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Operational control of water reservoir system with minimax objectives

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ABSTRACT: The optimal control problem for the Wupper Reservoir System case is presented. The aim of control is to realize many different tasks concerning the water resources distribution. However, multiobjective aspects of the problem are not the main subject of this paper. Our aim is rather to present a general concept of solving the problems formulated over the infinite time horizon and with minimax-type objectives (the worst case approach). The algorithm, based on the reachable sets concept and using the periodicity of control process, is discussed. We focus our attention on its theoretical aspects.

1 FORMULATION OF MINIMAX PERIODICAL CONTROL PROBLEM

A multireservoir system is considered, described over the infinite time horizon by the discrete state equation:

$$x_{t+1} = f_t(x_t, m_t, \varepsilon_t), \qquad t = 0,1,\ldots \qquad (1.1)$$

where $x_t \in \mathbf{X} = \mathbf{R}^n$ represents the process *state* value at instant **t**, $m_t \in \mathbf{M}$ - the *control* value and $\varepsilon_t \in \mathbf{E}$ - *disturbance* value within the stage **t** (i.e. between **t** and **t+1** time instants).

The precise description of both this model and its variables x_t, m_t, ε_t for the Wupper Reservoir System case is given in (Napiórkowski et al. 1997). It is worthwhile to underline here that the state vector x_t represents not only the physical system state (reservoir water contents). It represents also some artificial state variables, related to the calculation of some performance objectives values (see Sect.1.1), as well as to the representation of system operator knowledge of disturbances (scenario-type model of inflows (Karbowski et al. 1984)).

The aim is to minimize N performance objectives evaluating particular aspects of system performance. N functions $\mathbf{g}^j$ are given, depending on time **t** and current values of state, control and disturbance. The control goal is to minimize (in a multicriteria sense) the maximum (the worst) value of each function $\mathbf{g}^j$, j = 1, ... , N. This maximum value is taken over all time instants **t** = 0,1, ... , over every possible disturbance realization $\varepsilon_t \in \mathbf{E}_t(x_t) \subseteq \mathbf{E}$, **t** = 0,1, ... and over every initial state value $x_0 \in X_0 \subseteq \mathbf{X}$. The *decision variables* are the set X_0 and the controls; however, not values m_t, but the *control laws* $\mu_t \in \mathbf{M}^{\mathbf{X}}$, i.e. functions which connect the control value m_t and a state x_t. The following constraints are imposed on these control laws:

$$\mu_t(x_t) \in \mathbf{U}_t(x_t) \subseteq \mathbf{M}, \qquad t = 0,1,\ldots \qquad (1.2)$$

where $\mathbf{U}_t$ is a given mapping depending on **t**.

Thus, we search for the minimal (in Pareto sense) value of the vector **J** composed of N performance objectives:

$$\min_{x_0 \in X_0;\ \mu_t,\ t=0,1,\ldots} \mathbf{J} = [\mathbf{J}^1, \ldots, \mathbf{J}^N] \qquad (1.3)$$

where

$$\mathbf{J}^j = \max\{ \mathbf{g}^j_t(x_t, \mu_t(x_t), \varepsilon_t): x_0 \in X_0,\ \mu_t(x_t) \in \mathbf{U}_t(x_t),\ \varepsilon_t \in \mathbf{E}_t(x_t),\ t=0,1,\ldots \} \qquad (1.4)$$

$X_0 \subseteq \mathbf{X}$, $\mu_t \in \mathbf{M}^{\mathbf{X}}$ and x_t, ε_t satisfy the equation (1.1) with $m_t = \mu_t(x_t)$, $t = 0,1,\ldots$.

To solve the above problem we apply a special approach based on the following premises and concepts:

1) the reference point method for multiobjective optimization
2) the periodicity of control process

3) a general concept based on the reachable sets idea.

It will be shown that, using 1) and 2), one can transform our problem to a form which fits directly the concept 3) discussed in detail in Section 2.

1.1 *Application of the reference point method. Periodicity of the control process.*

According to the *reference point method* (Wierzbicki 1982), the problem (1.3) is now presented as:

$$\min \ s \tag{1.5}$$

subject to: $J^j \le s + \theta^j, \quad j = 1, \dots, N$

where $\theta = (\theta^1, \dots, \theta^N)$, called *reference point*, is a given element of the objective space.

Taking into account the definition (1.4) of J^j, we obtain the following *two-level* problem:

Find a set $X_0 \subseteq \mathbf{X}$ and a sequence of control laws $\mu_t: \mathbf{X} \to \mathbf{M}$, $t = 0,1, \dots$, which minimize the value s:
min s, subject to the condition:

$$\forall j=1,..,N \quad \forall x_0 \in X_0 \quad \forall t=0,1,.. \quad \forall \varepsilon_t \in \mathbf{E}_t(x_t): \\ [\mu_t(x_t) \in U_t(x_t) \wedge g^j_t(x_t, \mu_t(x_t), \varepsilon_t) \le s + \theta^j] \tag{1.6}$$

where x_t, ε_t satisfy the equation (1.1) with $m_t = \mu_t(x_t)$, $t = 0,1,\dots$.
Then, the *lower level*, (1.6), consists in solving a set of parametric constraints (with parameter **s**).

The *periodicity* of the problem (1.3) - (1.4) (and also, of the problem (1.6)) with respect to time **t**, plays a crucial role in its solution. It occurs in a water system case, due to the natural, physical features of such an object. Its natural *period* (cycle), denoted here by T, is one year. Notice, however, that the assumed periodicity of inflows does not have a direct, trivial character. The considered model respects the uncertainty of inflow realization in the following year, even if the previous year realization is known. The periodicity occurs then in a generalized state space, allowing us to make use of an *a priori* knowledge of the possible scenarios set and only this set is in fact a periodic one.
Formally, we assume the following periodicity relations in the considered control process:

$$\forall t: \quad f_t(x, y, z) = f_{t+T}(x, y, z)$$
$$\forall t: \quad U_t(x) = U_{t+T}(x)$$
$$\forall t: \quad \mathbf{E}_t(x) = \mathbf{E}_{t+T}(x)$$
$$\forall j = 1, \dots, N \ \forall t: \quad g^j_t(x, y, z) = g^j_{t+T}(x, y, z)$$

The following three objective functions $\mathbf{g}^j$, used in the numerical implementation, are a good example of the latter periodicity:

$$g^1_t = \bar{d}_t, \qquad g^2_t = \mathbf{d}_t^{max}, \qquad g^3_t = \tau_t$$

where $\bar{d}_t$, $\mathbf{d}_t^{max}$, τ_t are artificial state variables aimed at evaluating 3 aspects of water supply/deficit process: average annual deficit, maximum deficit value and continuous annual deficit time. These state variables are defined (periodically) as follows ($H(\cdot)$ is Heaviside's function):

$$\bar{d}_t = 0, \quad \mathbf{d}_t^{max} = 0, \quad \tau_t = 0 \qquad \text{if } t = k \cdot T;$$
$$\bar{d}_t = \bar{d}_{t-1} + d_t, \qquad \mathbf{d}_t^{max} = \max(\mathbf{d}_{t-1}^{max}, \mathbf{d}_t),$$
$$\tau_t = H(d_t) \cdot (\tau_{t-1} + 1) \qquad \text{otherwise}$$

where $d_t = (z_t - m_t)_+$, m_t, z_t are current deficit, current water supply and needs, respectively.

1.2 *Presentation of the problem in a regular, concise form.*

A formal manipulation will now be applied to present the problem (1.6) in a regular form allowing us to exploit the periodicity of the control process. Its idea is to transfer the constraints (1.6) from the state space **X** to the space of *state sets*, $\mathbf{2^X}$. Let us introduce the following notation and definitions:

$$\boldsymbol{M} = \{(\mu_0, \dots, \mu_{T-1}): \mu_i \in \mathbf{M}^{\mathbf{X}}, i = 0,1, \dots, T-1\}$$

is the set of all sequences of T control laws.

By $\{\mathbf{K}^i\}_0^T$ we denote the sequence of T+1 mappings $\mathbf{K}^i: \mathbf{2^X} \times \boldsymbol{M} \to \mathbf{2^X}$ (determining the sets of states reached in subsequent stages), defined, with $\boldsymbol{m} = (\mu_0, \dots, \mu_{T-1})$ as follows:

$$\mathbf{K}^0(X, \boldsymbol{m}) = X$$
$$\mathbf{K}^{i+1}(X, \boldsymbol{m}) = \{y: \exists x \in \mathbf{K}^i(X, \boldsymbol{m}) \ \exists \varepsilon \in \mathbf{E}_i(x) \\ [y = \mathbf{f}_i(x, \mu_i(x), \varepsilon)]\}, \quad i = 0,1, \dots, T-1 \tag{1.7}$$

The last mapping (corresponding to T-th stage) is denoted by **F**: $\mathbf{F} = \mathbf{K}^T$.

The sequence of T relations $\mathbf{P}_i \subseteq \mathbf{2^X} \times \boldsymbol{M}$, i=0,1,...,T-1, is defined, with $\boldsymbol{m} = (\mu_0, \dots, \mu_{T-1})$, by:

$$\mathbf{P}_i(X, \boldsymbol{m}) \equiv \forall j=1,..,N \ \forall x \in \mathbf{K}^i(X, \boldsymbol{m}) \ \forall \varepsilon \in \mathbf{E}_i(x) \\ [\mu_i(x) \in U_i(x) \wedge g^j_i(x, \mu_i(x), \varepsilon) \le s + \theta^j] \tag{1.8}$$

Finally, we define the relation **P** as the conjunction of the relations from the sequence $\{\mathbf{P}_i\}_0^{T-1}$:

$\mathbf{P}(X,\boldsymbol{m}) \equiv X \subseteq \mathbf{X} \wedge \boldsymbol{m} \in \boldsymbol{M} \wedge [\, \forall\, i = 0,1, \dots ,T\text{-}1: \mathbf{P}_i(X,\boldsymbol{m}) \,]$ (1.9)

It is seen that relations $\mathbf{P}_i$ describe the constraints (1.6) for subsequent stages, while **P** describes the constraints for the whole period. With the aid of this notation and due to periodicity of control process, the constraints (1.6) can be written in a concise form as:

$\mathbf{P}(X_0,\boldsymbol{m}_0) \wedge \mathbf{P}(X_1,\boldsymbol{m}_1) \wedge \dots \wedge \mathbf{P}(X_k,\boldsymbol{m}_k) \wedge \dots,$

where $\boldsymbol{m}_k$ is the sequence of control laws in the k-th period, i.e. $\boldsymbol{m}_k = (\mu_{kT}, \dots , \mu_{(k+1)T-1})$ and the state sets X_k are defined recursively:

$X_{k+1} = \mathbf{F}(X_k, \boldsymbol{m}_k), \qquad k = 0, 1, \dots$ (1.10)

Hence, the constraints (1.6) can be noted in the form:

$\forall \mathbf{k} = 0,1, \dots : \mathbf{P}(X_k,\boldsymbol{m}_k)$ (1.11)

where X_k, $\boldsymbol{m}_k$ satisfy the equation (1.10).

2. A GENERAL METHOD OF SOLUTION BASED ON THE REACHABLE SETS CONCEPT

A general scheme for solving the problems of the type (1.10) - (1.11), Sect. 1.2, will now be considered. This scheme is quite general in the sense that it doesn't make use of all specific properties of our original optimization problem (1.3) - (1.4) from Section 1. The idea of reachable sets (Bertsekas 1972), earlier applied to the problem (1.3) - (1.4) in (Karbowski et al. 1984), is used in this approach. This concept is based on elementary facts of the set and relations theory.

Only a very general characterization of the considered mathematical objects is assumed, inspired by specific properties of the problem from Section 1. The scheme itself is conceived as an abstraction of the original problem (1.6) structure, presented in the concise form (1.10) - (1.11).

2.1 *The simplified formulation.*

Consider the following infinite problem:

Find $\mathbf{X}^*$ which satisfies the sentential function $\boldsymbol{P}(X)$ of the form:

$\boldsymbol{P}(X) = \ulcorner \forall\, k = 0, 1, \dots : \mathbf{P}(X_k) \urcorner$ (2.1)

where X_k are recursively defined by the „state equation":

$X_0 = X; \quad X_{k+1} = \mathbf{F}(X_k), \qquad k = 0, 1, \dots$ (2.2)

$\mathbf{P}(X)$ is a *sentential function* and **F** a mapping, i.e. a function $\mathbf{F}: \mathbf{X} \rightarrow \mathbf{X}$.

We shall assume here that **P** and **F** satisfy the following conditions with respect to the relation $\subseteq$ of set inclusion:

$\mathbf{P}(X) \wedge X' \subseteq X \quad \rightarrow \mathbf{P}(X')$ **(a)**

$X' \subseteq X \rightarrow \mathbf{F}(X') \subseteq \mathbf{F}(X)$ **(b)**

The above relations are easily fulfilled in typical cases: **(a)** is satisfied if **P** is a *safe-type* description (Terlikowski 1990), i.e. if $\mathbf{P}(X) \equiv \forall x \in X: \mathbf{p}(x)$, where $\mathbf{p}(x)$ is a sentential function; **(b)** is satisfied if X is a set of states of a dynamic system and $\mathbf{F}(X)$ is the image of X in its state transformation.

Having introduced the following definition of the *largest set* from a given sets family $\{X_t\}$:

$X = \max_{\subseteq} \{X_t\} \equiv [\, X \in \{X_t\} \wedge \forall X' : X' \in \{X_t\} \rightarrow X' \subseteq X \,]$ (2.3)

we denote by $\mathbf{F}^{-1}(X)$ the *largest inverse* element to X in the mapping **F**.

The third general assumption notes the existence of the largest inverse element:

$\forall X \subseteq \mathbf{F}(\mathbf{X})\ \exists \mathbf{Z} : \mathbf{Z} = \max_{\subseteq} \{ Y: \mathbf{F}(Y) = X \}$ **(c)**

Under these assumptions the following theorems hold, which are essential for the proposed method of solving the problem (2.1). The first one however doesn't need the assumptions **(a)**, **(b)**, **(c)**.

Lemma 1.

If X is a solution of the problem (2.1), then $\mathbf{F}(X)$ is also a solution of this problem.

Moreover, the following relation holds:

$\boldsymbol{P}(X) \equiv \mathbf{P}(X) \wedge \boldsymbol{P}(\mathbf{F}(X))$ ■

The next lemma states some constructive properties of problem (2.1) solutions; the assumptions **(a)** and **(b)** are now important.

Lemma 2.

a) If X satisfies the relation:

$\mathbf{F}(X) \subseteq X$ (2.4)

then $X_k \subseteq X$ for every $k = 0, 1, \dots$

where X_k are defined by equation (2.2).

b) Moreover, if X satisfies the formula **P**, that is if:

$$\mathbf{P}(X) \wedge \mathbf{F}(X) \subseteq X \quad (2.5)$$

then X is a solution of problem (2.1), that is, X satisfies the formula $\boldsymbol{P}(X)$. ■

The relation (2.4) is called the *reachability condition*. Lemma 2 gives a „finite" construction (2.5) for a solution of an „infinite" problem (2.1). Two questions arise now: first, how general is this construction, i.e. does such a solution (satisfying (2.4)) always exist, if there exists any solution of (2.1); and, second, how can a solution of (2.5) be effectively constructed?

In the following lemma, related to the latter question, assumptions **(b)** and **(c)** are important.

Lemma 3.

The reachability condition (2.4), that is „$\mathbf{F}(X) \subseteq X$", is satisfied if and only if:

$$X = X \cap \mathbf{F}^{-1}(X) \quad ■ \quad (2.6)$$

Now, turning back to the generality question, observe that the relation (2.4) is *always* satisfied by some solutions of problem (2.1); namely, by X^* being the *largest set* from the family of all solutions of (2.1) - which is a simple consequence of Lemma 1.

Lemma 4

If X^* is the largest, w.r. to relation $\subseteq$, solution of problem (2.1), that is:

$$\boldsymbol{P}(X^*) \wedge [\ \forall X: \boldsymbol{P}(X) \rightarrow X \subseteq X^*\ ]$$

then X^* satisfies (2.4), and so we have:

$$\mathbf{P}(X^*) \wedge \mathbf{F}(X^*) \subseteq X^* \quad ■ \quad (2.7)$$

Whether this largest set X^* exists is itself another question; we only notice here that it depends on some topological features of function **F** and the set defined by **P**. The following lemma is a simple conclusion of the previous results of Lemmas 2, 3 and 4.

Lemma 5.

If X^* is the largest solution of problem (2.1), then X^* is the largest set X which satisfies the condition:

$$\mathbf{P}(X) \wedge [\ X = X \cap \mathbf{F}^{-1}(X)\ ] \quad ■ \quad (2.8)$$

The above results constitute a sufficient background for finding a solution of problem (2.1) in an effective way. The method which will be proposed is based on the *contraction mappings* concept and consists in the determination of some *fixed points* X of the following mapping **D**:

$$\mathbf{D}(X) = X \cap \mathbf{F}^{-1}(X) \quad (2.9)$$

namely, such that $X = \mathbf{D}(X)$
and at the same time $\mathbf{P}(X)$ (cp. (2.8))

The *algorithm* is as follows:

- Let $\mathbf{X_0}$ be *any* set satisfying the formula $\mathbf{P}(X)$. Starting from this set, define the sequence $\{\overline{X}_i\}$:

$$\overline{X}_0 = \mathbf{X_0}; \quad \overline{X}_{i+1} = \mathbf{D}(\overline{X}_i),\ i = 0, 1, \ldots \quad (2.10)$$

Since **D** has an obvious „contraction property", in the sense that $\mathbf{D}(X) \subseteq X$, the sequence $\{\overline{X}_i\}$ is a descending one: $\overline{X}_{i+1} \subseteq \overline{X}_i$, i = 0,1, .. ; and so $\overline{X}_i \subseteq \mathbf{X_0}$ for all i = 0, 1, ... Consider now the set $\overline{X}$ being the product of all sets $\overline{X}_i$:

$$\overline{X} = \cap\{\overline{X}_i: i = 0,1, \ldots\} \quad (2.11)$$

Due to $\overline{X}_i \subseteq \mathbf{X_0}$, $\mathbf{P}(\mathbf{X_0})$ and assumption **(a)**, we have $\mathbf{P}(\overline{X})$. On the other hand, under some assumptions which guarantee the continuity of **D**, the set $\overline{X}$ satisfies the relation $X = \mathbf{D}(X)$ and the sequence $\{\overline{X}_i\}$ converges (in a respective sense) to $\overline{X}$. From the first premise we conclude then, by Lemma 2.**b)** and 3, that $\overline{X}$ - which is the *final result* of our algorithm - is a solution of problem (2.1).

The next lemma finds that $\overline{X}$ is also an upper bound of the family of all sets satisfying the condition:

$$[X = \mathbf{D}(X)] \wedge [X \subseteq \mathbf{X_0}] \quad (2.12)$$

Lemma 6.

Any X satisfying (2.12) is included in the set

$$\overline{X} = \cap\{\overline{X}_i: i = 0,1, \ldots\}$$

with $\overline{X}_i$ defined by equations (2.10) and (2.11) ■

Hence, if we assume (as above) that $\overline{X} = \mathbf{D}(\overline{X})$, then $\overline{X}$ is also the largest solution of (2.12).

This result will be used to construct the *largest solution* X^* of problem (2.1). It is easy to see that for $\mathbf{X_0}$ being the largest set which satisfies the formula $\mathbf{P}(X)$:

$$\mathbf{X_0} = \overline{X}_0 = \mathbf{max}_{\subseteq}\ \{\ X: \mathbf{P}(X)\ \} \quad (2.13)$$

the relation (2.12) becomes, by assumption **(a)**, equivalent to (2.8).

Therefore, putting $\mathbf{X_0} = \tilde{X}_0$ in the algorithm (2.10), we obtain the sequence $\{\bar{X}_i\}$ which converges to the set $\tilde{X} = \cap\{\bar{X}_i\}$. This set is (by Lemma 6) the largest solution of (2.8) and hence, by Lemma 5, the largest solution of problem (2.1) (if this largest solution of (2.1) exists).

2.2. *The precise formulation with state and control variables.*

An analogous reasoning will now be proceeded for a more complicated problem which apparently fits the formulation (1.10) - (1.11) of the problem from Section 1:

Find $\mathbf{X}^*$ and an infinite sequence $\{m_k\}$ satisfying the formula $\boldsymbol{P}$:

$$\boldsymbol{P}(X, \{m_k\}) = \ulcorner \forall k = 0, 1, \ldots : \mathbf{P}(m_k, X_k) \urcorner \quad (2.14)$$

where X_k are defined by the state equation:

$$X_0 = X; \qquad X_{k+1} = F(m_k, X_k),\ k = 0, 1, \ldots \quad (2.15)$$

P is a given formula with two free variables and **F** is a mapping, $\mathbf{F}: \mathbf{X} \times \mathrm{M} \to \mathbf{X}$.

The two groups of variables: X_k and m_k have the sense of state sets and of controls.

For any given m_0 we denote by $\{\overline{\overline{m}}_0\}$ the *stationary* sequence of controls, i.e. such a sequence $\{m_k\}$ that $m_k = m_0$ for all $k = 0,1, \ldots$.

Let us introduce the following two notations:

family of sets $\mathbf{R}(X)$:

$$\mathbf{R}(X) = \{Y: \exists m\ [\mathbf{P}(Y,m) \wedge F(Y,m) \subseteq X\} \quad (2.16)$$

mapping **D**:

$$\mathbf{D}(X) = X \cap \mathbf{max}_{\subseteq}\ \mathbf{R}(X) \quad (2.17)$$

There is an analogy, not direct however, between $\mathbf{max}_{\subseteq}\mathbf{R}(X)$, $\mathbf{D}(X)$-(2.17) and, respectively, $\mathbf{F}^{-1}(X)$, $\mathbf{D}(X)$-(2.9) from Sect. 2.1.

We assume (similarly to **(c)** in Sect. 2.1) the existence of the largest „*P-inverse*" element of **F**:

$$\forall X \subseteq \mathbf{F}(\mathbf{X}, \mathrm{M}):\ \mathbf{R}(X) \neq \varnothing \rightarrow$$

$$[\exists \mathbf{Z} : \mathbf{Z} = \mathbf{max}_{\subseteq}\mathbf{R}(X)] \quad \mathbf{(C)}$$

where $\mathbf{F}(\mathbf{X},\mathrm{M})$ is the image of the „control space" M in the mapping $\mathbf{F}(\mathbf{X}, \cdot)$.

The following two additional assumptions (analogous to **(a)**, **(b)** of Sect. 2.1) will also be used:

$$\mathbf{P}(X,m) \wedge X' \subseteq X \quad \rightarrow \mathbf{P}(X') \quad \mathbf{(A)}$$

$$X' \subseteq X \ \rightarrow\ \mathbf{F}(X',m) \subseteq \mathbf{F}(X,m) \quad \mathbf{(B)}$$

Under these assumptions, which are true e.g. in the case of a dynamical process state equation (2.15) and a *safe-type* description **P**, (2.14), we can prove the following theorem:

Theorem 1.

a) Let the pair $(X,\{\overline{\overline{m}}_0\})$ be such that the pair (X, m_0) satisfies the relation:

$$\mathbf{P}(X, m_0) \ \wedge\ \mathbf{F}(X,m_0) \subseteq X \quad (2.18)$$

Then $(X, \{\overline{\overline{m}}_0\})$ is a solution of problem (2.14).

b) The following *extended P-reachability condition*:

$$\exists m:\ \mathbf{P}(m,X) \wedge \mathbf{F}(m,X) \subseteq X \quad (2.19)$$

is equivalent to the equation: $\quad X = \mathbf{D}(X)$

and is a sufficient condition for existence of a pair satisfying (2.18).

c) If X^* is the *largest* solution of problem (2.14) (with some $\{m_k\}$), then X^* is the *largest set* satisfying the equation:

$$X = \mathbf{D}(X) \qquad \blacksquare$$

The *algorithm* resulting from this theorem follows the idea of algorithm (2.10) from Section 2.1:

- we define the „starting" set

$$\tilde{X}_0 = \mathbf{max}_{\subseteq}\ \{Y: \exists m\ \mathbf{P}(Y, m)\} \quad (2.20)$$

-putting $\bar{X}_0 = \tilde{X}_0$ we determine the sequence $\{\bar{X}_i\}$:

$$\bar{X}_{i+1} = \mathbf{D}(\bar{X}_i) = \bar{X}_i \cap \mathbf{max}_{\subseteq}\ \mathbf{R}(\bar{X}_i) \quad i = 0,1,..(2.21)$$

This sequence - under some topological conditions (see Sect. 2.1) - converges, for $i \to \infty$, to the set:

$$\bar{X} = \cap\{\bar{X}_i : i = 0,1, \ldots\}, \quad (2.22)$$

which is, by Th. 1.a), b) and respective continuity conditions, a solution of problem (2.14). Moreover, if there exists the largest solution X^* of (2.14), $\bar{X}$ is just (by Th. 1.c)) this largest solution of (2.14). Then a stationary control sequence $\{\overline{\overline{m}}_0\}$ is determined, according to the relation (2.18).

3. THE ALGORITHM FOR THE MINIMAX PROBLEM OF THE RESERVOIR SYSTEM

We shall apply now the above method of solving the problem (2.14) to our original problem of Sect. 1, namely - to problem (1.6).
In the following two-level method, the algorithm (2.21) is used to solve the problem (1.10)-(1.11). In this case the definitions of function **F** and relation **P** are given by (1.7) and (1.9). Then, we can apply (2.21), since all sufficient conditions for applicability of the approach as described in Sect. 2.2 are fulfilled. This is quite evident for assumptions **(A)** and **(B)**.

(I) *Lower level.*

For a given value **s** (see (1.6)) the contraction algorithm (2.21) is applied to determine the largest set $X^*(\mathbf{s})$, being a solution of (1.10) - (1.11).
At each iteration **k** of this algorithm, T steps of the following *finite* „backward" iteration scheme are performed: for any set X -
- we put X as the starting point: $Y_T = X$

- for i = 1, ... , T-1 we calculate:

$$\mathbf{Y}_i = \{y: \exists m\in\mathbf{U}_i(y)\ \forall\varepsilon\in\mathbf{E}_i(y)\ [\ \mathbf{f}_i(y,m,\varepsilon)\in\mathbf{Y}_{i+1} \wedge \forall j=1,..,N\ \mathbf{g}^j{}_i(y,m,\varepsilon) \le \mathbf{s}+\theta^j\ ]\} \quad (3.1)$$

- finally, the set $\mathbf{Y}_0 = \mathbf{D}(X)$ is determined as:

$$\mathbf{Y}_0=\mathbf{D}(X)=X\cap\{y:\exists m\in\mathbf{U}_0(y)\forall\varepsilon\in\mathbf{E}_0(y)[\ \mathbf{f}_0(y,m,\varepsilon)\in\mathbf{Y}_1 \wedge \forall j=1,..,N\ \mathbf{g}^j{}_0(y,m,\varepsilon) \le \mathbf{s}+\theta^j\ ]\} \quad (3.2)$$

We use the above scheme (corresponding to one period of control process) as follows:

- for **k** = 0 we put X = **X** (the whole space of states); the resulting set $\mathbf{Y}_0$, (3.2), equal to $\tilde{X}_0$ (see (2.20), is taken as $\overline{X}_0$ (see (2.21))

- for **k** = 1,2, ... we take $X = \overline{X}_{k-1}$ and, after T steps (3.1)-(3.2) we obtain $\overline{X}_k$ as equal to $\mathbf{Y}_0$.

The whole algorithm is thus the algorithm (2.21) „multiplied" T times for every iteration.

Its result is the set $X^*(\mathbf{s})$, as well as the sets $\mathbf{Y}^*_i$, i = 0, 1, ... ,T, defined by (3.1), (3.2) with the initial condition $\mathbf{Y}^*_i = X^*(\mathbf{s})$.

The sets $\mathbf{Y}^*_i$ serve to determine a sequence $m_0 = (\mu_0, \ldots, \mu_{T-1})$ of control laws which satisfy, with the set of initial states $X_0 = X^*(\mathbf{s})$, all constraints (1.6) for **t** = 0, 1, ... , T-1.

These control laws are defined by the following formula - for each $\mu_i(\cdot)$, i = 0, 1, ... , T-1:

$$\forall x\in\mathbf{Y}^*_i\ \forall\varepsilon\in\mathbf{E}_i(x) : \mathbf{f}_i(y,\mu_i(x),\varepsilon)\in \mathbf{Y}^*_{i+1} \wedge [\ \forall j=1,..,N\ \mathbf{g}^j{}_i(y,\mu_i(x),\varepsilon) \le \mathbf{s}+\theta^j\ ] \quad (3.3)$$

As it is seen, our algorithm determines a *stationary* pair $(X^*(\mathbf{s}),\{\overline{m}_0\})$ as a solution of the problem (1.6) (see Th. 1.a)).

(II) *Upper level.*

The minimum value of parameter **s** is then searched (with the aid of a simple non-gradient method), for which there exists a solution of constraints (1.6). It is so (under some standard assumptions of continuity of mappings $\mathbf{U}_t$, $\mathbf{E}_t$ and $\mathbf{g}^j{}_t$), if and only if the lower-level algorithm, as described above, produces a non-empty set $X^*(\mathbf{s})$.
A stationary pair $(X^*(\mathbf{s}_{min}),\{\overline{m}_0\})$, which corresponds to the minimum value $\mathbf{s}_{min}$ is taken as a solution of our multiobjective optimization problem (1.3) - (1.4).

The reference point θ is selected by the so called „Pareto race" method, (Karbowski et al. 1984), (Korhonen et al. 1986), which makes it possible to translate the users/operator preferences in a clear, intuitive way. It is possible to modify interactively the value θ, if it produces an only *weakly effective* solution, (Wierzbicki 1982).

4. CONCLUSIONS

The solution method of a vector min-max design and control problem concerning multireservoir systems has been considered. It pertains to the case where the sets, to which the future inflows may belong are given, e.g. as the set of scenarios of all possible inflow trajectories.

The two main tasks of this paper are the following:
1) first, to develop the concept of *reachable sets* for a specific, but general problem defined over the infinite horizon (Section 2),

2) second, to show the relation of the above statement with the minimax periodical control problem for a water reservoir system (Section 1).

The latter problem has been transferred to the form of the general abstract statement mentioned above in 1), by the reference point method (Sect. 1.1 and 1.2). Due to the fact that all sufficient applicability conditions **(A)**, **(B)**, **(C)** from Sect. 2 are satisfied (under standard assumptions) in this optimal control problem, we obtain directly its complete solution fitting exactly the general approach of Section 2. Moreover, an effective algorithm presented in Sect. 2 appears to be entirely applicable to our control problem.

The whole algorithm described in Sect. 3 has two-level character. The actual optimization is performed in a higher level through a scalar minimization procedure, while the lower level task consists in a recursive contraction of sets.

Due to the fact that iterations are performed on the sets, this method reduces the „curse of dimensionality" of the traditional discrete dynamic programming. Another advantage of the proposed approach is that the obtained control policies are more general than in the classical approach. Instead of a single control law, the algorithm delivers a collection of control laws. This may be especially useful in a DSS environment, giving the possibility to take into account some additional criteria (e.g. hardly formalizable), not considered in the performance vector.

The presented approach has been applied and computationally verified in a few cases of real water systems, e.g. Wupper Reservoir System. However, our attention has been focused here mostly on the theoretical aspects of the proposed solution. It is worthwhile to mention that the relation between these two formulations: the minimax periodical control problem (Sect. 1) and the general, abstract form of Sect. 2, is not a secondary result, found as an application of a primarily developed general concept. On the contrary, the attempt to solve the original optimal control problem (Karbowski et al. 1994) has led us to these generalizations.

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