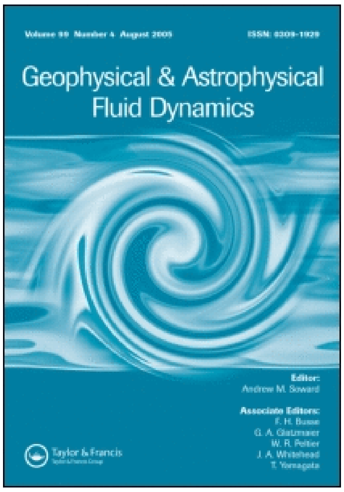

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Compressible Ekman–Hartmann boundary layers

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We consider the effect of compressibility on mixed Ekman–Hartmann boundary layers on an infinite plane ($z=0$), in the presence of an external magnetic field oblique to the boundary. The aim is to investigate the influence of the magnetic pressure on the fluid density, and hence, via mass conservation, on the mass flow into or out of the boundary layer. We find that if the z-component of vorticity in the main flow, immediately above the boundary layer, is negative, then there is a competition between Ekman suction and the magnetic pressure effect. Indeed, as the magnetic field strength is increased, the magnetic pumping may overcome the Ekman suction produced by anti-cyclonic main flow vortices. Such a mechanism, based on the competition between these effects, may be of importance for understanding the dynamics of the magnetic field in stellar (or planetary) interiors. For the solar tachocline, we find that the analysed magnetic pressure effect is unlikely to play a significant role; however, we give examples of what changes in the assumed scalings would be necessary for the effect to become important.

Keywords: Ekman–Hartmann boundary layers; Compressibility; Magnetic pressure

1. Introduction

Motivated partly by geophysical considerations and partly by the intrinsic magnetohydrodynamical interest of the problem, several studies have investigated the nature of Ekman–Hartmann boundary layers, which may arise in rapidly rotating fluids influenced by strong magnetic fields. The influence of a magnetic field on the classical Ekman boundary layer was first considered by Gilman and Benton (1968), who considered steady Ekman–Hartmann layers formed from imposing a magnetic field aligned with the rotation axis. Gilman and Benton (1968) produced two main findings: that Ekman suction (ES) is inhibited by the magnetic field, and that an induced "Hartmann current" in the main flow could be significant. The complementary, time-dependent spin-up problem was first addressed by Benton and Loper (1969), who showed that the characteristic spin-up time is reduced by the presence of a magnetic field. The linear stability of Ekman–Hartmann layers was first studied by Gilman (1971) and later extended, and applied to the core-mantle boundary, by Desjardins *et al.* (2001). The nonlinear stability problem was first considered by Desjardins *et al.* (1999)

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and has been generalised by Rousset (2004). A full bibliography of related early work can be found in Acheson and Hide (1973).

Studies to date of Ekman–Hartmann boundary layers have considered incompressible fluids. In this article, motivated by astrophysical considerations, we investigate the additional influence of compressibility, with a view to investigating the role of magnetic pressure in compressible Ekman–Hartmann layers. The plane-layer model we consider, described in section 2, represents the simplest possible configuration in which the problem can be investigated. Ekman–Hartmann layers form when both the Hartmann number and Taylor number are large, with the effects of the Coriolis and Lorentz forces being comparable. For our study, we also require that the density is influenced, at leading order, by magnetic pressure variations. The simplest ordering of the various parameters that allows an investigation of compressible Ekman–Hartmann layers, together with the ensuing analysis, is contained in section 3. In section 3.1, we consider the simplified problem in the absence of rotation and consider the influence of compressibility on pure Hartmann layers; the full problem of Ekman–Hartmann layers is considered in section 3.2. Our main result is the identification of a new effect, magnetic pressure pumping (MPP), which can compete with ES and indeed, for sufficiently strong magnetic fields, can dominate.

In astrophysical situations, one can conceive that the role of magnetic pressure on Ekman–Hartmann layers may be of importance in gas planets with a solid core, or in stellar interiors where, although there are no strictly rigid boundaries, somewhat analogous layers may form. The most exciting and important recent solar observational finding is the determination, via helioseismology, of the solar internal rotation rate (Schou *et al.* 1998; see also Christensen-Dalsgaard and Thompson 2007 for a review of all the observational results). This revealed that the surface latitudinal rotation profile is essentially maintained throughout the convection zone, and that the radiative zone rotates as a solid body, at a rate equal to that on the surface at latitude approximately 30°. Furthermore, the differing angular velocity profiles $\Omega(r, \theta)$ at the base of the convection zone and in the radiative zone are accommodated by a thin region of strong radial variation in angular velocity, a region known as the *tachocline*; for low latitudes $\partial\Omega/\partial r > 0$, for high latitudes $\partial\Omega/\partial r < 0$. Estimates of the radial extent of the tachocline are in the range 2–5% of the solar radius, although it is too thin to be resolved precisely with current seismic data (Gough 2007). Although it represents a simplification of the true physics, the tachocline can sometimes profitably be modelled as an Ekman or Ekman–Hartmann layer (see e.g. Ponty *et al.* 2001, Garaud 2007). In section 4, we examine the parameter scalings that pertain to the tachocline and address the applicability of our model. We show, possibly not surprisingly, that the MPP effect is small in the tachocline; however, we then consider what differences would be necessary for this effect to be astrophysically relevant.

2. Mathematical formulation

We consider a plane layer, compressible, magnetohydrodynamic system, with gravity taken to act in the negative z-direction, and with rotation about the z-axis. The governing equations are those of momentum (the Navier–Stokes equation), magnetic induction and mass conservation, together with the solenoidal constraint for the magnetic field. If length, velocity, magnetic field and density are scaled with representative values L, U, B_s and ρ_s, then the equations may be expressed in dimensionless form as

$$\mathrm{Re}\,\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p+\frac{M^2}{\mathrm{Rm}}(\nabla\times\mathbf{B})\times\mathbf{B}-\tau^{1/2}\rho\hat{\mathbf{e}}_z\times\mathbf{u}-G\rho\hat{\mathbf{e}}_z+\nabla^2\mathbf{u}+\frac{1}{3}\nabla(\nabla\cdot\mathbf{u}), \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}(\nabla\cdot\mathbf{u})+\frac{1}{\mathrm{Rm}}\nabla^2\mathbf{B}, \tag{2}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0, \tag{3}$$

$$\nabla\cdot\mathbf{B}=0, \tag{4}$$

where

$$M=\frac{B_sL}{\sqrt{\mu_0\mu\eta}},\quad \tau=\frac{4\rho_s^2\Omega^2L^4}{\mu^2},\quad \mathrm{Re}=\frac{\rho_sUL}{\mu},\quad \mathrm{Rm}=\frac{UL}{\eta} \tag{5}$$

are the Hartmann number, Taylor number, Reynolds number and magnetic Reynolds number respectively; μ is the (constant) shear viscosity, μ_0 is the magnetic permeability and η is the (constant) magnetic diffusivity. The pressure has been scaled with $\mu U/L$ and the parameter G is given by

$$G=\frac{\rho_s g L^2}{\mu U}. \tag{6}$$

The fluid is described by the perfect gas law and, for simplicity, is assumed to be isothermal; hence, the (dimensionless) equation of state becomes

$$p=\alpha\rho, \tag{7}$$

where

$$\alpha=\frac{k_B\rho_sLT}{m_A\mu U}, \tag{8}$$

with k_B the Boltzmann constant, m_A the atomic mass of the gas atoms and T the temperature of the gas.

In the analysis that follows, we shall pursue the traditional boundary layer approach and decompose the field, flow, density and pressure into main flow and boundary layer components. We shall assume the presence of an external magnetic field, oblique to the boundary, and choose its z-component as the scale of the magnetic field (B_s); hence,

$$\mathbf{B}=\mathfrak{B}\hat{\mathbf{e}}_x+\hat{\mathbf{e}}_z+\mathbf{b}, \tag{9}$$

where $\mathfrak{B}=B_{0x}/B_{0z}$ is the ratio of the horizontal to vertical components of the external field, and $\mathbf{b}$ is the induced magnetic field. Furthermore, we shall assume that the lower boundary, which we take to be at $z=0$, is rigid and non-slip, and that the horizontal component of the induced magnetic field vanishes (an illustrative condition commonly used in magnetoconvection studies). The solutions of the above equations are therefore subject to the following set of boundary conditions,

$$\mathbf{u}|_{z=0} = 0, \quad \mathbf{b}_H|_{z=0} = 0, \quad \left.\frac{\partial b_z}{\partial z}\right|_{z=0} = 0, \tag{10}$$

where the subscript H denotes the horizontal component.

3. Magnetic pressure pumping

In order to provide a clean analytical example of MPP – which is essentially the mass flow out of or into the Hartmann (or mixed Ekman–Hartmann) boundary layer, caused by compressibility effects and the influence of magnetic pressure on the density of the system – we assume the following scalings:

$$\tau^{1/2} \sim M^2 \gg 1; \quad \mathrm{Re} \sim M; \quad \mathrm{Rm} \sim 1; \quad \alpha \sim M; \quad G \sim M. \tag{11}$$

The first assumption, $\tau^{1/2} \sim M^2 \gg 1$, that the Coriolis and Lorentz forces are strong and comparable in magnitude, is crucial for the formation of the mixed Ekman–Hartmann boundary layer; assuming $\alpha \sim M$ allows for the density to be influenced at the leading order by the magnetic pressure variations, which is essential for our analysis. The orderings of the Reynolds numbers and of the gravitational term are adopted in order to simplify the analysis. If we denote the thickness of the boundary layer by δ and define the boundary layer variable ξ by

$$\xi = \frac{z}{\delta}, \tag{12}$$

then the approximate solution of equations (1)–(4) may be expressed as

$$\mathbf{u} \approx \sum_{n\geq 0} \epsilon^n \left[\mathbf{u}_n^M(x,y,z) + \mathbf{u}_n^B(x,y,\xi) - \mathbf{u}_n^{\mathrm{match}}(x,y)\right], \tag{13}$$

$$\mathbf{b} \approx \sum_{n\geq 0} \epsilon^n \left[\mathbf{b}_n^M(x,y,z) + \mathbf{b}_n^B(x,y,\xi) - \mathbf{b}_n^{\mathrm{match}}(x,y)\right], \tag{14}$$

$$\rho \approx \sum_{n\geq 0} \epsilon^n \left[\rho_n^M(x,y,z) + \rho_n^B(x,y,\xi) - \rho_n^{\mathrm{match}}(x,y)\right], \tag{15}$$

$$p \approx \sum_{n\geq -1} \epsilon^n \left[p_n^M(x,y,z) + p_n^B(x,y,\xi) - p_n^{\mathrm{match}}(x,y)\right], \tag{16}$$

where the superscripts M and B refer to the main flow and the boundary layer respectively, and where ϵ is a small parameter, with

$$\epsilon = \mathrm{O}\left(M^{-1}\right), \quad \delta = \mathrm{O}\left(M^{-1}\right). \tag{17}$$

In the standard form of the singular perturbation series, the solution is divided into three parts: the main flow, the boundary layer and the matching region (defined by $\xi \to \infty$, $z \to 0$). It follows that all dependent variables must satisfy matching relations of the form $\mathbf{u}^B(\xi \to \infty) = \mathbf{u}^M(z \to 0) = \mathbf{u}^{\mathrm{match}}(x,y)$; furthermore, the boundary conditions are applied to the boundary layer variables, i.e. $\mathbf{u}^B|_{\xi=0} = 0$, $\mathbf{b}_H^B\big|_{\xi=0} = 0$ and $\partial_\xi b_z^B\big|_{\xi=0} = 0$.

We seek steady solutions of the governing equations. The orderings assumed above then lead to the following two sets of equations.

In the main flow,

$$0 = \frac{M^2}{\mathrm{Rm}}\left(\nabla \times \mathbf{b}_0^M\right) \times \left(\mathfrak{B}\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z + \mathbf{b}_0^M\right) - \tau^{1/2}\rho_0^M \hat{\mathbf{e}}_z \times \mathbf{u}_0^M, \tag{18}$$

$$\mathbf{u}_0^M \cdot \nabla \mathbf{b}_0^M = \mathfrak{B}\partial_x \mathbf{u}_0^M + \partial_z \mathbf{u}_0^M + \mathbf{b}_0^M \cdot \nabla \mathbf{u}_0^M - \left(\mathfrak{B}\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z + \mathbf{b}_0^M\right)\left(\nabla \cdot \mathbf{u}_0^M\right) + \frac{1}{\mathrm{Rm}}\nabla^2 \mathbf{b}_0^M, \tag{19}$$

$$\nabla \cdot \left(\rho_0^M \mathbf{u}_0^M\right) = 0, \quad \nabla \cdot \mathbf{b}_0^M = 0; \tag{20}$$

and in the boundary layer, after introducing the approximate forms of variables (13)–(16) into the equations and taking advantage of the main flow balance (18)–(20),

$$b_{0x}^B = 0, \quad b_{0y}^B = 0, \quad b_{0z}^B = 0, \quad u_{0z}^B = 0, \tag{21}$$

$$\frac{\epsilon}{\delta}\partial_\xi\left(\rho_0^B u_{1z}^B - \varrho W\right) = -\partial_x\left(\rho_0^B u_{0x}^B - \varrho V_x\right) - \partial_y\left(\rho_0^B u_{0y}^B - \varrho V_y\right), \tag{22}$$

$$\mathrm{Rm}\,\partial_\xi \mathbf{u}_0^B + \frac{\epsilon}{\delta}\partial_\xi^2 \mathbf{b}_1^B = 0, \tag{23}$$

$$-\frac{\tau^{1/2}}{M^2}\varrho\hat{\mathbf{e}}_z \times \mathbf{V} = -\frac{\tau^{1/2}}{M^2}\rho_0^B \hat{\mathbf{e}}_z \times \mathbf{u}_0^B + \frac{\epsilon}{\mathrm{Rm}\,\delta}\partial_\xi \mathbf{b}_1^B + \frac{1}{\delta^2 M^2}\partial_\xi^2 \mathbf{u}_0^B, \tag{24}$$

$$\partial_\xi p_{-1}^B = -\frac{\epsilon^2 M^2}{\mathrm{Rm}}\mathfrak{B}\partial_\xi b_{1x}^B, \tag{25}$$

together with

$$p_{-1}^{M,B} = \epsilon\alpha\rho_0^{M,B}, \tag{26}$$

where the boundary conditions (10) were used to derive (21); to simplify the notation, we have introduced $\varrho \equiv \rho_0^M(z=0)$, $\mathbf{V} \equiv \mathbf{u}_0^M(z=0)$ and $W = u_{1z}^M(z=0)$. The latter is the actual *boundary layer pumping* (BLP), which can be either positive (with fluid pumped out of the boundary layer) or negative (when the boundary layer sucks in fluid from the main flow). The pumping term, i.e. the z-component of the velocity u_{1z}^B, can be easily found by the use of equation (22), after first resolving (23)–(26). From the last two equations, (25) and (26), we infer that the density ρ_0^B is a function of b_{1x}^B, and that, therefore, the problem is nonlinear, with the nonlinearity mainly due to the presence of the Coriolis force.

3.1. *Compressible Hartmann layer*

In this section we neglect the background rotation and concentrate solely on the effect of MPP. Thus, we set the Taylor number to zero, $\tau = 0$, and seek an analytical solution of the boundary layer equations (21)–(26). From (25) and (26) we obtain,

$$\rho_0^B(x,y,\xi) = \varrho - \frac{\epsilon M^2}{\alpha \mathrm{Rm}}\mathfrak{B}\left[b_{1x}^B(x,y,\xi) - b_{1x}^{\mathrm{match}}(x,y)\right]. \tag{27}$$

Furthermore, in this case, $\delta = \epsilon = M^{-1}$, and equations (23) and (24) can be easily solved to give

$$\mathbf{u}_0^B = -\mathbf{V}e^{-\xi} + \mathbf{V}, \tag{28}$$

$$\mathbf{b}_1^B = -\frac{\mathrm{Rm}\,\delta}{\epsilon}\mathbf{V}e^{-\xi} + \frac{\mathrm{Rm}\,\delta}{\epsilon}\mathbf{V}. \tag{29}$$

Using (27) we now introduce the density jump across the layer,

$$\triangle\rho \equiv \varrho - \rho(z=0) = -\frac{M^2\delta}{\alpha}\mathfrak{B}V_x, \tag{30}$$

where $\rho(z=0)$ is the density at the base of the layer. Using (27), (28) and (29), equation (22) can be integrated with respect to ξ to give the vertical momentum component,

$$\rho_0^B u_{1z}^B = \frac{M}{\alpha}\mathfrak{B}(\mathbf{V}\cdot\nabla V_x + V_x\nabla_H\cdot\mathbf{V})\left(e^{-\xi} - \frac{1}{2}e^{-2\xi}\right) - \nabla_H\cdot(\varrho\mathbf{V})e^{-\xi} + \varrho W. \tag{31}$$

With the use of the matching condition $u_{1z}^B(\xi\to\infty) = u_{1z}^M(z\to 0)$ and the boundary condition $u_{1z}^B(\xi=0)=0$, we obtain the pumping term,

$$\begin{aligned} W &= -\frac{1}{2\varrho}\frac{M}{\alpha}\mathfrak{B}(\mathbf{V}\cdot\nabla V_x + V_x\nabla_H\cdot\mathbf{V}) + \frac{1}{\varrho}\nabla_H\cdot(\varrho\mathbf{V}) \\ &= \frac{1}{2\varrho}\frac{\triangle\rho}{V_x}(\mathbf{V}\cdot\nabla V_x + V_x\nabla_H\cdot\mathbf{V}) + \frac{1}{\varrho}\nabla_H\cdot(\varrho\mathbf{V}). \end{aligned} \tag{32}$$

If the density jump across the layer is zero, then the MPP term,

$$\mathrm{MPP} = \frac{1}{2\varrho}\frac{\triangle\rho}{V_x}(\mathbf{V}\cdot\nabla V_x + V_x\nabla_H\cdot\mathbf{V}), \tag{33}$$

is zero, and we are left with only one term in the pumping expression, namely

$$\frac{1}{\varrho}\nabla_H\cdot(\varrho\mathbf{V}), \tag{34}$$

which we shall call the *basic pumping* (BP). Furthermore, if we assume that $\varrho=\mathrm{const}$ and that there is no vertical component in the leading order main flow velocity (or that it does not depend on z), the BP term is zero. However, we emphasise that the BP does not require compressibility, since even if the medium is incompressible it can still be non-zero, provided that the vertical main flow velocity depends on z. If the medium is compressible, no such requirement is needed, and the BP is, in general, non-zero. Moreover, almost any type of boundary layer, not only a Hartmann (or an Ekman) layer, would generate the BP, since it is a direct outcome of integration of equation (22) and the dependence of $\mathbf{V}$ and/or ϱ on x and y. It is also instructive to point out at this stage that BP and MPP do not necessarily have the same signs and can either cooperate (e.g. $\varrho=1$, $V_x=\mathrm{const}<0$, $V_y=-y$, $V_z=z \Longrightarrow \mathrm{MPP}=(M\mathfrak{B}/2\alpha)V_x<0$ and $\mathrm{BP}=-1$) or compete (e.g. $\varrho=1$, $V_x=\mathrm{const}>0$, $V_y=-y$, $V_z=z\Longrightarrow \mathrm{MPP}=(M\mathfrak{B}/2\alpha)V_x>0$ and $\mathrm{BP}=-1$) with each other.

To give a better physical description of the new phenomenon found here, namely the MPP, we calculate the pressure jump across the boundary layer,

$$\triangle p \equiv p_{-1}^B(\xi=0) - p_{-1}^M(z=0) = -\frac{\epsilon^2 M^2}{\mathrm{Rm}}\mathfrak{B}b_{1x}^{\mathrm{match}}. \tag{35}$$

This pressure difference, which is a result of the influence of strong magnetic pressure, accounts for a decrease (or increase) of the fluid density in the boundary layer, which via mass conservation drives the mass flow out of (or into) the boundary layer.

Not surprisingly, because of mass conservation, the pumping is non-zero only if there are horizontal gradients of the magnetic field, and thus also of the velocity.

3.2. *The competition between Ekman suction and magnetic pressure pumping*

We now proceed to an analysis of MPP in a mixed Ekman–Hartmann boundary layer and examine its competition with ES. As already pointed out, the full system of equations (21)–(26) is nonlinear, and thus in general we can only provide its numerical solution. Some analytical progress can, however, be made by simplifying the problem via a suitable linearisation of equation (24). Equations (25) and (26) lead to the same expression for the density as in the previous subsection,

$$\rho_0^B(x,y,\xi) = \varrho - \frac{\epsilon M^2}{\alpha\,\mathrm{Rm}}\mathfrak{B}\big[b_{1x}^B(x,y,\xi) - b_{1x}^{\mathrm{match}}(x,y)\big]. \tag{36}$$

Thus, under our linearisation, equation (24) becomes

$$-\frac{\tau^{1/2}}{M^2}\varrho\hat{\mathbf{e}}_z\times\mathbf{V} = -\frac{\tau^{1/2}}{M^2}\varrho\hat{\mathbf{e}}_z\times\mathbf{u}_0^B + \frac{\epsilon}{\mathrm{Rm}\,\delta}\partial_\xi\mathbf{b}_1^B + \frac{1}{\delta^2 M^2}\partial_\xi^2\mathbf{u}_0^B, \tag{37}$$

where in the linearisation procedure we have neglected terms only of the type $\mathbf{u}_0\mathbf{b}_1$ (i.e. ρ_0^B is replaced by ϱ). The neglect of the term $\mathbf{u}_0^B b_{1x}^B$ is crucial for analytical progress since its retention makes the boundary layer problem strongly nonlinear. On the other hand, it is only for the sake of simplicity that we neglect the term $\mathbf{u}_0^B b_{1x}^{\mathrm{match}}$, and hence our solution is only valid provided that the term $(\epsilon M^2/\alpha\,\mathrm{Rm})\mathfrak{B}\big[b_{1x}^B - b_{1x}^{\mathrm{match}}\big]\hat{\mathbf{e}}_z\times\mathbf{u}_0^B$ remains small compared with the other terms in equation (24). However, even with these simplifications, the model captures the essence of non-zero pumping. We specifically choose not to linearise equation (22), since here the nonlinear terms are necessary for a detailed study of the effects of compressibility and magnetic pressure.

The solutions of equations (23), (36) and (37) now take the form

$$u_{0x}^B = -e^{-\xi}\big[V_x\cos(\chi\xi) + V_y\sin(\chi\xi)\big] + V_x, \tag{38}$$

$$u_{0y}^B = -e^{-\xi}\big[V_y\cos(\chi\xi) - V_x\sin(\chi\xi)\big] + V_y, \tag{39}$$

$$b_{1x}^B = -\frac{\mathrm{Rm}\,\delta}{\epsilon(1+\chi^2)}e^{-\xi}\big[-V_x(\chi\sin(\chi\xi) - \cos(\chi\xi)) + V_y(\sin(\chi\xi) + \chi\cos(\chi\xi))\big] + \frac{\mathrm{Rm}\,\delta}{\epsilon(1+\chi^2)}\big(V_x + \chi V_y\big), \tag{40}$$

$$b_{1y}^B = -\frac{\mathrm{Rm}\,\delta}{\epsilon(1+\chi^2)}e^{-\xi}\big[-V_y(\chi\sin(\chi\xi) - \cos(\chi\xi)) - V_x(\sin(\chi\xi) + \chi\cos(\chi\xi))\big] + \frac{\mathrm{Rm}\,\delta}{\epsilon(1+\chi^2)}\big(V_y - \chi V_x\big), \tag{41}$$

$$\delta = \left(\frac{2}{M^2 + (M^4 + \varrho^2\tau)^{1/2}}\right)^{1/2}, \quad \chi = \frac{(M^4 + \varrho^2\tau)^{1/2} - M^2}{\varrho\tau^{1/2}} = \frac{1}{2}\varrho\tau^{1/2}\delta^2, \tag{42}$$

where $\varrho \equiv \rho_0^M(z=0)$, $\mathbf{V} \equiv \mathbf{u}_0^M(z=0)$, as previously, and $\triangle\rho \equiv \varrho - \rho(z=0) = -[M^2\delta/\alpha\big(1+\chi^2\big)]\mathfrak{B}\big(V_x + \chi V_y\big)$. Since ϱ enters the expressions both for the boundary layer thickness δ and for χ we will assume from now on that $\varrho = \mathrm{const}$. Having the solutions for $\mathbf{u}_0^B$ and b_{1x}^B, and using $u_{1z}^B\big|_{\xi=0} = 0$ and $u_{1z}^B(\xi \to \infty) = u_{1z}^M(z \to 0)$, we can easily obtain the BLP term,

$$\begin{aligned} W = &-\frac{\delta^2 M^2}{\epsilon\alpha(1+\chi^2)}\frac{\mathfrak{B}}{\varrho}\left[\frac{1-\chi^2}{2(1+\chi^2)}V_x\nabla_H\cdot\mathbf{V} + \frac{\chi}{1+\chi^2}V_y\nabla_H\cdot\mathbf{V} + \left(\frac{1-\chi^2}{1+\chi^2}-\frac{1}{2}\right)\mathbf{V}\cdot\nabla V_x\right.\\ &\left. + \chi\left(\frac{2}{1+\chi^2}-\frac{1}{2}\right)\mathbf{V}\cdot\nabla V_y + \left(\frac{\chi}{2}\left(\frac{1-\chi^2}{1+\chi^2}\right)V_x + \frac{\chi^2}{1+\chi^2}V_y\right)\omega_{0z}^M\right]\\ &+\frac{\delta}{\epsilon}\left(\frac{1}{1+\chi^2}\right)\nabla_H\cdot\mathbf{V} + \frac{\delta}{\epsilon}\left(\frac{\chi}{1+\chi^2}\right)\omega_{0z}^M, \end{aligned} \tag{43}$$

where ω_{0z}^M is the main flow vorticity at $z=0$. We can readily recover the result of the previous section by setting the Taylor number to zero: $\tau = 0 \Longrightarrow \chi = 0$, $\delta = M^{-1}$.

There are three effects that contribute to the BLP in expression (43). The general structure of the BLP can therefore be expressed as

$$\mathrm{BLP} = \mathrm{MPP} + \mathrm{BP} - \mathrm{ES}. \tag{44}$$

The basic pumping term,

$$\mathrm{BP} = \frac{\delta}{\epsilon}\left(\frac{1}{1+\chi^2}\right)\nabla_H\cdot\mathbf{V}, \tag{45}$$

results from the fact that u_z^M depends on z or that the medium is compressible (or both); it is non-zero even if the magnetic pressure does not influence the density at the leading order. Moreover, the expression for BP would be more complicated if we had not assumed $\varrho = \text{const}$. The magnetic pressure pumping term,

$$\mathrm{MPP} \sim -\frac{\delta^2 M^2}{\epsilon\alpha(1+\chi^2)}\frac{\mathfrak{B}}{\varrho} = \frac{\delta\triangle\rho}{\epsilon\varrho(V_x+\chi V_y)}, \tag{46}$$

is an outcome of the magnetic pressure influencing the density distribution, an effect somewhat reminiscent of magnetic buoyancy. The final term,

$$\mathrm{ES} = \frac{\delta}{\epsilon}\left(\frac{\chi}{1+\chi^2}\right)\omega_{0z}^M, \tag{47}$$

is simply the Ekman suction, which can be positive or negative depending on the sign of the vorticity of the main flow. In general, there can be competition between these effects. In particular, when the main flow vorticity ω_{0z}^M is negative and MPP is positive, the latter may overcome the ES and produce positive BLP if the magnetic field is sufficiently strong. To provide an example we may take,

$$\left.\begin{aligned} &V_x = \lambda y,\ V_y = \text{const},\ V_z = 0,\\ &b_{1x}^M|_{z=0} = \kappa\big(\lambda y + \chi V_y\big),\ b_{1y}^M|_{z=0} = \kappa\big(V_y - \chi\lambda y\big),\ b_{1z}^M|_{z=0} = \chi\kappa\lambda z + \text{const},\\ &\varrho = \text{const}, \end{aligned}\right\} \tag{48}$$

where $\kappa = \mathrm{Rm}\,\delta/\epsilon\big(1+\chi^2\big)$, and for which the main flow vorticity

$$\omega_{0z}^M = -\lambda, \tag{49}$$

and hence is negative, provided $\lambda > 0$. This leads to the following pumping expression,

$$W = -\frac{\delta^2 M^2}{\epsilon\alpha(1+\chi^2)}\frac{\mathfrak{B}}{\varrho}\left[\lambda V_y\left(\frac{1-\chi^2}{1+\chi^2}-\frac{1}{2}\right)-\chi\lambda\left(\frac{1-\chi^2}{2(1+\chi^2)}\lambda y+\frac{\chi V_y}{1+\chi^2}\right)\right]-\frac{\delta}{\epsilon}\left(\frac{\chi\lambda}{1+\chi^2}\right), \tag{50}$$

which clearly demonstrates the competition between magnetic pressure pumping and Ekman suction. The linearisation procedure requires that the term $(\epsilon M^2/\alpha\,\mathrm{Rm})\mathfrak{B}\big(b_{1x}^B - b_{1x}^{\mathrm{match}}\big)\mathbf{u}_0^B$ be small compared with $\mathbf{u}_0^B$ and $1/\mathrm{Rm}\mathbf{b}_1^B$; thus V_x and V_y must be small and hence the MPP term in expression (50) is much smaller than that representing ES. However, the influence of the magnetic pressure on the BLP and the fact that MPP can either compete or cooperate with ES can clearly be seen. Furthermore, the gradients of velocity need not necessarily be small, which means that, for example, we could assume the main flow velocity to be $V_x = \lambda y + \tilde{\lambda} x$, $V_y = -\tilde{\lambda} y$, with $\lambda \ll 1$, but $\tilde{\lambda} \sim 1$, and consider small values of the coordinates x and y. This would lead to comparable MPP and ES and an actual competition between those effects, even in the linear regime.

For comparison with the nonlinear system examined below, we consider a specific numerical example. Taking $M^2 = 10^6$, $\epsilon = \delta$, $\alpha = M = \tau^{1/4}$, and $\varrho = 1$, $\lambda = 0.65$, $V_y = 1$ at $y = 1$, we obtain $W \approx 0.0174\mathfrak{B} - 0.2298$, which is positive (i.e. the fluid is pumped out of the boundary layer) for a strong horizontal component of the external magnetic field ($\mathfrak{B} > 13.2$), and negative (with the fluid being sucked into the boundary layer) for weaker horizontal fields. Varying the strength of the shear, λ, gives a similar effect, i.e. the MPP increases with the strength of the shear. In other words, a strong enough shear may lead to MPP overcoming ES and producing an outward mass flow from the boundary layer. The dependence on shear though, as is clear from expression (50), is not monotonic. For the above choice of parameter values and with $\mathfrak{B} = 1.0$, the BLP is negative for $\lambda = 1$, with $W = -0.2869$. If the shear is increased to $\lambda = 2$, the suction is increased and now $W = -0.3463$. However, for even stronger shear, the BLP again becomes positive (e.g. $W = 0.2177$ when $\lambda = 4$) and stays positive for all $\lambda > \lambda_c$ where $\lambda_c \approx 3.5$.

To demonstrate this effect more clearly, we have resolved numerically the full nonlinear system of the boundary layer equations,

$$\partial_\xi \mathbf{u}_0^B = -\frac{1}{\mathrm{Rm}\,\delta}\partial_\xi^2 \mathbf{b}_0^B, \tag{51}$$

$$-\frac{\tau^{1/2}}{M^2}\varrho\hat{\mathbf{e}}_z \times \mathbf{V} = -\frac{\tau^{1/2}}{M^2}\left[\varrho - \frac{M^2\mathfrak{B}}{\alpha\,\mathrm{Rm}}\big(b_{0x}^B - b_{0x}^M\big)\right]\hat{\mathbf{e}}_z \times \mathbf{u}_0^B + \frac{1}{\mathrm{Rm}\,\delta}\partial_\xi \mathbf{b}_0^B + \frac{1}{\delta^2 M^2}\partial_\xi^2 \mathbf{u}_0^B, \tag{52}$$

$$\frac{\epsilon}{\delta}\partial_\xi\big(\rho_0^B u_{1z}^B - \varrho W\big) = -\partial_x\big(\rho_0^B u_{0x}^B - \varrho V_x\big) - \partial_y\Big(\rho_0^B u_{0y}^B - \varrho V_y\Big), \tag{53}$$

$$\rho_0^B = \varrho - \frac{M^2}{\alpha\,\mathrm{Rm}}\mathfrak{B}\big[b_{0x}^B - b_{0x}^{\mathrm{match}}\big], \tag{54}$$

with the boundary conditions $\mathbf{u}_0^B(\xi = 0) = 0$, $\mathbf{u}_0^B(\xi \to \infty) = \mathbf{V}$, $\mathbf{b}_0^B(\xi = 0) = 0$, $\mathbf{b}_0(\xi \to \infty) = \mathbf{b}_0^{\mathrm{match}}$, in the rectangular domain $0 \le y \le 1.2$, $0 \le z \le 10$. The main flow velocity and magnetic field were chosen as in (48), and the parameter values were taken as above, i.e. $M^2 = 10^6$, $\epsilon = \delta$, $\mathrm{Rm} = \alpha = M = \tau^{1/4}$, and $\lambda = 0.65$, $\varrho = 1$, $V_y = 1$. The solution for the horizontal magnetic field strength $\mathfrak{B} = 10$ is presented in figure 1(a). We see that for small values of the independent variable y, in the area where the vertical

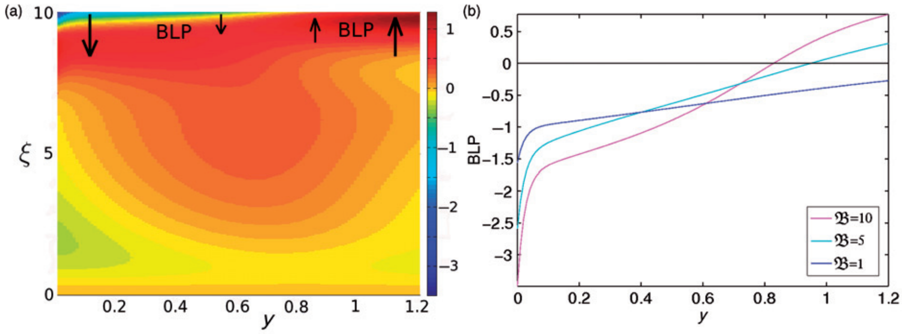

Figure 1. BLP for $M^2 = 10^6$, $\epsilon = \delta$, $\mathrm{Rm} = M$, $\alpha = M = \tau^{1/4}$, $\lambda = 0.65$, $\varrho = 1$ and the main flow velocity field chosen as in (48). (a) Colour map of the vertical momentum $\rho_0^B u_{1z}^B$, with the horizontal component of the external magnetic field $\mathfrak{B} = 10$. The ES, caused by the anti-cyclonic vortex in the main flow, dominates, and the fluid is sucked into the boundary layer for $y \lesssim 0.83$. For larger values of y, however, MPP becomes stronger and overcomes ES. (b) Comparison of the BLP for different values of the horizontal magnetic field. Only for sufficiently strong $\mathfrak{B}$ is the MPP able to overcome the ES, and produce positive pumping for $y \gtrsim 0.83$.

vorticity of the main flow is negative, the boundary layer sucks in the outside fluid. However, as y increases, the MPP overcomes the ES and in the region where $\omega_{0z}^M < 0$ the fluid is pumped out of the boundary layer into the main flow. Figure 1(b) shows the increase in the BLP strength with the increase in the horizontal magnetic field intensity $\mathfrak{B}$. The comparison with the linear results in (50) highlights the importance of the nonlinear effects in the problem, since for $\mathfrak{B} = 10$ at $y = 1$, BLP calculated from (50) is still negative. In this example, no BP is present, since for our choice of the main flow solution, as in (48), $\nabla_H \cdot \mathbf{V} = 0$. This effect, however, is in general present for a compressible medium.

One more word of comment seems to be necessary since, in the parameter regime chosen for this section, defined by (11), the main flow leading-order momentum equation (18) constitutes a balance between the Coriolis and Lorentz forces without the inclusion of the pressure gradient (changing the order of the pressure gradient and including it in the main flow balance by setting $\alpha \sim M^2$ and leaving other scalings in (11) unchanged, would lead to a decrease in the MPP by an order of magnitude in M). Although such a limit may seem a little unusual, our simple model provides a clear example of the MPP. Clearly, in applications to stellar interiors, for example, further, more complex, effects must be taken into account, rendering the main flow balance more physical. The MPP would of course then be modified; however, it is unlikely to be completely suppressed, since the leading order density would still be affected by the magnetic pressure variations. In the following section, we consider different parameter regimes, more closely related to that of the solar tachocline, to investigate the role of MPP in more astrophysically realistic situations.

4. The solar tachocline regime

In section 3, in order to illustrate the effect of MPP, we chose a very simple model, characterised by an isothermal equation of state, and governed by the specific ordering (11). Moreover, the boundary conditions for the induced magnetic field and the velocity, i.e. (10), may be regarded as idealised. However, the simplifications of our model were introduced only so as to present a clear example of the analysed phenomenon, and relaxing them will not necessarily lead to the suppression or elimination of MPP. In fact it is likely to persist, although in a modified form, for many types of boundary conditions leading to the formation of a boundary layer.

One possible astrophysical application of the ideas developed in this article is to the solar tachocline, which may, in a simplification of the true physics, be described as an Ekman or Ekman–Hartmann layer (see Garaud 2007). However, as stressed above, MPP requires the magnetic pressure to influence the density at leading order, and hence the magnetic pressure has to be of the same order as the gas pressure (i.e. plasma β of order unity). Given that β at the base of the convection zone far exceeds unity, it may be anticipated therefore that the role of MPP in the dynamics of the solar tachocline is unlikely to be significant. That said, it is of interest to examine the estimates of the magnitudes of the governing parameters for the tachocline, and then to discuss under what circumstances MPP may be important in an astrophysical context.

The various important dimensional quantities, evaluated at the base of the convection zone, may be estimated as follows:

$$g \approx 5.4 \times 10^2\,\mathrm{m\,s^{-2}}, \qquad \rho_s \approx 210\,\mathrm{kg\,m^{-3}}, \qquad T \approx 2.3 \times 10^6\,\mathrm{K},$$

$$\Omega \approx 2.7 \times 10^{-6}\,\mathrm{s^{-1}}, \qquad U \approx 10\,\mathrm{m\,s^{-1}}, \qquad L \approx 2.1 \times 10^8\,\mathrm{m}, \tag{55}$$

$$\nu \equiv \frac{\mu}{\rho_s} \approx 2.7 \times 10^{-3}\,\mathrm{m^2\,s^{-1}}, \qquad \eta \approx 4.1 \times 10^{-2}\mathrm{m^2\,s^{-1}}, \qquad B_T \approx 10^5 G.$$

The values of g, ρ_s, T, Ω, L, ν and η at the base of the convection zone, taken from Gough (2007), are reasonably well-determined. We have adopted for L the depth of the whole convection zone; however, it could plausibly be chosen to be smaller by about one order of magnitude. The estimates for U, the magnitude of the toroidal velocity, and B_T, the strength of the toroidal magnetic field, taken from Gough (2007) and Garaud (2007), are much more uncertain. For consistency with the foregoing analysis, when estimating the parameter values, we have adopted the strength of the poloidal component of the magnetic field, which we have assumed to be two orders of magnitude weaker than the strength of the toroidal field. In addition, we make the assumption that the gas is mainly hydrogen, and that therefore

$$m_A \approx 1.7 \times 10^{-27}\,\mathrm{kg}. \tag{56}$$

This leads to the following estimates of the parameters in equations (1) and (2),

$$M \approx 10^{11}, \quad \tau^{1/2} \approx 10^{14}, \quad \mathrm{Re} \approx 10^{12}, \quad \mathrm{Rm} \approx 10^{11}, \quad G \approx 10^{20}, \quad \alpha \approx 10^{20}. \tag{57}$$

This parameter regime is thus suggestive of the orderings,

$$\tau^{1/2} \sim M, \quad \mathrm{Re} \sim M, \quad \mathrm{Rm} \sim M, \quad \alpha \sim M^2, \quad G \sim M^2, \tag{58}$$

although it can be seen that terms that differ fairly considerably, such as M and $\tau^{1/2}$, are assigned comparable magnitudes. Under the ordering (58), the governing equations

take the following form:
in the main flow,

$$\partial_z p_{-2}^M = -G\rho_0^M \quad \text{and} \quad \partial_x p_{-2}^M = \partial_y p_{-2}^M = 0, \tag{59}$$

$$\begin{aligned} \mathrm{Re}\, \rho_0^M \big(\mathbf{u}_0^M \cdot \nabla\big)\mathbf{u}_0^M = -\nabla p_{-1}^M + \frac{M^2}{\mathrm{Rm}} \big(\nabla \times \mathbf{b}_0^M\big) \times \big(\mathfrak{B}\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z + \mathbf{b}_0^M\big) \\ - \tau^{1/2} \rho_0^M \hat{\mathbf{e}}_z \times \mathbf{u}_0^M - \epsilon G \rho_1^M \hat{\mathbf{e}}_z, \end{aligned} \tag{60}$$

$$\big(\mathbf{u}_0^M \cdot \nabla\big)\mathbf{b}_0^M = \mathfrak{B}\partial_x \mathbf{u}_0^M + \partial_z \mathbf{u}_0^M + \big(\mathbf{b}_0^M \cdot \nabla\big)\mathbf{u}_0^M - \big(\mathfrak{B}\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_z + \mathbf{b}_0^M\big)\nabla \cdot \mathbf{u}_0^M, \tag{61}$$

$$\nabla \cdot \rho_0^M \mathbf{u}_0^M = 0, \quad \nabla \cdot \mathbf{b}_0^M = 0, \tag{62}$$

$$p_{-2}^M = \epsilon^2 \alpha \rho_0^M, \quad p_{-1}^M = \epsilon^2 \alpha \rho_1^M; \tag{63}$$

and in the boundary layer,

$$b_{0z}^B = 0, \quad u_{0z}^B = 0, \tag{64}$$

$$\frac{\epsilon}{\delta} \partial_\xi \big(\rho_0^B u_{1z}^B - \varrho W\big) = -\partial_x \big(\rho_0^B u_{0x}^B - \varrho V_x\big) - \partial_y \Big(\rho_0^B u_{0y}^B - \varrho V_y\Big), \tag{65}$$

$$\mathrm{Rm}\, \delta \partial_\xi \mathbf{u}_0^B + \partial_\xi^2 \mathbf{b}_0^B = 0, \tag{66}$$

$$-\nabla_H \big(p_{-2}^B - p_{-2}^{\mathrm{match}}\big) + \frac{1}{\mathrm{Rm}\, \delta} \partial_\xi \mathbf{b}_0^B + \frac{1}{\delta^2 M^2} \partial_\xi^2 \mathbf{u}_0^B = 0, \tag{67}$$

$$\partial_\xi p_{-2}^B = 0, \quad \partial_\xi p_{-1}^B = -\frac{\epsilon M^2}{\mathrm{Rm}} \mathfrak{B} \partial_\xi b_{0x}^B - \epsilon \delta G \big(\rho_0^B - \varrho\big), \tag{68}$$

$$p_{-2}^B = \epsilon^2 \alpha \rho_0^B, \quad p_{-1}^B = \epsilon^2 \alpha \rho_1^B, \tag{69}$$

where, as in section 3, $\varrho \equiv \rho_0^M(z=0)$, $\mathbf{V} \equiv \mathbf{u}_0^M(z=0)$ and $W = u_{1z}^M(z=0)$. The boundary layer equations lead to $p_{-2}^B = \text{const}$, which means that the leading order density in the boundary layer ρ_0^B would also have to be constant. The magnetic pressure influences the density at first order, which would lead to much weaker MPP and BP at second order. In other words, the fact that the plasma β, the ratio of gas pressure to magnetic pressure, is very large in the Sun means that the density cannot be influenced by magnetic pressure variations at leading order.

To present some physical examples in which the MPP would play a significant role, let us now consider slightly modified scalings to those applicable to the solar tachocline, given by (58). Assuming, for example, as in section 3,

$$\alpha \sim M \quad \text{and} \quad G \sim M, \tag{70}$$

leads to changes only in the last three equations, (67)–(69), which become

$$\frac{1}{\mathrm{Rm}\, \delta} \partial_\xi \mathbf{b}_0^B + \frac{1}{\delta^2 M^2} \partial_\xi^2 \mathbf{u}_0^B = 0, \tag{71}$$

$$\partial_\xi p_{-1}^B = -\frac{\epsilon M^2}{\mathrm{Rm}} \mathfrak{B} \partial_\xi b_{0x}^B, \tag{72}$$

$$p_{-1}^B = \epsilon \alpha \rho_0^B. \tag{73}$$

We now have the desired influence of the magnetic pressure on the leading order density, and thus non-zero first order MPP. The BP term is, in general, also non-zero.

However, the Coriolis force does not enter the leading order equations, and therefore the ES influences the BLP only at higher orders. As a result, no competition can occur between ES and MPP; however, there is still the possibility of competition between BP and MPP. The set of equations (64)–(66), (71)–(73) is in fact the same as that describing the pure Hartmann layer, discussed in section 3.1, except that the magnetic field is stronger by an order of magnitude. Hence the formula for the BLP, expression (32), derived in that section is exactly valid here. Furthermore, in the parameter regime chosen here, the main flow momentum equation (60) no longer represents a balance of just two forces (Coriolis and Lorentz), but also includes the pressure gradient.

Finally, with respect to this ordering, we should like to point out that the assumption $G \sim M$, which eliminates gravity from our theory, is not strictly necessary, and that taking $G \sim M^2$ would not violate the conclusions. The density distribution would be different, mainly giving stable stratification in the lower parts of the layer; however, all three effects discussed in the pumping expression, i.e. MPP, BP and, in general, ES, would still be present.

The scalings (58) proposed above lead to exclusion of the Coriolis effect in the BLP at leading order, since the Sun's rotation is not very rapid. However, if we consider rapid rotation, i.e. $\tau^{1/2} \sim M^2$ and leaving all the other relations in (58) unchanged, an additional assumption of very strong horizontal magnetic field with respect to the vertical field, i.e.

$$\tau^{1/2} \sim M^2, \quad \mathrm{Re} \sim M, \quad \mathrm{Rm} \sim M, \quad \alpha \sim M^2, \quad \mathfrak{B} \sim M, \tag{74}$$

leads to a system of equations similar to those in section 3.2.

In the main flow,

$$0 = -\epsilon^{-2}\nabla p_{-2} - \tau^{1/2}\rho_0^M \hat{\mathbf{e}}_z \times \mathbf{u}_0^M + \frac{M^2\mathfrak{B}}{\mathrm{Rm}}\left(\nabla \times \mathbf{b}_0^M\right) \times \hat{\mathbf{e}}_x, \tag{75}$$

$$0 = \mathfrak{B}\partial_x \mathbf{u}_0^M - \left(\mathfrak{B}\nabla \cdot \mathbf{u}_0^M\right)\hat{\mathbf{e}}_x, \tag{76}$$

$$\nabla \cdot \left(\rho_0^M \mathbf{u}_0^M\right) = 0, \quad \nabla \cdot \mathbf{b}_0^M = 0; \tag{77}$$

and in the boundary layer,

$$b_{0z}^B = 0, \quad u_{0z}^B = 0, \tag{78}$$

$$\frac{\epsilon}{\delta}\partial_\xi\left(\rho_0^B u_{1z}^B - \varrho W\right) = -\partial_x\left(\rho_0^B u_{0x}^B - \varrho V_x\right) - \partial_y\left(\rho_0^B u_{0y}^B - \varrho V_y\right), \tag{79}$$

$$\mathrm{Rm}\,\mathfrak{B}\delta\partial_x\left(\mathbf{u}_0^B - \mathbf{V}\right) + \mathrm{Rm}\,\partial_\xi \mathbf{u}_0^B - \mathrm{Rm}\,\mathfrak{B}\delta\Big[\nabla_H \cdot \left(\mathbf{u}_0^B - V\right) + \frac{\epsilon}{\delta}\partial_\xi u_{1z}^B\Big]\hat{\mathbf{e}}_x + \frac{1}{\delta}\partial_\xi^2 \mathbf{b}_0^B = 0, \tag{80}$$

$$-\frac{\tau^{1/2}}{M^2}\varrho\hat{\mathbf{e}}_z \times \mathbf{V} = -\frac{1}{\epsilon^2 M^2}\nabla_H p' - \frac{\tau^{1/2}}{M^2}\rho_0^B \hat{\mathbf{e}}_z \times \mathbf{u}_0^B + \frac{1}{\mathrm{Rm}\,\delta}\partial_\xi \mathbf{b}_0^B + \frac{1}{\delta^2 M^2}\partial_\xi^2 \mathbf{u}_0^B, \tag{81}$$

$$\partial_\xi p_{-2}^B = -\frac{\epsilon^2 M^2}{\mathrm{Rm}}\mathfrak{B}\partial_\xi b_{0x}^B, \tag{82}$$

together with

$$p_{-2}^{M,B} = \epsilon^2 \alpha \rho_0^{M,B}, \tag{83}$$

where

$$p' = p_{-2} - p_{-2}^{\mathrm{match}} + \frac{\epsilon^2 M^2 \mathfrak{B}}{\mathrm{Rm}} \left(b_{0x}^B - b_{0x}^{\mathrm{match}} \right), \tag{84}$$

and where, for simplicity, we have again assumed that $G \sim M$. The most important differences between the above set of equations and the system (18)–(26) are that the leading order, main flow, momentum equation now simply constitutes a magnetostrophic balance, and also that additional terms now appear in the boundary layer induction equation. These terms significantly change the character of the boundary layer equations, from ordinary to partial differential equations. Also, the boundary layer induction and continuity equations are now coupled via the term $(\epsilon/\delta)\partial_\xi u_{1z}^B$, which makes the problem strongly nonlinear. These additional terms do not, however, act to suppress the MPP, which is still present in this case, since the density continues to be influenced by the magnetic pressure at leading order (cf. equations (82) and (83)) and the continuity equation (79) remains unchanged. Moreover, if we follow the linearisation procedure introduced in section 3.2, the simple assumption that the boundary layer solution and the main flow velocity at $z = 0$ depend only on y yields precisely the same system of boundary layer equations as obtained in section 3.2; hence, the results obtained there are also valid here. It is important to emphasise that a crucial feature of the model with the parameter regime given by (74) is that the magnetic field $\mathfrak{B} \sim M$ has to be very strong in order for the MPP to enter the dynamics and to be able to overcome the ES.

5. Concluding remarks

Through our study of the influence of compressibility on magnetic boundary layers of Hartmann and mixed Ekman–Hartmann type, we have identified a new and potentially important effect, which can influence the boundary layer pumping. The fluid can be pumped out of or sucked into the boundary layer not only via the well-known mechanism of Ekman suction, but also as a result of the change in density distribution caused by the magnetic pressure (MPP). The compressibility of the main flow can also create non-zero pumping (BP, which may also be due to the presence of the vertical main flow velocity and its dependence on the distance from the boundary, and hence may also be non-zero for an incompressible medium). The net result of these three effects (ES, MPP and BP) can be to cause either boundary layer suction or pumping, depending on the characteristics of the solution and the relations between the parameters describing the system. In particular, as we have shown in section 3.2, in the region of negative main flow vorticity, ES can be overcome by MPP if the horizontal magnetic field is strong enough, causing the fluid to be pumped out of the boundary layer into the main flow. The crucial characteristic of the analysed problem is that the magnetic field has to become strong before the fluid can be pumped out of the layer (dragging the magnetic field with it). This is because, first, the Hartmann numbers must be large in order for the boundary layers to form, and, second, the intensity of the MPP increases with the strength of the horizontal magnetic field, which, in the regions where

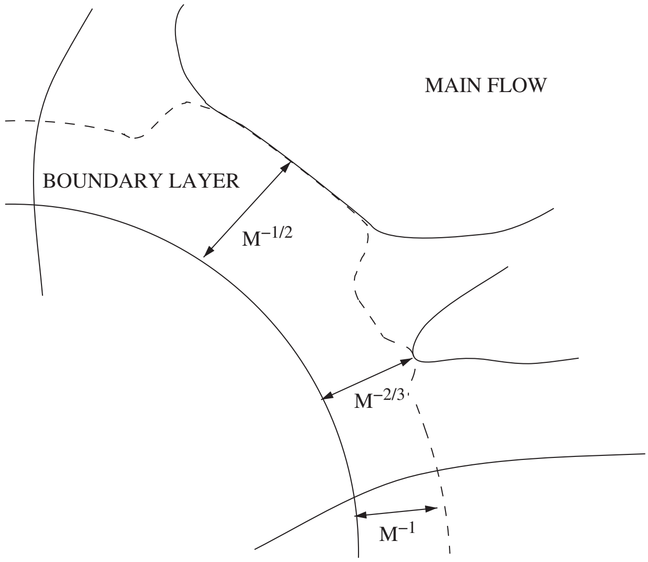

Figure 2. The width of the magnetic boundary layer in three cases: (a) $\delta \sim M^{-1}$ if the vertical component of the magnetic field is of the same order as the value of the whole magnetic field vector; (b) $\delta \sim M^{-2/3}$ in the neighbourhood of an isolated point, where a magnetic field line is tangent to the boundary; (c) $\delta \sim M^{-1/2}$ in a region where the magnetic field is tangent to the boundary.

other effects produce suction, has to be sufficiently strong to overcome the ES and/or the basic suction.

We have studied here only the case for which the vertical component of the external magnetic field is significant. However, the structure and properties of any magnetic boundary layer are strongly dependent on the magnitude of the vertical component of the field. In the case where the magnetic field is tangent to the boundary, the width of the boundary layer is $\delta \sim M^{-1/2}$ (Roberts 1967a), whereas if there exists only one isolated point at which the external magnetic field is tangent to the boundary, the width of the layer is $\delta \sim M^{-2/3}$ (Roberts 1967b). The three possible cases, including that analysed here, are shown schematically in figure 2. Since the BLP appears at the first order of the perturbation analysis, it is proportional to the parameter of perturbative expansion, δ, and hence we could expect it to be stronger in those cases when the vertical component of the magnetic field vanishes. Furthermore, the presence of the vertical component of the field and also the conductivity of the boundary strongly influence the currents flowing out of or into the boundary layer (Dormy *et al.* 2002, Mizerski and Bajer 2007, Soward and Dormy 2010), which may also have important consequences for the dynamics of the flow at the base of the solar convective zone.

The Sun is not a very rapidly rotating star, and therefore the Coriolis force enters the boundary layer dynamics only at higher orders (at first order, to be precise). This means that ES occurs at $O(M^{-2})$ and hence, in the solar tachocline, ES does not enter the leading order BLP. Moreover, the magnetic pressure is about five orders of magnitude smaller than the gas pressure, and thus the MPP is weak and unlikely to play any significant role in the dynamics. Competition between ES and MPP could, however, happen in more rapidly rotating stellar objects, where the proposed scalings are more likely to occur.

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