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APPLICABILITY OF FLOW PREDICTION BASED ON ATTRACTORS

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A b s t r a c t

The methods of dynamical systems theory are applied to analyse hydrological historical records (daily inflows). The dimension of attractor and predictability period were reconstructed using the time series trajectories in the embedding phase space. The dimension of attractor is 2.1 and the predictability period is not more than 50 days. The choice of parameters necessary to perform proper reconstruction of the characteristics is detailed.

Key words: flow prediction, attractors.

1. Introduction

One of the most promising methods to examine the nature of hydrological data (time series) uses the concepts pertaining to deterministic chaos approach. This method consists in the reconstruction of certain properties of the attractor in the phase space based on the time-dependence of only one component presented in the form of a time series. The aim of this paper is to present the utilisation of such a method for the hydrological records representing daily inflows based on the data from the Wupper Reservoir System in Germany.

Let us briefly define the problem. It is obvious that the fundamental physical laws can be described by differential equations. In case of the existence and uniqueness of the solutions, a dynamical system is well defined. The time evolution of the system may be represented by the set of n ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i = f_i(x_1, ..., x_n), \qquad i = 1, ..., n. \tag{1}$$

It was shown by Packard *et al.* (1980) and Takens (1981) that equation (1) can be transformed to a single non-linear differential equation of order n:

$$x^{(n)} = f\left(x(t), x'(t), \dots x^{(n-1)}(t)\right). \tag{2}$$

In practice we are very often limited to the observations of only one variable given in the form of a time series. This leads to the problem of a reliable method to obtain information about the original dynamical system knowing only one observable. Let us create a state vector $x(t)$ in the following form (Ruelle, 1981):

$$x(t) = x(t), x(t+\tau), \dots x\left(t+(n-1)\tau\right), \tag{3}$$

where t is the delay time. After choosing proper values of τ (guaranteeing non-linear independence of the delay coordinates) equation (3) defines the embedding in a new phase space homeomorphic with the original one.

An important characteristic of the dynamical system is an invariant, closed and bounded set "attracting" all the trajectories from its neighbourhood. The precise definition of attractors can be found in (Grassberger and Procaccia, 1983b; Takens, 1981). The Hausdorff dimension of the attractor D is smaller than the number of degrees of freedom of the dynamical system. The estimate of D for our test case study will be given further in the paper. The divergence of state trajectories allows us to obtain the limit of predictability and consequently the predictability period of time T. The algorithm developed by Grassberger and Procaccia (1983b) is often used to reconstruct those characteristics from time series. Such approach has been applied to analyse weather and climate data (Nicolis and Nicolis, 1984; Fraedrich, 1986), wind records (Tsonis and Elsner, 1988), rainfall and storm rainfall data (Rodriguez-Iturbe *et al.*, 1989), river turbulence (Nikora *et al.*, 1994). Recently the applications of the method to analyse streamflow data were shown by Jayawardena and Feizhou Lai (1994) and Beauvais and Dubois (1995).

2. Phase space and characteristics of attractors

An estimate of the Hausdorff dimension from a single time series is not an easy task, especially when D appears to be larger than 2. Therefore another measure, called correlation dimension ν, was proposed for the estimate of the attractor dimension (Packard *et al.*, 1980; Grassberger and Procaccia, 1983b). It may be defined by

$$\nu = \lim_{r\to 0} \frac{\log C(r)}{r}, \tag{4}$$

where $C(r)$ is the correlation integral

$$C(r)_m \equiv \lim_{N\to\infty} \frac{1}{N^2}\sum_{i,j=1}^{N} \Theta(r-|X_i-X_j|) \approx \frac{1}{M}\sum_{j=1}^{M}\frac{1}{N}\sum_{i=1}^{N} \Theta(r-|X_i-X_j|), \tag{5}$$

X_i $(i=1, ..., N)$ is the set of points on the attractor obtained from the series $X_i = x(t+i\tau)$ with a fixed time increment τ between successive measurements, r is the radius of sphere centred on X_j, m is the current dimension of the phase space, N is the total number of pairs (i, j), Θ is the Heaviside function, M is the number of X_j points and $|\bullet|$ denotes the Euclidean norm. It is important to note here that the correlation dimension cannot be larger than the Hausdorff measure. Assuming a straight-line dependence of $\log C(r)$ on $\log r$ the estimate of ν is straightforward from the given data.

The measure of predictability for a dynamical system can be characterised as a period of time T during which the forecast is possible with a certain degree of accuracy. The values of T are taken as the inverse of Kolmogorov entropy $K_{2,m}$ (Fraedrich, 1986) defined by Grassberger and Procaccia (1983a) as

$$K_{2,m} \approx \frac{1}{\tau k} \ln \frac{C(r)_m}{C(r)_{m+k}} . \tag{6}$$

The distance r should be selected on the interval where $\ln C(r)$ can be approximated by straight lines of identical slopes.

Let us illustrate implementation of the method for the artificial chaotic system, the Henon map (Henon and Pomeau, 1976):

$$\left.\begin{aligned} x_{n+1} &= 1-\alpha x_n^2 + y_n \\ y_{n+1} &= \beta x_n \end{aligned}\right\} . \tag{7}$$

Figure 1 shows the time evolution of the Henon map in the original phase space (a) and in the phase space reconstructed from time series $x(t)$ by delay coordinates method (b).

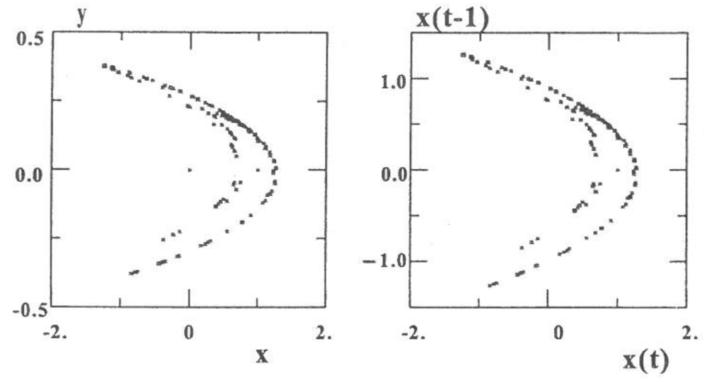

Fig. 1. Phase space for the Henon map ($\alpha = 1.4$, $\beta = 0.3$).

It is evident that the reconstructed phase space has the same topological properties as the original one. This system has been studied theoretically and numerically in detail by Grassberger and Procaccia (1983b). The dimension of the Henon map is equal to 1.26. Figure 2a shows the correlation integrals calculated for m = 2, 3 and 4. The variation of the slope of correlation integrals for growing r is presented in Fig. 2b. The plot of the function $\nu(r)$ allows to select the interval of the values of r on which this function behaves as a constant. This constant is interpreted to be a dimension of attractor by means of description (4) and its value is approximately 1.21. Having calculated $C(r)$ and knowing the value of τ (for Henon map $\tau = 1$) it is easy to calculate T for pairs $[C(r)_2, C(r)_3]$, $[C(r)_3, C(r)_4]$.

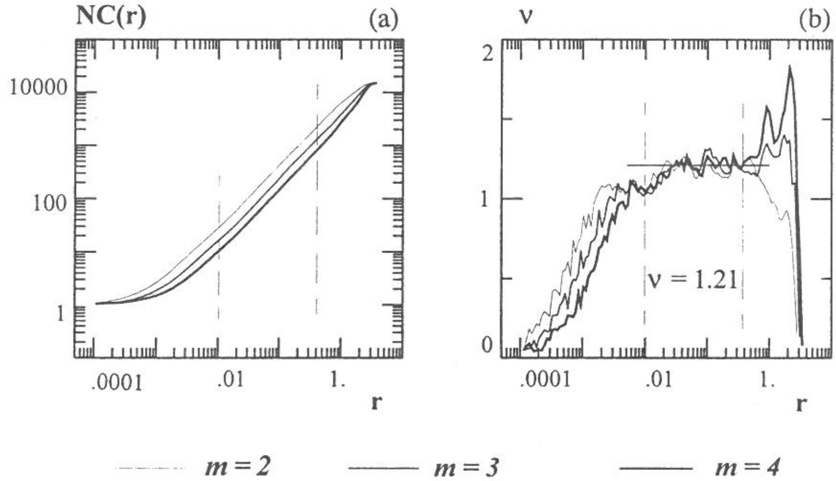

Fig. 2. (a) Correlation integrals NC(r), and (b) function $\nu(r)$ for the Henon map calculated for m = 2, 3 and 4.

3. Implementation to daily inflows

The presented method has been applied in a number of studies, especially in physics. Recently, Jayawardena and Feizou Lai (1994) have confirmed the usefulness of the method to analyse and to forecast records of inflow. It was shown that the inflow data series are better modelled by this method than by traditional autoregressive moving average approach. However, when the method is applied for natural processes, important questions arise as to the proper choice of the delay time τ and the correct length of the considered time series. At present the general recipes are unknown and they must be specially considered for every particular case.

Object and data

We implement the method to analyse the daily inflows of the Wupper Reservoir System in Germany. The catchment of river Wupper is located in the southern part of North Rhine Westfalia. Annual rainfall is about 1300 mm/yr. Because the catchment lithology is only a thin soil layer overlying a solid rock, hydraulic storage properties are poor and the catchment is "floody" yielding severe floods, and poorly sustained low flows. When flow at the Wuppertal gauge exceeds 5 m^3/s, the water quality standard II (stable β-mesoaprobic) is generally met; flow less than 1 m^3/s is disastrous ecologically.

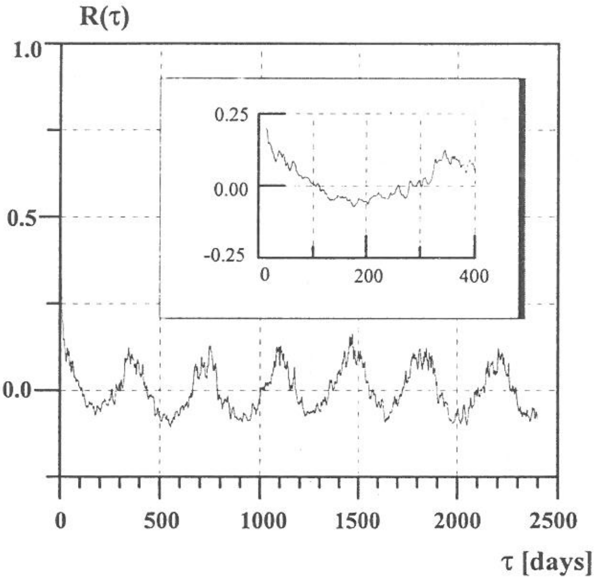

Fig. 3. Correlation function for Wupper Reservoir System gage record.

There are two control gages at the system on which 38 years long historical records have been recorded. Additionally the lateral inflow to the system has been reconstructed. For all three records the correlation functions demonstrate practically the same behaviour. The example of correlation function is shown in Fig. 3. The inset enlarges the area close to the first zero-value of the correlation function. Strong seasonal periodicity and nonstationarity of the records are evident. The function $R(\tau)$ assumes the first zero value at $\tau = 100$ days.

Results

A number of trial calculations with $M = 100, 250, 500, 750, 1\,000, 2\,000, 5\,000$ and $10\,000$ were carried out in order to choose an appropriate value of M (see eq. 5) (Jayawardena and Feizhou Lai, 1994).

Figure 4 shows the typical errors obtained in trial calculations with the use of the formula $Error\ (r) = 100[C(r)_N - C(r)_M]\ /C(r)_N$, with $N = 10\ 000$. The accuracy ranging from 10 to 15% was obtained for $M = 750$. This significantly reduced the time of calculations and allowed us to carry out the analysis with the use of personal computers.

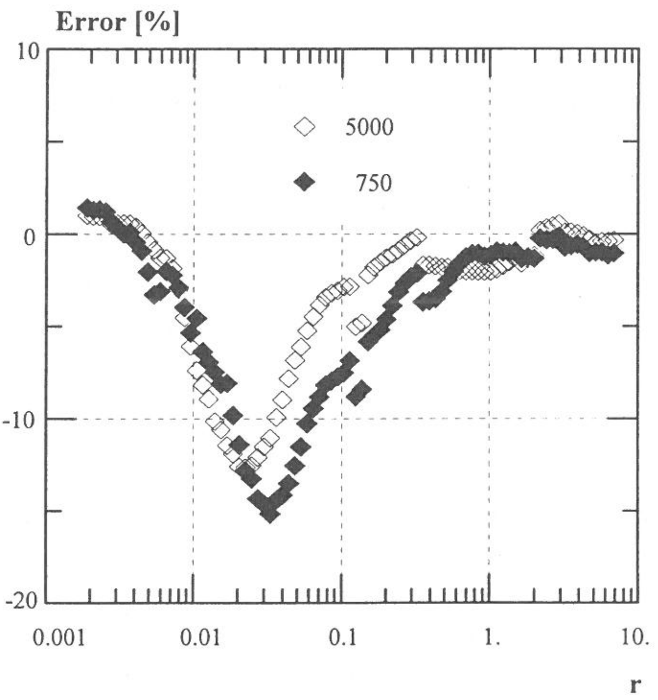

Fig. 4. Example of the accuracy of correlation integral calculations.

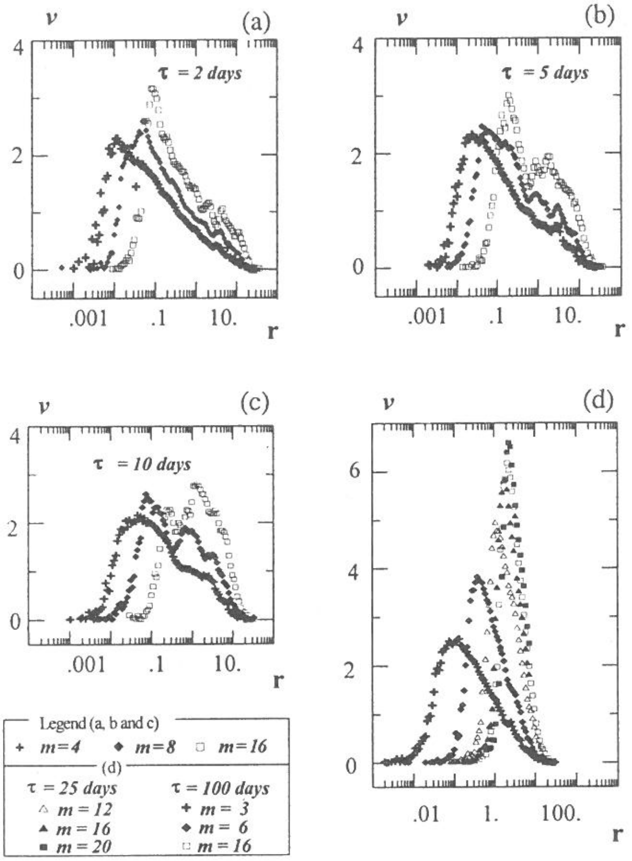

Fig. 5. Examples of functions $v(r)$ for various times of delay.

We needed an appropriate value of the time delay τ to guarantee the mutual independence of coordinates in phase space. The dependence of the coordinates corresponds to the rotating of the space as it was shown by Fraser and Swinney (1986). A possible choice of τ is to take the time of zero value of autocorrelation function or the time at which the autocorrelation function falls below $1/e$ (Tsonis and Elsner, 1988). This is, however, only justified in case of a stationary process and when the autocorrelation function assumes an exponential form.

A set of trials was used to choose the delay time. The correlation integrals were calculated for values of τ equal to 2, 5, 10, 25 and 100 days for phase spaces with dimensions m ranging from 2 to 20. Figure 5 presents the typical functions $v(r)$ calculated for different τ and m. It is clear that if τ is too small (Figs. 5a and 5b) then the coordinates are dependent and the range of r values in which $v(r) \sim$ const is too narrow for appropriate estimate of v. On the other hand, if τ is too large (Fig. 5d), the correlation between the state vectors in phase space would be lost and the system behaves like a random process and the attractor dimension keeps increasing with m. Figure 5a, b, c show that the range of r values where $v(r) \sim$ const widens for certain values of the delay time. For the daily inflows data of the Wupper System, the optimal value of τ was established to be 10 days.

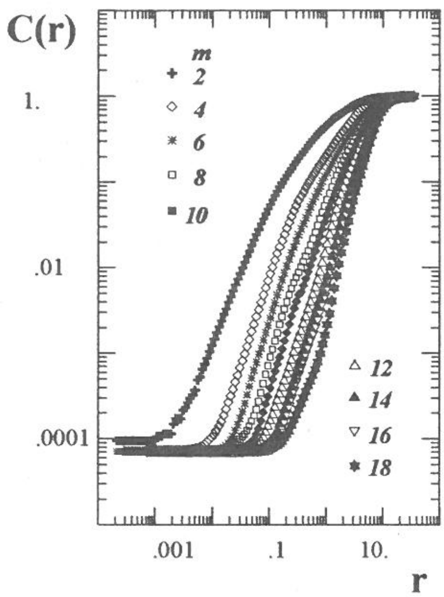

Fig. 6. Correlation integral for gage Biver, $\tau = 10$ days.

Examplary results of calculations performed for $M = 750$, $\tau = 10$ days and m ranging from 2 to 20 are presented in Fig. 6. The values of correlation dimensions for various embedding dimensions are presented in Fig. 7. It is evident that at some value of embedding dimension the increases in function $\nu(m)$ cease. This saturation value for functions $\nu(m)$ is interpreted as the estimate of the attractor dimension (Grassberger and Procaccia, 1983b).

For the case of the daily inflow data, the correlation dimension of attractor is equal to 2.1. Thus, the Wuppertal Reservoir System behaves as a dynamical system with a low dimensional attractor. This fact supports the idea that the time evolution of hydrologic system can be predicted based on the method of local approximation in the phase space. The dimension of this phase space cannot be less than 3.

The correlation integrals were also used to determine the entropy and consequently the predictability period of time T (inverse of Kolmogorov entropy $K_{2,m}$, eq. 6). The predictability time T varied in the range from 40 to 50 days.

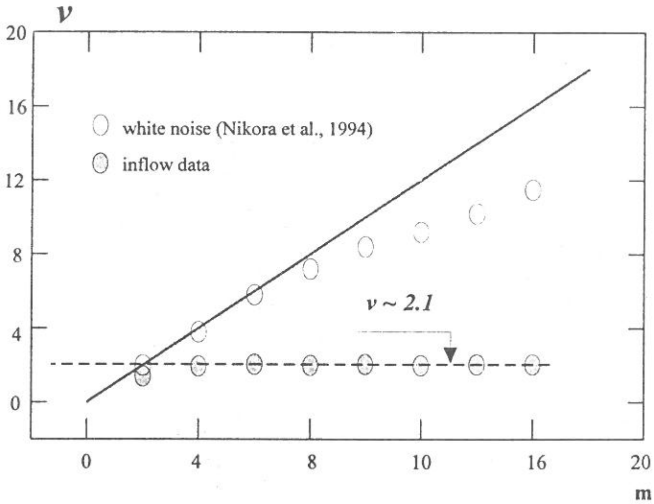

Fig. 7. Estimation of attractor dimension for daily inflows.

4. Conclusions

The major observations and conclusions of the study are summarised as follows:

- Daily inflows of Wupper Reservoir System behave as the chaotic dynamical system with low dimensional attractor of the dimension equal to 2.1;
- The time evolution of this system could be predicted using the method of local approximation in the phase space of delayed coordinates. The dimension of this phase space is not less than 3;
- The long time prediction for this system is limited to 40–50 days;
- The available methods do not allow to estimate the value of the delay time. The trial calculations to establish the proper value of τ have to be performed for each case.

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