hat{\mathbf{e}}_z\times\mathbf{u} + \sigma\frac{1}{\bar{\rho}}\nabla\cdot\varsigma, \tag{2}$$

$$\nabla\cdot(\bar{\varrho}\mathbf{u}) = 0, \tag{3}$$

$$\bar{\rho}\bar{T}\left[\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla(\bar{s}+s)\right] = \nabla\cdot\left[\lambda(\bar{\rho})\bar{T}\nabla(\bar{s}+s)\right] - \frac{\theta}{2\mathcal{R}}\frac{1}{\bar{\rho}}\varsigma:\varsigma, \tag{4}$$

$$\frac{p}{\bar{p}} = \frac{T}{\bar{T}} + \frac{\rho}{\bar{\rho}}, \quad s = \frac{1}{\gamma}\frac{p}{\bar{p}} - \frac{\rho}{\bar{\rho}}, \tag{5}$$

where we choose d^2/κ_r, d and κ_r/d as units of time, length and velocity respectively, with κ_r being the reference value (i.e. taken at $z=0$) of the thermal diffusivity. The ratio of specific heats is denoted by $\gamma=c_p/c_v$, $\varsigma_{ij}=\bar{\rho}\left[\partial_j u_i + \partial_i u_j - (2/3)(\nabla\cdot\mathbf{u})\delta_{ij}\right]$ is the symmetric stress tensor and in dyadic notation $\varsigma:\varsigma=\varsigma_{ij}\varsigma_{ij}$. The dimensionless parameters in the above equations are the Prandtl number $\sigma=\nu/\kappa_r$, the Rayleigh number $\mathcal{R}=gd^3\epsilon/\kappa_r\nu$ and the Taylor number $\tau=4\Omega^2 d^4/\nu^2$, where ν and Ω are the kinematic viscosity (assumed constant) and the rotation rate, respectively.

Furthermore, we have assumed here that the conductive heat flux due to molecular diffusion is an order of magnitude in ϵ smaller than the turbulent heat flux due to unresolved small scale turbulence expressed in terms of entropy as $k_T T\nabla s$

(cf. Braginsky and Roberts 1995). Hence only this turbulent heat flux is included in the entropy equation (4). We will consider two cases: in the first case (Case 1), the turbulent thermal conductivity $k_T = c_p \bar{\rho} \kappa_T$ is constant, i.e. $\lambda(\bar{\rho}) = 1$, whilst in the second case (Case 2), the turbulent thermal diffusivity κ_T is constant and then $\lambda(\bar{\rho}) = \bar{\rho}$.

We consider a basic state, for which the temperature at the bottom of the layer is T_0 and the temperature jump across the layer is $\triangle T < 0$:

Case 1

$$\bar{T} = 1 + \theta z, \tag{6a}$$

$$\bar{\rho} = (1 + \theta z)^m, \tag{6b}$$

$$\bar{p} = -\frac{\mathcal{R}\sigma_m}{\sigma_\eta \theta (m+1)} (1 + \theta z)^{m+1}, \tag{6c}$$

$$\bar{s} = \frac{m + 1 - \gamma m}{\gamma \epsilon} \ln(1 + \theta z) + \text{const.} \qquad \text{with} \quad \frac{m + 1 - \gamma m}{\gamma} = -\frac{\epsilon}{\theta} = \mathrm{O}(\epsilon), \tag{6d}$$

where $\theta = \triangle T / T_0$ $(-1 < \theta \leq 0)$.

Case 2

$$\bar{T} = 1 + \theta z + \epsilon \frac{\gamma - 1}{\theta(\gamma - 2)} \left[1 - (1 + \theta z)^{(\gamma-2)/(\gamma-1)}\right], \tag{7a}$$

$$\bar{\rho} = (1 + \theta z)^{1/(\gamma-1)} + \epsilon \frac{1}{\theta(\gamma - 2)} \left[(1 + \theta z)^{(2-\gamma)/(\gamma-1)} - (\gamma - 1)^2\right], \tag{7b}$$

$$\bar{p} = -\frac{\mathcal{R}\sigma_m(\gamma - 1)}{\sigma_\eta \theta \gamma} \left\{(1 + \theta z)^{\gamma/(\gamma-1)} + \epsilon \frac{\gamma(1 + \theta z)}{\theta(\gamma - 2)} \left[(1 + \theta z)^{(2-\gamma)/(\gamma-1)} - (\gamma - 1)\right]\right\}, \tag{7c}$$

$$\bar{s} = \frac{\gamma - 1}{\theta} (1 + \theta z)^{1/(1-\gamma)} + \text{const.}, \tag{7d}$$

where $\theta = -gd/c_p T_0 = \triangle T / T_0 + \mathrm{O}(\epsilon)$ $(-1 < \theta \leq 0)$.

Since the analysis of both Cases 1 and 2 is very similar, we will only present the analysis for Case 1 in detail, and only the results for Case 2 will be given. Upon introducing the form of the Case 1 basic state into the equations (2)–(4), we obtain

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\left(\frac{p}{\bar{\varrho}}\right) + \mathcal{R}\sigma s \hat{\mathbf{e}}_z - \tau^{1/2} \sigma \hat{\mathbf{e}}_z \times \mathbf{u} + \sigma \nabla^2 \mathbf{u} \\ + \frac{\sigma m \theta}{1 + \theta z} \left[\frac{\partial \mathbf{u}}{\partial z} + \frac{2}{3} \nabla u_z + \frac{1}{3}(1 + 2m)\theta \frac{u_z}{1 + \theta z} \hat{\mathbf{e}}_z\right], \end{aligned} \tag{8}$$

$$\nabla \cdot \mathbf{u} = -\frac{m\theta}{1 + \theta z} u_z, \tag{9}$$

$$\begin{aligned} \frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = \frac{u_z}{1 + \theta z} + \frac{1}{\bar{\varrho}} \nabla^2 s + \frac{\theta}{(1 + \theta z)^{m+1}} \frac{\partial s}{\partial z} \\ - \frac{\theta}{\mathcal{R}} \frac{1}{(1 + \theta z)} \left[2 \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial x_i}\right)^2 + \sum_{i<j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{2}{3}(\nabla \cdot \mathbf{u})^2\right], \end{aligned} \tag{10}$$

where, again, the kinematic viscosity ν is assumed constant. The momentum and mass-conservation equations for Case 2 are obtained by simple exchange $m \to 1/(\gamma-1)$. and the entropy equation is

$$\frac{\partial s}{\partial t}+\mathbf{u}\cdot\nabla s = u_z(1+\theta z)^{\gamma/(1-\gamma)}+\nabla^2 s+\frac{\theta\gamma}{(\gamma-1)(1+\theta z)}\frac{\partial s}{\partial z} -\frac{\theta}{\mathcal{R}}\frac{1}{(1+\theta z)}\left[2\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2+\sum_{i<j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2-\frac{2}{3}(\nabla\cdot\mathbf{u})^2\right]. \qquad (11)$$

The parameter θ, which measures the temperature jump across the fluid layer is the measure of stratification and compressibility of the system. Indeed, from (9) we get $\nabla\cdot\mathbf{u}\sim\theta u_z$, so that if $|\theta|\ll 1$ the fluid is weakly compressible. In the following section we will study the influence of the compressibility of the medium, defined in the above way, on the onset of convective instability, in a plane, infinite layer both analytically for weak compressibility and strong rotation and numerically for arbitrary compressibility and rotation. However, it is crucial to understand that since we have already incorporated the effects of small scale turbulence into the expression for conductive heat flux and thus into all the diffusivity coefficients, the threshold of convection in our analysis is approached from a developed convection state by decreasing the Rayleigh number to a value close to critical but still above it, so that the unresolved small scale turbulence is still present (cf. Jones *et al.* 2009). We are simply interested in the linear theory as the small amplitude limit of the fully nonlinear problem, which is typically dominated by turbulence. Of course, for very small amplitudes, at very small supercriticality, the turbulence would not be present and then the molecular, not turbulent diffusion, would be relevant. However, physical intuition suggests that even a very little amount of turbulence would dominate the heat transport and thus the turbulent diffusion would swamp the molecular diffusion. Therefore it is expected that in compressible convection, i.e. for $|\theta|\gg\epsilon$, the range of Rayleigh numbers where the flow is laminar and the molecular diffusion is significant, is most likely small. Since we are interested in relating our results to fully developed, nonlinear convection, the inclusion of turbulent effects seems appropriate, however, to distinguish between the actual critical value of the Rayleigh number (and the wavenumber) for laminar convection and the threshold value for large scale convection throughout this article we call the latter a characteristic (as opposed to critical) Rayleigh number (and characteristic wavenumber).

3. The onset of weakly compressible convection in a plane layer

3.1. *Analytical results*

We first present an asymptotic expansion for small θ and rapid rotation. We linearize the equations and assume the following form of the perturbations,

$$\mathbf{u}=\hat{\mathbf{u}}(z)\mathrm{e}^{\mathrm{i}(k_1x+k_2y)}\mathrm{e}^{\mu t},\quad \xi=\hat{\xi}(z)\mathrm{e}^{\mathrm{i}(k_1x+k_2y)}\mathrm{e}^{\mu t},\quad s=\hat{s}(z)\mathrm{e}^{\mathrm{i}(k_1x+k_2y)}\mathrm{e}^{\mu t}, \qquad (12)$$

and

$$\hat{\mathbf{u}}=\hat{\mathbf{u}}^0+\theta\hat{\mathbf{u}}^1+\mathrm{O}(\theta^2),\quad \hat{\xi}=\hat{\xi}^0+\theta\hat{\xi}^1+\mathrm{O}(\theta^2),\quad \hat{s}=\hat{s}^0+\theta\hat{s}^1+\mathrm{O}(\theta^2), \qquad (13)$$

where $\xi = \partial_x u_y - \partial_y u_x$ is the z-component of the vorticity and the z-component of the velocity field $\mathbf{u}$ will be denoted by w. Taking the z-components of the curl and the double curl of equation (8) and linearizing, we obtain at the lowest two orders: Order θ^0:

$$\mu\hat{\xi}^0 = \tau^{1/2}\sigma\frac{\mathrm{d}\hat{w}^0}{\mathrm{d}z} - \sigma k^2\hat{\xi}^0 + \sigma\frac{\mathrm{d}^2\hat{\xi}^0}{\mathrm{d}z^2}, \tag{14a}$$

$$\mu\left(k^2\hat{w}^0 - \frac{\mathrm{d}^2\hat{w}^0}{\mathrm{d}z^2}\right) = \mathcal{R}^0\sigma k^2\hat{s}^0 + \tau^{1/2}\sigma\frac{\mathrm{d}\hat{\xi}^0}{\mathrm{d}z} - \sigma\left(k^4\hat{w}^0 - 2k^2\frac{\mathrm{d}^2\hat{w}^0}{\mathrm{d}z^2} + \frac{\mathrm{d}^4\hat{w}^0}{\mathrm{d}z^4}\right), \tag{14b}$$

$$\mu\hat{s}^0 - \hat{w}^0 = \frac{\mathrm{d}^2\hat{s}^0}{\mathrm{d}z^2} - k^2\hat{s}^0. \tag{14c}$$

Order θ^1:

$$\mu\hat{\xi}^1 - \tau^{1/2}\sigma\frac{\mathrm{d}\hat{w}^1}{\mathrm{d}z} + \sigma k^2\hat{\xi}^1 - \sigma\frac{\mathrm{d}^2\hat{\xi}^1}{\mathrm{d}z^2} = \tau^{1/2}\sigma m\hat{w}^0 + \sigma m\frac{\mathrm{d}\hat{\xi}^0}{\mathrm{d}z}, \tag{15a}$$

$$\begin{aligned} &\mu\left(k^2\hat{w}^1 - \frac{\mathrm{d}^2\hat{w}^1}{\mathrm{d}z^2}\right) - \mathcal{R}^0\sigma k^2\hat{s}^1 - \tau^{1/2}\sigma\frac{\mathrm{d}\hat{\xi}^1}{\mathrm{d}z} + \sigma\left(k^4\hat{w}^1 - 2k^2\frac{\mathrm{d}^2\hat{w}^1}{\mathrm{d}z^2} + \frac{\mathrm{d}^4\hat{w}^1}{\mathrm{d}z^4}\right) \\ &= m\left(\mu + 2\sigma k^2\right)\frac{\mathrm{d}\hat{w}^0}{\mathrm{d}z} - 2m\sigma\frac{\mathrm{d}^3\hat{w}^0}{\mathrm{d}z^3} + \mathcal{R}^1\sigma k^2\hat{s}^0, \end{aligned} \tag{15b}$$

$$\mu\hat{s}^1 - \hat{w}^1 - \frac{\mathrm{d}^2\hat{s}^1}{\mathrm{d}z^2} + k^2\hat{s}^1 = -z\left(\hat{w}^0 + m\frac{\mathrm{d}^2\hat{s}^0}{\mathrm{d}z^2} - mk^2\hat{s}^0\right) + \frac{\mathrm{d}\hat{s}^0}{\mathrm{d}z}. \tag{15c}$$

The boundaries are assumed to be impermeable and stress-free. Condition of isentropy is imposed at $z = 0$, 1, i.e.

$$\hat{w}|_{z=0,1} = 0, \quad \hat{s}|_{z=0,1} = 0, \quad \left.\frac{\mathrm{d}\hat{\xi}}{\mathrm{d}z}\right|_{z=0,1} = 0. \tag{16}$$

The leading order (in θ) solution at the instability threshold, i.e. at $\mu = 0$ is, therefore (cf. Chandrasekhar 1961)

$$\hat{w}^0 = A\sin(n\pi z), \quad \hat{s}^0 = \frac{A}{k^2 + n^2\pi^2}\sin(n\pi z), \quad \hat{\xi}^0 = \tau^{1/2}\frac{An\pi}{k^2 + n^2\pi^2}\cos(n\pi z), \tag{17}$$

$$\mathcal{R}^0 = \frac{1}{k^2}\left[\left(k^2 + n^2\pi^2\right)^3 + n^2\pi^2\tau\right], \tag{18}$$

where A is an arbitrary constant and in general $n \in \mathbb{N}$, however, at the instability threshold $n = 1$. To facilitate the analysis, we assume that the system is rapidly rotating, thus $\tau \gg 1$ (keeping $\sigma_m = \mathrm{O}(1)$ and $\sigma_\eta = \mathrm{O}(1)$), which leads to the following scalings (Chandrasekhar 1961)

$$\mathcal{R}^0 = \tau^{2/3}\tilde{\mathcal{R}}^0, \quad k = \tau^{1/6}\tilde{k} \quad \text{and} \quad \tilde{\mathcal{R}}^0 = \frac{1}{\tilde{k}^2}\left[\tilde{k}^6 + n^2\pi^2\right]. \tag{19}$$

This yields the following set of equations for $\hat{w}^1$, $\hat{\xi}^1$ and $\hat{s}^1$ to leading order in τ and order θ (cf (15a)):

$$-\tau^{1/2}\frac{\mathrm{d}\hat{w}^1}{\mathrm{d}z}+\tau^{1/3}\tilde{k}^2\hat{\xi}^1=\tau^{1/2}m\hat{w}^0, \tag{20}$$

$$-\tau\mathcal{R}^0\tilde{k}^2\hat{s}^1-\tau^{1/2}\frac{\mathrm{d}\hat{\xi}^1}{\mathrm{d}z}+\tau^{2/3}\tilde{k}^4\hat{w}^1=\mathcal{R}^1\hat{w}^0, \tag{21}$$

$$-\hat{w}^1+\tau^{1/3}\tilde{k}^2\hat{s}^1=(m-1)z\hat{w}^0, \tag{22}$$

(here we have assumed, in addition, that $|\theta|\gg\tau^{-1/12}$) from which we obtain

$$\hat{s}^1=\frac{\tau^{-1/3}}{\tilde{k}^2}\left[(m-1)z\hat{w}^0+\hat{w}^1\right], \tag{23}$$

$$\pi^2\hat{w}^1+\frac{\mathrm{d}^2\hat{w}^1}{\mathrm{d}z^2}=-m\frac{\mathrm{d}\hat{w}^1}{\mathrm{d}z}-\tilde{k}^2\left[\tau^{-2/3}\mathcal{R}^1+\tilde{\mathcal{R}}^0(m-1)z\right]\hat{w}^0, \tag{24}$$

and hence

$$\hat{w}^1=C\sin(\pi z)+A\mathcal{D}_1z(z-1)\cos(\pi z)+A\mathcal{D}_2z\sin(\pi z), \tag{25}$$

$$\hat{s}^1=\frac{\tau^{-1/3}}{\tilde{k}^2}\left\{A[\mathcal{D}_3z\sin(\pi z)+\mathcal{D}_1z(z-1)\cos(\pi z)+\mathcal{D}_2z\sin(\pi z)]+C\sin(\pi z)\right\}, \tag{26}$$

where C is a new arbitrary constant. For Case 1 defined in (6a–d), $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are defined in the following way:

$$\mathcal{D}_1=\frac{\tilde{k}^2\tilde{\mathcal{R}}^0(m-1)}{4\pi},\quad \mathcal{D}_2=-\frac{1}{2\pi}\left[\frac{\tilde{k}^2\tilde{\mathcal{R}}^0(m-1)}{2\pi}+m\pi\right],\quad \mathcal{D}_3=m-1, \tag{27}$$

and in Case 2:

$$\mathcal{D}_1=-\frac{\tilde{k}^2\tilde{\mathcal{R}}^0\gamma}{4\pi(\gamma-1)},\quad \mathcal{D}_2=-\frac{1}{2\pi}\left[\frac{\pi}{\gamma-1}-\frac{\tilde{k}^2\tilde{\mathcal{R}}^0\gamma}{2\pi(\gamma-1)}\right],\quad \mathcal{D}_3=-\frac{\gamma}{\gamma-1}. \tag{28}$$

Defining $\tau^{-2/3}\mathcal{R}^1=\tilde{\mathcal{R}}^1$, the first order correction to the characteristic Rayleigh number ($\mathcal{R}\approx\tau^{2/3}\tilde{\mathcal{R}}^0+\theta\tau^{2/3}\tilde{\mathcal{R}}^1$) resulting from the compressibility of the medium is

$$\tilde{\mathcal{R}}^1=-\frac{1}{2}(m-1)\tilde{\mathcal{R}}^0 \tag{29}$$

in Case 1, and

$$\tilde{\mathcal{R}}^1=\frac{\gamma}{2(\gamma-1)}\tilde{\mathcal{R}}^0 \tag{30}$$

in Case 2. Since $\theta\le 0$, these results suggest that it is easier to excite the convective instability in a compressible medium with constant thermal diffusivity (Case 2) than in an incompressible one at high rotation rates, in the sense that in the former case the characteristic Rayleigh number for the onset of instability is smaller. On the other hand, in a compressible medium with constant thermal conductivity (Case 1), the Rayleigh number has to reach higher values than in the incompressible case for the instability to settle in.

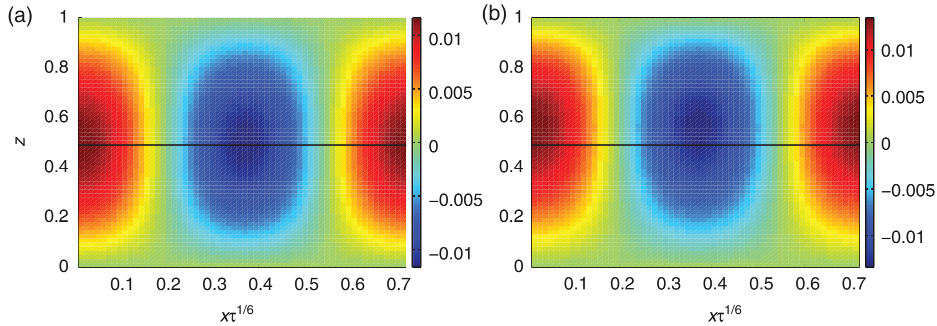

Figure 1. The solution $s=(\hat{s}^0+\theta\hat{s}^1)\cos kx$ (see (17) and (26)) for the entropy on an XZ plane with $m=2$ and $\theta=-0.1$ for (a) Case 1 and (b) Case 2. The wave number is set to its characteristic value $\tilde{k}=(\pi/\sqrt{2})^{1/3}$ and the constants A and C are taken equal to unity. The continuous lines are drawn at $z=0.5$. At the onset at large Taylor numbers the compressibility shifts the convective rolls downwards in Case 1 and upwards in Case 2.

Figure 1 shows the effect of small compressibility on the entropy at the onset of convection. The convective rolls are shifted in the vertical direction, in Case 1 towards the bottom boundary and in Case 2 towards the top boundary. The value of the shift is $\Delta z=(\theta/\pi^2)[\pi/4\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3]$, thus for $\tilde{k}=(\pi/\sqrt{2})^{1/3}$ (the characteristic value minimizing $\tilde{\mathcal{R}}^0$) and $A=C=1$ it becomes $\Delta z\approx-0.0056$ in Case 1 and $\Delta z\approx 0.057$ in Case 2. Evidently, in Case 2 this shift is a manifestation of what appears to be a more general fact, that the compressibility tends to decrease the size of the convective structures (see, e.g. Glatzmaier and Gilman 1981a, Jones *et al*. 2009, Jones and Kuzanyan 2009), by confining the convection to the region of small density (in Case 1 the size of convective structures is also decreased with respect to the symmetric Boussinesq case, however, on the contrary the region of convection is very slightly shifted towards larger densities; it seems that such downward shift appears only for Case 1 rapidly rotating systems close to onset, cf. figure 4 in section 3.2, and figure 6 in section 4.1). This can be clearly seen for larger compressibilities and in nonlinear regime, cf. figure 6 in section 4.1. The introduction of smaller scales into the dynamics can, in principle, have important consequences for the evolution of the system.

3.2. *Numerical linear results*

We compare the analysis above with the numerical solutions of the linear eigenvalue problem, which can be performed with no restrictions on θ or τ.

Linearization of the full system about the basic state yields

$$-\frac{\partial}{\partial t}\left[\nabla^2 w+\frac{\partial}{\partial z}\left(\frac{m\theta}{1+\theta z}w\right)\right]=-\mathcal{R}\sigma\nabla_H^2 s+\tau^{1/2}\sigma\frac{\partial\xi}{\partial z}-\sigma\nabla^2\nabla^2 w+\frac{3\sigma m(2-m)\theta^4}{(1+\theta z)^4}w$$
$$-\frac{2\sigma m\theta}{1+\theta z}\frac{\partial}{\partial z}\nabla_H^2 w-\frac{2\sigma m^2\theta^2}{3(1+\theta z)^2}\nabla_H^2 w-\frac{2\sigma m\theta}{1+\theta z}\frac{\partial^3 w}{\partial z^3}$$
$$+\frac{\sigma m(4-m)\theta^2}{(1+\theta z)^2}\frac{\partial^2 w}{\partial z^2}-\frac{3\sigma m(2-m)\theta^3}{(1+\theta z)^3}\frac{\partial w}{\partial z},\tag{31}$$

$$\frac{\partial\xi}{\partial t}=\tau^{1/2}\sigma\left(\frac{\partial w}{\partial z}-\nabla\cdot\mathbf{u}\right)+\sigma\nabla^2\xi+\frac{\sigma m\theta}{1+\theta z}\frac{\partial\xi}{\partial z},\tag{32}$$

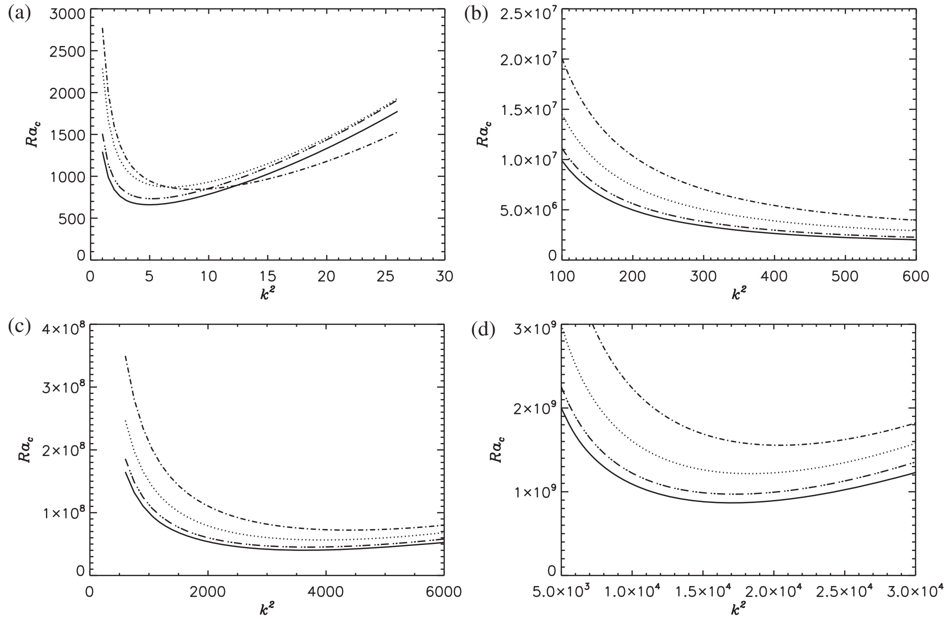

Figure 2. Characteristic Rayleigh number versus k^2 for $\theta = 0$ (solid), $\theta = -0.4$ (dot-dot-dashed), $\theta = -0.784$ (dotted), $\theta = -0.99898$ (dot-dashed) and (a) $\tau^{1/2} = 0$, (b) $\tau^{1/2} = 10^4$, (c) $\tau^{1/2} = 10^5$, (d) $\tau^{1/2} = 10^6$. Note that in all the cases compressibility increases the Rayleigh number optimized over k.

$$\frac{\partial s}{\partial t} - \frac{w}{1+\theta z} = \frac{1}{(1+\theta z)^m}\nabla^2 s + \frac{\theta}{(1+\theta z)^{m+1}}\frac{\partial s}{\partial z}, \tag{33}$$

$$\nabla \cdot \mathbf{u} = -\frac{m\theta}{1+\theta z} w \tag{34}$$

in Case 1, where $\nabla_H = (\partial_x, \partial_y, 0)$ is the horizontal part of the differential operator. Again, by simply substituting $m \to 1/(\gamma - 1)$ in the above equation, we obtain an analogous system of equations for Case 2 with the entropy equation replaced by

$$\frac{\partial s}{\partial t} = \nabla^2 s + \frac{\theta\gamma}{(\gamma-1)(1+\theta z)}\frac{\partial s}{\partial z}. \tag{35}$$

We seek solutions $\propto \exp(\mathrm{i}k_1 x + \mathrm{i}k_2 y + \mu t)$. The characteristic Rayleigh number can then be calculated as occurring when $\mathrm{Re}\{\mu\} = 0$. If $\mathrm{Im}\{\mu\} \neq 0$ then the bifurcation is a Hopf bifurcation and we set $\mathrm{Im}\{\mu\} = \omega$. We consider, however, $\sigma = 1$ (fixed) and find that $\omega = 0$. The polytropic index is set to $m = 1.4$. Owing to the symmetry between x and y we may calculate the characteristic Rayleigh number Ra_c as a function of $k^2 = k_1^2 + k_2^2$. This calculation is repeated at a number of different rotation rates for a few choices of θ. The results are shown in figure 2. In each case, the minimum value of the characteristic Rayleigh number is found at $\theta = 0$. This is in agreement with the analytic results for small θ at high rotation rates.

The characteristic Rayleigh number and wavelength scale are as expected, as a function of rotation rate, for both the unstratified and most stratified cases. We find that the wavenumber scales as $k_{\min} \sim \tau^{1/6}$ whilst $Ra_{c_{\min}} \sim \tau^{2/3}$, even when the layer is highly stratified, as shown on figure 3 (we have also checked that the ‘‘compensated’’

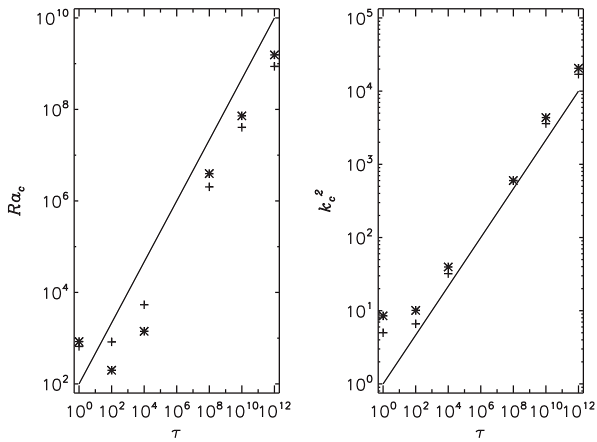

Figure 3. The scalings of the characteristic Rayleigh number Ra_c and the square of the characteristic wavenumber k_c^2 with the Taylor number Ta. The stars correspond to $\theta = 0$ and the crosses to $\theta = -0.99$. The solid lines indicate the analytic scalings obtained for small $|\theta|$, thus their slopes are 2/3 for the Rayleigh number and 1/3 for the square of the wavenumber.

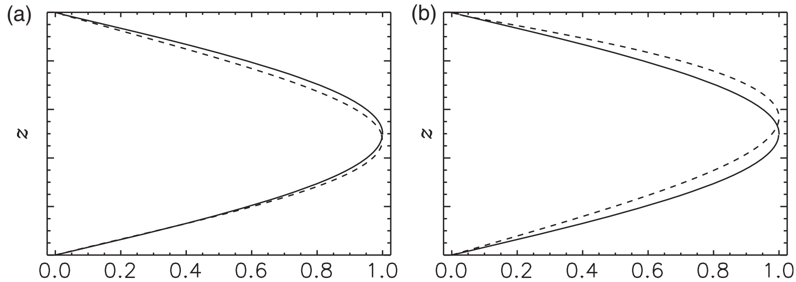

Figure 4. Eigenfunctions for $\theta = 0$ (solid) and $\theta = -0.99898$ (dashed) for (a) $\tau^{1/2} = 10^6$ and (b) $\tau^{1/2} = 0$.

Rayleigh number $\tilde{\mathcal{R}}$ and wavenumber $\tilde{k}$ approach a constant value as the Taylor number increases). This motivates our choice of scalings in the next section for the nonlinear regime. The eigenfunctions for a selection of values of θ for the non-rotating and most rapidly rotating cases are shown in figure 4. Note that as θ is decreased from zero the eigenfunctions become weakly asymmetric as expected in both the cases. The degree of asymmetry for rapidly rotating cases is less pronounced than that for weakly rotating cases, and the sense is different with the eigenfunctions for the stratified rapidly rotating case being shifted down relative to the non-stratified case (this shift was also observed in the analytical results for small compressibility, cf. figure 1(a)).

4. Nonlinear anelastic convection in the rapid rotation limit

We now proceed by analysing nonlinear effects in rapidly rotating convection under the anelastic approximation. After Julien and Knobloch (1999), who studied rapidly rotating Boussinesq convection, we develop expansions in terms of $\varepsilon = \tau^{-1/6} = E^{1/3}$, where E is the Ekman number, of the form

$$\mathbf{x} = (\varepsilon x', \ \varepsilon y', \ z), \qquad t = \varepsilon^2 t', \tag{36}$$

$$\mathbf{u} = \varepsilon^{-1}\mathbf{u}'(x', y', z, t'), \qquad s = S(z) + \varepsilon s'(x', y', z, t'), \qquad \mathcal{R} = \varepsilon^{-4}\tilde{\mathcal{R}}, \tag{37}$$

in which $S(z)$ is the value of the entropy averaged over horizontal variables and time. The primes will be dropped in what follows. From (8)–(10) we derive the equations for the vertical components of velocity w and vorticity ξ, which in Case 1 are

$$\begin{aligned} -\frac{\partial}{\partial t}\left[\nabla^2 w + \frac{\partial}{\partial z}\left(\frac{m\theta}{1+\theta z} w\right)\right] = & -\left\{\nabla \times \nabla \times [(\mathbf{u}\cdot\nabla)\mathbf{u}]\right\}\cdot\hat{\mathbf{e}}_z - \mathcal{R}\sigma\nabla_H^2 s + \tau^{1/2}\sigma\frac{\partial \xi}{\partial z} \\ & - \sigma\nabla^2\nabla^2 w - \frac{2\sigma m\theta}{1+\theta z}\frac{\partial}{\partial z}\nabla_H^2 w - \frac{2\sigma m^2\theta^2}{3(1+\theta z)^2}\nabla_H^2 w \\ & - \frac{2\sigma m\theta}{1+\theta z}\frac{\partial^3 w}{\partial z^3} + \frac{\sigma m(4-m)\theta^2}{(1+\theta z)^2}\frac{\partial^2 w}{\partial z^2} \\ & - \frac{3\sigma m(2-m)\theta^3}{(1+\theta z)^3}\frac{\partial w}{\partial z} + \frac{3\sigma m(2-m)\theta^4}{(1+\theta z)^4} w, \end{aligned} \tag{38}$$

$$\frac{\partial \xi}{\partial t} + \left\{\nabla \times [(\mathbf{u}\cdot\nabla)\mathbf{u}]\right\}\cdot\hat{\mathbf{e}}_z = \tau^{1/2}\sigma\left(\frac{\partial w}{\partial z} - \nabla\cdot\mathbf{u}\right) + \sigma\nabla^2\xi + \frac{\sigma m\theta}{1+\theta z}\frac{\partial \xi}{\partial z}, \tag{39}$$

$$\begin{aligned} \frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s = & \frac{w}{1+\theta z} + \frac{1}{(1+\theta z)^m}\nabla^2 s + \frac{\theta}{(1+\theta z)^{m+1}}\frac{\partial s}{\partial z} \\ & - \frac{\theta}{\mathcal{R}}\frac{1}{(1+\theta z)}\left[2\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2 + \sum_{i<j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{2}{3}(\nabla\cdot\mathbf{u})^2\right], \end{aligned} \tag{40}$$

$$\nabla\cdot\mathbf{u} = -\frac{m\theta}{1+\theta z} w, \tag{41}$$

while in Case 2, as previously stated, $m \to 1/(\gamma-1)$ and the entropy equation should be replaced by

$$\begin{aligned} \frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s = & \, w(1+\theta z)^{\gamma/(1-\gamma)} + \nabla^2 s + \frac{\theta\gamma}{(\gamma-1)(1+\theta z)}\frac{\partial s}{\partial z} \\ & - \frac{\theta}{\mathcal{R}}\frac{1}{(1+\theta z)}\left[2\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2 + \sum_{i<j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{2}{3}(\nabla\cdot\mathbf{u})^2\right]. \end{aligned} \tag{42}$$

After introducing the above scalings (36) and (37), into the Case 1 equations (38)–(41), we obtain at leading order

$$-\frac{\partial}{\partial t}\nabla_H^2 w = \nabla_H^2\mathcal{N}_z - \sigma\tilde{\mathcal{R}}\nabla_H^2 s + \sigma\frac{\partial \xi}{\partial z} - \sigma\nabla_H^4 w, \tag{43}$$

$$\frac{\partial \xi}{\partial t} = -\left(\frac{\partial \mathcal{N}_y}{\partial x} - \frac{\partial \mathcal{N}_x}{\partial y}\right) + \sigma\left(\frac{\partial w}{\partial z} + \frac{m\theta}{1+\theta z} w\right) + \sigma\nabla_H^2\xi, \tag{44}$$

$$\frac{\partial s}{\partial t} = -\mathbf{u}\cdot\nabla_H s + \frac{w}{1+\theta z} + \frac{1}{(1+\theta z)^m}\nabla_H^2 s - w\frac{\mathrm{d}S}{\mathrm{d}z}, \tag{45}$$

$$\frac{1}{(1+\theta z)^m}\frac{\mathrm{d}^2 S}{\mathrm{d}z^2} + \frac{\theta}{(1+\theta z)^{m+1}}\frac{\mathrm{d}S}{\mathrm{d}z} - \frac{\mathrm{d}}{\mathrm{d}z}\langle ws\rangle_{x,y,t} - \frac{m\theta}{1+\theta z}\langle ws\rangle_{x,y,t} - \frac{\theta}{\tilde{\mathcal{R}}}\frac{1}{1+\theta z}\langle \mathrm{VD}\rangle_{x,y,t} = 0 \tag{46}$$

and finally

$$\varepsilon^{-1}\nabla_H \cdot \mathbf{u} + \frac{\partial w}{\partial z} = -\frac{m\theta}{1+\theta z} w, \tag{47}$$

where $\langle \cdot \rangle_{x,y,t}$ denotes average over the horizontal variables and time. We have also assumed that $\mathbf{u}$ and s are periodic in x, y and t with zero average. Here the viscous dissipation term is

$$\mathrm{VD} = \frac{4}{3}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{4}{3}\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2, \tag{48}$$

where u and v are the x and y components of the velocity field, respectively, and the nonlinear terms are

$$\mathcal{N}_x = (\mathbf{u} \cdot \nabla_H)u, \quad \mathcal{N}_y = (\mathbf{u} \cdot \nabla_H)v, \quad \mathcal{N}_z = (\mathbf{u} \cdot \nabla_H)w. \tag{49}$$

Following Julien and Knobloch (1999) we now consider steady and oscillatory solutions with square planform – although it may be that these are not the preferred solutions when stratification is included – i.e. we set

$$[w,\ \xi,\ s] = [\hat{w}(z),\ \hat{\xi}(z),\ \hat{s}(z)]\mathrm{e}^{\mathrm{i}\varpi t}(\cos kx + \cos ky) + \text{c.c.} + \mathrm{O}(\varepsilon), \tag{50}$$

$$[u,\ v] = k^{-1}[-\sin ky,\ \sin kx]\hat{\xi}(z)\mathrm{e}^{\mathrm{i}\varpi t} + \text{c.c.} + \mathrm{O}(\varepsilon) \tag{51}$$

for which

$$\langle \mathrm{VD} \rangle_{x,y,t} = 2k^2|\hat{w}(z)|^2 + 2|\hat{\xi}(z)|^2, \tag{52}$$

$$\langle ws \rangle_{x,y,t} = \hat{w}(z)\hat{s}^*(z) + \hat{w}^*(z)\hat{s}(z). \tag{53}$$

The great advantage of this choice of planform is that the nonlinear terms generated by $\mathcal{N}_x$, $\mathcal{N}_y$ and $\mathcal{N}_z$ (see (43), (44) and (49)) vanish identically, considerably simplifying the analysis. Hence equations (43)–(46) take the form

$$\mathrm{i}\varpi k^2 \hat{w} = \tilde{\mathcal{R}}\sigma k^2 \hat{s} + \sigma \frac{\mathrm{d}\hat{\xi}}{\mathrm{d}z} - \sigma k^4 \hat{w}, \tag{54}$$

$$\mathrm{i}\varpi \hat{\xi} = \sigma\left(\frac{\mathrm{d}\hat{w}}{\mathrm{d}z} + \frac{m\theta}{1+\theta z}\hat{w}\right) - \sigma k^2 \hat{\xi}, \tag{55}$$

$$\mathrm{i}\varpi \hat{s} = \left(\frac{1}{1+\theta z} - \frac{\mathrm{d}S}{\mathrm{d}z}\right)\hat{w} - \frac{1}{(1+\theta z)^m} k^2 \hat{s}, \tag{56}$$

$$\frac{1}{(1+\theta z)^m}\frac{\mathrm{d}^2 S}{\mathrm{d}z^2} + \frac{\theta}{(1+\theta z)^{m+1}}\frac{\mathrm{d}S}{\mathrm{d}z} - \frac{\mathrm{d}}{\mathrm{d}z}(\hat{w}\hat{s}^* + \hat{w}^*\hat{s}) - \frac{m\theta}{1+\theta z}(\hat{w}\hat{s}^* + \hat{w}^*\hat{s})$$
$$-\frac{2\theta}{\tilde{\mathcal{R}}}\frac{1}{1+\theta z}\left(k^2|\hat{w}|^2 + |\hat{\xi}|^2\right) = 0. \tag{57}$$

It follows from (54)–(56) that the amplitude $\hat{w}(z)$ obeys

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} + \frac{m\theta}{1+\theta z}\frac{\mathrm{d}\hat{w}}{\mathrm{d}z} + \left[\tilde{\mathcal{R}}k^2 g(z)\frac{i\varpi/\sigma + k^2}{i\varpi + k^2/(1+\theta z)^m} - k^2(i\varpi/\sigma + k^2)^2 - \frac{m\theta^2}{(1+\theta z)^2}\right]\hat{w} = 0. \tag{58}$$

Here $g(z)=1/(1+\theta z)-(\mathrm{d}S/\mathrm{d}z)$ satisfies the following first-order non-homogeneous equation:

$$\frac{\mathrm{d}}{\mathrm{d}z}[g(z)\varsigma(\hat{w},z)]+\frac{\theta}{(1+\theta z)}g(z)=-\mathcal{V}(\hat{w},z), \tag{59}$$

with

$$\varsigma(\hat{w},z)=1+\frac{2k^2\left|\hat{w}(z)\right|^2}{\varpi^2+k^4/(1+\theta z)^{2m}}, \tag{60}$$

$$\mathcal{V}(\hat{w},z)=\frac{2\theta}{\tilde{\mathcal{R}}}(1+\theta z)^{m-1}\left[k^2|\hat{w}|^2+\frac{\sigma^2}{\sigma^2k^4+\varpi^2}\left(\left|\frac{\mathrm{d}\hat{w}}{\mathrm{d}z}\right|^2+\frac{m\theta}{1+\theta z}\frac{\mathrm{d}|\hat{w}|^2}{\mathrm{d}z}+\frac{m^2\theta^2}{(1+\theta z)^2}|\hat{w}|^2\right)\right], \tag{61}$$

which can be readily obtained by using (55) and (56) to express $\hat{\xi}$ and $\hat{s}$ in terms of $\hat{w}$ and introducing these expressions into (57). There is also a condition, which results from imposing the conditions of isentropy at the boundaries, namely

$$\int_0^1 g(z)\mathrm{d}z=-\triangle\bar{s}=\frac{1}{\theta}\ln(1+\theta), \tag{62}$$

where $\bar{s}(z)$ denotes the basic state entropy (6d). From equations (59)–(62) we can express the function $g(z)$ by the amplitude $\hat{w}(z)$ as

$$g(z)=\left\{-\int_0^z\mathcal{V}(z')e^{I(z')}\mathrm{d}z'+\frac{I_1(z)-\triangle\bar{s}}{I_2(z)}\right\}\frac{\exp(-I(z))}{\varsigma(z)}, \tag{63}$$

where

$$I(z)=\theta\int_0^z\frac{\mathrm{d}z'}{(1+\theta z')\varsigma(z')}, \tag{64}$$

$$I_1(z)=\int_0^1\left[\frac{\exp(-I(z'))}{\varsigma(z')}\int_0^{z'}\mathcal{V}(z'')e^{I(z'')}\mathrm{d}z''\right]\mathrm{d}z', \tag{65}$$

$$I_2(z)=\int_0^1\frac{\exp(-I(z'))}{\varsigma(z')}\mathrm{d}z'=\frac{1}{\theta}\{1-(1+\theta)\exp[-I(1)]\}+\int_0^1\exp[-I(z')]\mathrm{d}z'. \tag{66}$$

In the next section we give the numerical solutions of the above set of equations for steady states, but first we discuss some properties of the equations. (For the sake of completeness the necessary equations for Case 2 are listed in the appendix.)

In the Boussinesq limit, when $\theta=0$, the above equations simplify significantly ($I(z)=0$, $\int_0^1 g(z)\mathrm{d}z=1$, $\mathcal{V}(\hat{w},z)=0$ and $g(z)=1/[\varsigma(z)\int_0^1(\varsigma(z'))^{-1}\mathrm{d}z']$) and the amplitude equation (58) becomes

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2}+\left[\tilde{\mathcal{R}}k^2\frac{(\mathrm{i}\varpi/\sigma+k^2)(-\mathrm{i}\varpi+k^2)}{\varpi^2+k^4+2k^2\left|\hat{w}(z)\right|^2}\,\frac{1}{\int_0^1(\varsigma(z',\hat{w}))^{-1}\mathrm{d}z'}-k^2\left(\mathrm{i}\varpi/\sigma+k^2\right)^2\right]\hat{w}=0, \tag{67}$$

which agrees with the equation obtained by Julien and Knobloch (1999) in the Boussinesq case (their equation (39) is for the amplitude of the velocity poloidal potential which has to by multiplied by k^2 in order to obtain the vertical velocity amplitude $\hat{w}(z)$). On the other hand, in linear regime $|\hat{w}(z)| \ll 1$ we get $\varsigma(z) \approx 1$, $\mathcal{V}(z) \approx 0$, $g(z) \approx 1/(1+\theta z)$, thus in the stationary case ($\varpi = 0$) we obtain the following amplitude equation

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} + \frac{m\theta}{1+\theta z}\frac{\mathrm{d}\hat{w}}{\mathrm{d}z} + \left[\tilde{\mathcal{R}}k^2(1+\theta z)^{m-1} - k^6 - \frac{m\theta^2}{(1+\theta z)^2}\right]\hat{w} = 0. \tag{68}$$

Note that, as expected, the amplitude of the solution is not determined at this order. Furthermore, if we introduce the sum of the convective and conductive vertical heat flux,

$$f_C(z) = \bar{T}\frac{\mathrm{d}}{\mathrm{d}z}(\bar{s}+S) - \bar{\rho}\bar{T}\langle ws\rangle_{x,y,t} = -(1+\theta z)\,g(z) - (1+\theta z)^{m+1}\langle ws\rangle_{x,y,t}, \tag{69}$$

equation (46) can be rewritten in the following form:

$$\frac{\mathrm{d}f_C}{\mathrm{d}z} = -\theta(1+\theta z)^m\langle ws\rangle_{x,y,t} + \frac{\theta}{\tilde{\mathcal{R}}}(1+\theta z)^m\langle \mathrm{VD}\rangle_{x,y,t}. \tag{70}$$

This, however, is not the total heat flux in the system at a given $z \neq 0, 1$, since it does not include the viscous heating and the work done by the buoyancy force. The latter two, represented by the two terms on the right-hand side of equation (70), in a stationary state, are globally in balance. However, the local heat flux would, in general, be influenced by viscous and buoyancy effects. Hence we may define the Nusselt number in the following way:

$$Nu = \left(f_T\big|_{z=1} - f_A\right)\big/(f_{BC} - f_A) \quad \text{or} \quad Nu = f_T\big|_{z=1}\big/f_{BC}, \tag{71}$$

where $f_A = Kg/c_p$ is the heat flux at the adiabatic state, $f_{BC} = -K\triangle T/d$ is the conductive heat flux of the basic, static state and $f_T\big|_{z=1} = -\varepsilon k_T\bar{T}\mathrm{d}(\bar{s}+S)/\mathrm{d}z$ is the conductive heat flux at $z=1$ in a convective state, which in a stationary state must be equal to the heat flux at $z=0$, $f_T|_{z=0}$, and is the total heat flux of the system. Here K is the molecular thermal conductivity coefficient (as opposed to the turbulent thermal conductivity coefficient k_T). When deriving the entropy equation (4), we have assumed that the turbulent heat flux is much greater then the molecular, thus $K/k_T = \epsilon = -(d/c_p)(\mathrm{d}\bar{s}/\mathrm{d}z)_r \ll 1$. This means that

$$Nu = \frac{(1+\theta)\,g(1) + \epsilon + \theta}{\epsilon} = \frac{\gamma\left(1+\theta^{-1}\right)g(1) + (\gamma-1)(m+1)}{\gamma m - m - 1} \quad \text{or} \quad Nu = -\left(1+\frac{1}{\theta}\right)g(1). \tag{72}$$

Such a definition, however, leads to very large Nusselt number in the Boussinesq limit $\theta = \mathrm{O}(\epsilon)$. Hence we may yet use a different definition, where the Nusselt number is a ratio of the total heat flux $f_T\big|_{z=1} = -\varepsilon k_T\bar{T}\mathrm{d}(\bar{s}+S)/\mathrm{d}z$ to the conductive heat flux of the basic state expressed in terms of entropy $f_{BCT} = -\varepsilon k_T\bar{T}\mathrm{d}\bar{s}/\mathrm{d}z$, namely

$$Nu = (1+\theta)\,g(1), \tag{73}$$

which in the Boussinesq limit simply reduces to $Nu = g(1)|_{\theta=0}$ (because we shall seek stationary solutions $(1+\theta)g(1) = g(0)$). Hence in what follows we adopt the latter definition (73) for Nu. It should be noted, that since the expression for conductive heat flux in terms of entropy is valid only by incorporating small scale turbulence, the flux f_{BCT} is not a conductive heat flux of a static state. It should rather be interpreted as a heat flux in a state approached from developed convection by decreasing the Rayleigh number to value close to critical but still above it, in which small scale turbulence is still present, and is incorporated into the turbulent heat flux coefficient $k_T \gg K$.

The weakly nonlinear theory (WNT) performed in the stationary (and hence real) case, under the assumptions that $\hat{w} = (a\hat{w}_1 + a^2\hat{w}_2 + a^3\hat{w}_3)(\cos kx + \cos ky) + \mathrm{O}(\varepsilon, a^4)$, $g = g_0 + ag_1 + a^2 g_2 + \mathrm{O}(\varepsilon, a^3)$ and $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_c + a^2\tilde{\mathcal{R}}_2 + \mathrm{O}(\varepsilon^2, a^3)$, where $a \ll 1$ is the perturbation amplitude, leads to equation (68) with additional constraints for the amplitude, which could also be used to find $\hat{w}$ (it also gives the result $\tilde{\mathcal{R}}_1 = 0$, and hence $a \approx \sqrt{(\tilde{\mathcal{R}} - \tilde{\mathcal{R}}_c)}$). Furthermore, under additional assumption of weak compressibility the solvability conditions provide constraints for the constants A and C in the equation (25) and also allow the calculation of $g(1)$ and hence the Nusselt number in terms of $\tilde{\mathcal{R}}_{c,0}(k)$ and $\tilde{\mathcal{R}}_{2,0}(k)$ (the characteristic Rayleigh number and its order two WNT correction in the Boussinesq case $\theta = 0$). The simplest assumption of $1 \gg |\theta| \sim a^2 \gg \varepsilon$ leads to

$$Nu = 1 + 2a^2 \frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}} + \theta(m-1) \quad \text{in Case 1,} \tag{74a}$$

$$Nu = 1 + 2a^2 \frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}} - \theta\frac{\gamma}{\gamma - 1} \quad \text{in Case 2.} \tag{74b}$$

If, however, we assume slightly stronger compressibility (but still weak), i.e $a \ll |\theta| \ll 1$, having two independent parameters we must modify the expansions, i.e. $\hat{w} = (a\sum_{i,j=0} a^i\theta^j\hat{w}_{i+1,j})(\cos kx + \cos ky)$, $g = \sum_{i,j=0} a^i\theta^j g_{i,j}$ and $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_{c,0} + \sum_{i,j=0} a^i\theta^j\tilde{\mathcal{R}}_{i,j}$. Interestingly, with stronger compressibility the first non-zero correction to the Nusselt number resulting from the compressibility of the medium is smaller, of order $a^2\theta$, i.e.

$$Nu = 1 + 2a^2 \frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}} \left[1 + \frac{1}{2}\theta(m-1) + \theta\frac{\tilde{\mathcal{R}}_{2,1}}{\tilde{\mathcal{R}}_{2,0}}\right] \quad \text{in Case 1,} \tag{75a}$$

$$Nu = 1 + 2a^2 \frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}} \left[1 - \frac{1}{2}\theta\frac{\gamma}{\gamma - 1} + \theta\frac{\tilde{\mathcal{R}}_{2,1}}{\tilde{\mathcal{R}}_{2,0}}\right] \quad \text{in Case 2.} \tag{75b}$$

From the above expressions (74) and (75) we infer that for all values of γ in Case 2 and only for $m < 1$ (i.e. $\gamma > 2 + \mathrm{O}(\epsilon)$) in Case 1, weak compressibility increases the efficiency of heat transfer by a convective system. In other words the total amount of heat transferred by a convective system is greater in the compressible case than for a Boussinesq fluid for the same departure from characteristic value $\tilde{\mathcal{R}} - \tilde{\mathcal{R}}_c \approx a^2\tilde{\mathcal{R}}_{2,0}$. [On the other hand, if we compare the Nusselt numbers in the Boussinesq and anelastic convection for the same departure from characteristic value defined as $(\tilde{\mathcal{R}} - \tilde{\mathcal{R}}_c)/\tilde{\mathcal{R}}_c$, for $a \ll |\theta| \ll 1$ the $a^2\theta$ correction in (75) vanishes for both compressible cases considered and the heat transfer turns out to be independent of compressibility, at least at the considered level of accuracy; for $|\theta| \sim a^2$, exactly as above, the heat transfer is more efficient in the compressible Case 2 for all γ and in Case 1 only for $\gamma > 2 + O(\epsilon)$.] Furthermore systems with constant turbulent thermal diffusivity κ_T are, in general, more efficient in heat transfer than those with constant turbulent thermal conductivity k_T.

4.1. *Numerical nonlinear results*

In this section we consider the effects of stratification and compressibility for nonlinear steady convection in the rapid rotation limit. In particular, we solve the system described by equations (58)–(62) as a nonlinear two-point boundary problem in z. For steady solutions we set the frequency to be zero and solve the system using a Newton–Raphson–Kantorovich (NRK) solver for a range of choices of stratification θ. As for the linear case we fix $\sigma = 1$ and $m = 1.4$. We also fix the planform so that $k_x = k_y = k = 1/\sqrt{2}$ and $k_x^2 + k_y^2 = 1$ – hence the wavenumber has not been optimized over (or we are considering a domain of fixed aspect ratio).

Figure 5 shows the convective heat transport (as measured by $Nu - 1$, with the Nusselt number defined in equation (73)) as a function of Ra for a selection of θ. As $|\theta|$ is increased (at fixed Ra) the heat transport decreases (in line with the analytical results). The z-dependence of the nonlinear solutions for the vertical velocity w is shown in figure 6. When $\theta = 0$ the solutions are symmetric about the mid-plane and grow in amplitude as Ra is increased as expected (Julien and Knobloch 1999). However, as $|\theta|$ is increased, the nonlinear solutions become asymmetric, particularly at large amplitude, with the solution becoming more and more localized near the low density region where the basic state density is small. This is to be expected and is in agreement with similar calculations in spherical geometry (Glatzmaier and Gilman 1981a, b, Jones *et al.* 2009, Jones and Kuzanyan 2009).

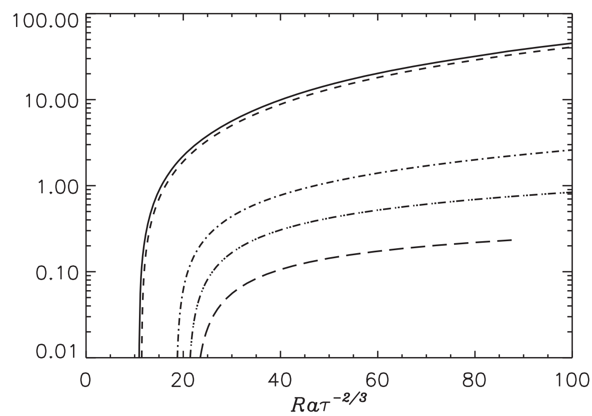

Figure 5. $Nu - 1$ versus Ra for $\theta = 0$ (solid), $\theta = -0.2$ (dashed), $\theta = -0.893$ (dot-dashed), $\theta = -0.965$ (dot-dot-dashed) and $\theta = -0.998$ (long-dashed).

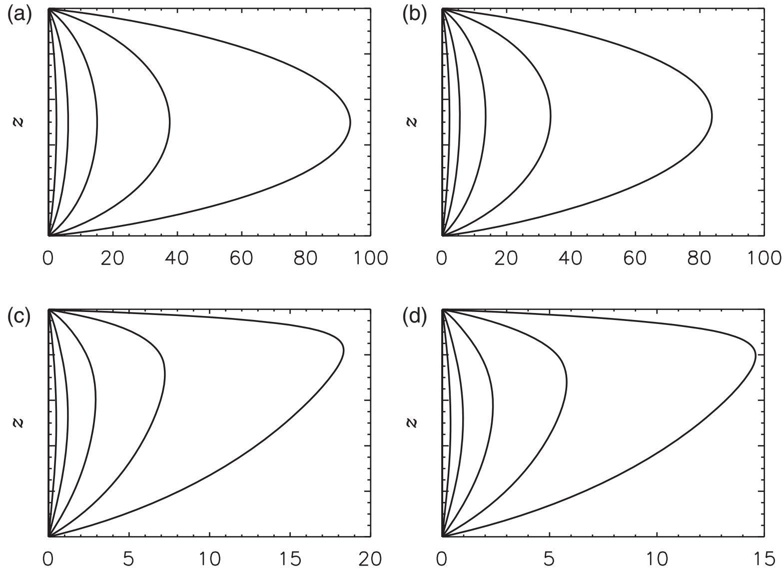

Figure 6. Nonlinear solutions for $w(z)$ for (a) $\theta = 0$ and $\mathcal{R} = 10.8697$, 10.8703, 23.5669, 43.5010, 86.8195, 182.095, 395.166, (b) $\theta = -0.2$ and $\mathcal{R} = 11.4002$, 11.4006, 21.3462, 38.2638, 75.2033, 156.024, 336.153, (c) $\theta = -0.965$ and $\mathcal{R} = 20.9684$, 20.9684, 22.2286, 27.8354, 50.9217, 128.793, 369.396 and (d) $\theta = -0.977$ and $\mathcal{R} = 21.4638$, 21.4638, 22.3498, 26.4895, 45.0281, 112.353, 339.724.

5. Concluding remarks

In this article, we have studied the influence of compressibility and stratification on the convection threshold and on the nonlinear convective heat transfer in a rotating plane layer. We utilized the simplest formulation of the anelastic approximation with the thermal energy flux expressed in terms of the entropy gradient (Braginsky and Roberts 1995). We performed an asymptotic analysis in the limit of small compressibility and rapid rotation $\tau^{-1/12} \ll |\theta| \ll 1$. Firstly we reported that in compressible systems with constant thermal diffusivity κ_T convective instability is more easily excited (in other words the characteristic Rayleigh number is smaller) than in incompressible systems. On the other hand, if the thermal conductivity $k_T = c_p\bar{\rho}\kappa_T$ is constant, the characteristic Rayleigh number for convection is often larger than in the incompressible case (smaller only if the polytropic index m is smaller than unity). Moreover, we also calculated the Nusselt numbers, i.e. the total heat transfer in both the compressible cases considered in the limit of rapid rotation and small compressibilities in the weakly nonlinear regime. We found that systems with constant turbulent thermal conductivity k_T are less efficient in transporting the heat than those with constant κ_T and more efficient than Boussinesq systems only if the polytropic index $m < 1$ (which corresponds to $\gamma > 2 + \mathrm{O}(\epsilon)$, where ϵ is the departure from adiabaticity). On the other hand constant diffusivity κ_T always leads to more efficient heat transfer by a compressible system than a Boussinesq analogue. We compared these analytical results with numerical solutions of the equations, which can be performed for arbitrary stratification and rotation rate in the linear regime and arbitrary stratification and rapid rotation in the nonlinear regime. The numerical results are in good agreement with the asymptotic expansion and demonstrate that the conclusions can be carried through to larger stratifications (thus larger values of $|\theta|$).

We studied the dependence of the convective heat flux on the compressibility in the nonlinear regime for the case of constant k_T. With the choice of the polytropic index greater than unity, namely $m=1.4$ (and hence $\gamma\approx 1.7+\mathrm{O}(\epsilon)$), we reported, that the total heat transferred by the system decreases with increasing compressibility $|\theta|$. Furthermore, the numerical results for the characteristic Rayleigh number obtained also for case 1 with $m=1.4$ suggest that in non-rotating or weakly rotating systems the dependence of $\mathcal{R}_c$ on the compressibility is not monotonic. It increases with increase in the stratification parameter $|\theta|$ up to a certain threshold value, and then starts to decrease, however, it never seems to drop below the Boussinesq value.

Glatzmaier and Gilman (1981a) studied the role of the $k_T=$const versus $\kappa_T=$const assumption in spherical geometry and reported that because the basic state in the $\kappa_T=$const case has much larger entropy gradients close to the outer (upper in our plane layer model) boundary, small scale convective structures develop in that region. This is so, since in the case $\kappa_T=$const the conductivity coefficient $k_T = c_p\bar{\rho}\kappa_T$ and thus the diffusive heat flux increases with depth. Therefore, in a stationary state the convective heat flux must increase with z to compensate for the loss of the diffusive flux at every $z=$const. However, a more general statement seems to be possible that in stratified cases convection tends to localize in the low density region near the upper boundary (Jones *et al.* 2009, Jones and Kuzanyan 2009), which was, in general, confirmed by our analytical and numerical results for both cases $k_T=$const and $\kappa_T=$const, although in rapidly rotating systems with constant k_T at the convection threshold, i.e. in linear regime, a slight downward shift of the convective rolls, towards larger densities, was found. Nevertheless all our results seem to suggest that at least in a non-chaotic regime, compressibility introduces smaller scales into the flow, which can have important consequences for the dynamics of the system, in particular in the magnetohydrodynamic case.

We conclude by outlining future directions for our research. Clearly it is important to compare our nonlinear results for asymptotically large rotation rates with those obtained at finite rotation. To this end a nonlinear anelastic time-stepping code has been developed and is currently being benchmarked. Furthermore it is of interest to determine the dynamo properties of the nonlinear flows that we have generated in this article, continuing the results of Soward (1974) into cases where stratification is included. This is the subject of a follow-up article Mizerski and Tobias (in preparation).

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Appendix A: Equations for Case 2

The nonlinear equations for Case 2 are given by

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2}+\frac{m\theta}{1+\theta z}\frac{\mathrm{d}\hat{w}}{\mathrm{d}z}+\left[\tilde{\mathcal{R}}k^2 g(z)\frac{\mathrm{i}\varpi/\sigma+k^2}{\mathrm{i}\varpi+k^2}-k^2\left(\mathrm{i}\varpi/\sigma+k^2\right)^2-\frac{m\theta^2}{(1+\theta z)^2}\right]\hat{w}=0, \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}[(1+\theta z)^m g(z)\varsigma(\hat{w},z)]+\theta(1+\theta z)^{m-1}g(z)=-\mathcal{V}(\hat{w},z), \tag{A.2}$$

where $m=1/(\gamma-1)$, $g(z)=1/(1+\theta z)^{m+1}-(\mathrm{d}S/\mathrm{d}z)$ and

$$\varsigma(\hat{w},z)=1+\frac{2k^2|\hat{w}(z)|^2}{\varpi^2+k^4}, \tag{A.3}$$

$$\mathcal{V}(\hat{w},z)=\frac{2\theta}{\tilde{\mathcal{R}}}(1+\theta z)^{m-1}\left[k^2|\hat{w}|^2+\frac{\sigma^2}{\sigma^2k^4+\varpi^2}\left(\left|\frac{\mathrm{d}\hat{w}}{\mathrm{d}z}\right|^2+\frac{m\theta}{1+\theta z}\frac{\mathrm{d}|\hat{w}|^2}{\mathrm{d}z}+\frac{m^2\theta^2}{(1+\theta z)^2}|\hat{w}|^2\right)\right], \tag{A.4}$$

and $g(z)$ must satisfy the following condition

$$\int_0^1 g(z)\mathrm{d}z=-\triangle\bar{s}=\frac{1}{m\theta}\left[1-\frac{1}{(1+\theta z)^m}\right]. \tag{A.5}$$

These equations, obviously, have the same Boussinesq limit as (58)–(62) and in the linear regime $|\hat{w}(z)|\ll 1$ when $\varsigma(z)\approx 1$, $\mathcal{V}(z)\approx 0$, $g(z)\approx 1/(1+\theta z)^{m+1}$ the amplitude equation becomes

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2}+\frac{m\theta}{1+\theta z}\frac{\mathrm{d}\hat{w}}{\mathrm{d}z}+\left[\tilde{\mathcal{R}}k^2(1+\theta z)^{-m-1}-k^6-\frac{m\theta^2}{(1+\theta z)^2}\right]\hat{w}=0. \tag{A.6}$$

Of course in Case 2 $Nu=(1+\theta)^{m+1}g(1)=g(0)$. The constants A and C, defining the amplitude of the velocity field in both cases are

$$A^2=2k^2\frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}}\quad\text{and}\quad C=\frac{A}{2}\left[\frac{\tilde{\mathcal{R}}_{2,1}}{\tilde{\mathcal{R}}_{2,0}}-\frac{1}{2}(m+1)-\mathcal{D}_2\right]. \tag{A.7}$$