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THE DISTRIBUTED MUSKINGUM MODEL

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ABSTRACT

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This paper investigates the limiting form of the multiple Muskingum model when the number of reaches increases to infinity, while maintaining finite values for the first and second moments. Both the cumulants, and the amplitude and phase characteristics of this distributed Muskingum model (DMM) are derived. The model is compared to the solution of the linearised Saint-Venant equation for a semi-infinite uniform channel (LSV). The error of the DMM in predicting the third central moment of the LSV is shown to be independent of channel length in contrast to the classical Muskingum model in which the error increases rapidly with length of channel.

1. INTRODUCTION

The Muskingum method of flood routing has been widely used in applied hydrology since its first use in connection with a flood control project in the Muskingum County of Ohio some fifty years ago (McCarthy, 1939). It has long been recognised that the method runs into difficulties as the length of channel reach to which it is applied increases. This difficulty can be overcome by dividing the channel reach into a number of shorter reaches of equal length and treating each of these as a Muskingum reach (Laurenson, 1959). This multiple Muskingum model gives an improvement in fitting field data at the cost of introducing as a third parameter the number of reaches into which the original long channel is divided.

The present paper investigates the properties of the distributed model obtained when the process of dividing the long channel into shorter reaches is carried to the limit of a very large number of very short reaches. In the limit the three-parameter multiple Muskingum model reduces to a two-parameter distributed model. The approximation of this two-parameter model to the linearised St. Venant solution for a uniform channel (which has four parameters) does not show any improvement over such traditional two-parameter conceptual models as the lag-and-route method or the cascade of equal linear reservoirs but does avoid the large errors of the classical Muskingum model.

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2. MUSKINGUM TYPE MODELS

The classical Muskingum model

The original Muskingum model (McCarthy, 1939) combines the lumped continuity equation:

$$Q_1(t) - Q_2(t) = \frac{\mathrm{d}S}{\mathrm{d}t} \tag{1}$$

with the simple linear expression for storage:

$$S(t) = K[aQ_1(t) + (1 - a)Q_2(t)] \tag{2}$$

where $Q_1(t)$ is the upstream inflow, $Q_2(t)$ the downstream outflow, $S(t)$ the volume of storage in the channel reach; K is a parameter reflecting the lag in the reach and a is a parameter reflecting the degree of influence of the upstream inflow on the storage volume.

In the method as originally proposed (McCarthy, 1939; Linsley et al., 1949), data were used from a historical flood, for which records of $Q_1(t)$ and $Q_2(t)$ were available, in order to determine the values of the parameters a and K by trial and error. $S(t)$ was plotted against the weighted value of $Q(t)$ for various values of a to determine the value which minimized the divergence between the plotted positions for the rising hydrograph and the falling hydrograph; the value of K was then given as the slope of the best linear fit of the storage against the weighted flow for this value of a. The more objective approach based on the use of moments, introduced by Nash (1959) in the study of catchment response, was applied by Dooge and Harley (1967b) to parameter estimation in conceptual models of linear channel response. Their result for a wide rectangular channel with Chezy friction was later generalised by the present authors to the case of any shape of channel and any friction law (Dooge et al., 1982).

By substituting from eqn. (2) into eqn. (1) and gathering the inflow and outflow terms, we obtain the equation:

$$Q_2(t) + (1 - a)K\frac{\mathrm{d}Q_2}{\mathrm{d}t} = Q_1(t) - aK\frac{\mathrm{d}Q_1}{\mathrm{d}t} \tag{3}$$

Consequently the system function $H_1(s)$, i.e. the Laplace transform of the impulse response, for this model is given by:

$$H_1(s) = \frac{Q_2(s)}{Q_1(s)} = \frac{1 - aKs}{1 + (1 - a)Ks} \tag{4}$$

This can be inverted to the time domain to give the linear channel response $h(t)$ as:

$$h(t) = \frac{1}{(1 - a)^2K}\exp\left[-\frac{t}{(1 - a)K}\right] - \left(\frac{a}{1 - a}\right)\delta(t) \tag{5}$$

where $\delta(t)$ is the Dirac delta function. The practical difficulty of negative ordinates in some cases of the use of the Muskingum model is reflected in the sign of the second term on the right-hand side of eqn. (5).

The moments and the cumulants of the impulse response of any linear model can be found from the system response (Dooge and Harley, 1967a; Dooge, 1973). It can be shown that for the classical Muskingum model (CMM) the Rth cumulant is given by:

$$k_R(\mathrm{CMM}) = (R - 1)![(1 - a)^R - (-a)^R]K^R \tag{6a}$$

It can be readily verified that in particular the first cumulant (which is the same as the first moment about the origin) is given by:

$$k_1(\mathrm{CMM}) = K \tag{6b}$$

and the second cumulant (which is the same as the second moment about the centre) as:

$$k_2(\mathrm{CMM}) = (1 - 2a)K^2 \tag{6c}$$

and so on.

Matching of the first and second cumulants given by eqn. (6) to the corresponding cumulants of the linearised St. Venant equation (given later in section 4) leads to:

$$k_1 = K = \frac{L}{c_k} \tag{7}$$

$$k_2 = (1 - 2a)K^2 = \frac{1}{m}[1 - (m - 1)^2F_0^2]\left(\frac{y_0}{S_0L}\right)\left(\frac{L}{c_k}\right)^2 \tag{8}$$

so that we have the relationship:

$$a = \frac{1}{2} - \frac{1}{2m}[1 - (m - 1)^2F_0^2]\frac{y_0}{S_0L} \tag{9}$$

where L is the length and S_0 the slope of the channel, m is the ratio of kinematic wave speed (c_k) to reference velocity, y_0 and F_0 are values of depth and of Froude number at the reference conditions. Accordingly the optimum value of the parameter a will vary from minus infinity for a very short channel to 0.5 for a very long channel.

The multiple Muskingum model

The inability of the classical Muskingum model to model the flood routing behaviour of long channels can be deduced from the basic model assumption represented by eqn. (2). From that equation it can be clearly seen that after inflow $Q_1(t)$ has ceased, the model reduces to a linear reservoir and the outflow $Q_2(t)$ will decline exponentially with a time constant of $(1 - a)K$. Consequently the model is unable to reproduce the crest of the outflow hydrograph if the channel length is such that the time taken to reach maximum outflow is greater than the duration of inflow.

The linking of the length of the channel to the appearance of negative outflows from the Muskingum model can be seen from a comparison of eqns. (6a) and (9). As the total length of channel (L) increases, the value of a approaches closer and closer to 1/2 so that the values of a and $(1 - a)$ approach closer and closer. Consequently the even-order cumulants defined by eqn. (6a) approach zero while the odd-order cumulants have the limit:

$$\lim_{a \to \frac{1}{2}} [k_R(\mathrm{CMM})] = (R - 1)!\,(\tfrac{1}{2})^{R-1} K^R \tag{10}$$

which is always nonzero and positive. Such contrasting behaviour between the even-order and odd-order cumulants is not consistent with a response function whose ordinates are all nonnegative.

By dividing the total channel length in a number of equal reaches the above undesirable features in the response function of the classical Muskingum model can be avoided (Laurenson, 1959). For the multiple Muskingum model the system function $H_n(s)$ is given by:

$$H_n(s) = \left[\frac{1 - aKs}{1 + (1 - a)Ks}\right]^n \tag{11}$$

where n is the number of shorter channel reaches into which the total channel length (L) has been divided. The general expression for Rth cumulant of this multiple Muskingum model (MMM) is given by (Dooge, 1973):

$$k_R(\mathrm{MMM}) = n(R - 1)![(1 - a)^R - (-a)^R]K^R \tag{12}$$

For any given value of n the remaining parameters K and a can be optimised (in the moments sense) by adapting eqns. (7) and (9) to the length of each individual reach, i.e.:

$$K = \frac{L}{nc_k} \tag{13a}$$

and:

$$a = \frac{1}{2} - \frac{1}{2m}[1 - (m - 1)^2 F_0^2]\left(\frac{ny_0}{S_o L}\right) \tag{13b}$$

As the value of n is increased the value of K decreases in accordance with eqn. (13a) and the value of a decreases in accordance with eqn. (13b) below the value given by eqn. (9) where the total channel length is treated as a single reach. The value of the reach length (L/n) for which the parameter a reduces to zero is the characteristic reach length as defined by Kalinin and Milyukov (1957) and used as the basis of a method of flood routing based on a cascade of linear reservoirs analogous to the Nash model (1958) for catchment response.

The distributed Muskingum model

It was shown by Strupczewski and Napiorkowski (1986) that, if the two other second-order terms in the linear St. Venant equation are expressed through the kinematic wave solution in terms of the mixed second-derivative term, then the resulting linear channel response can be interpreted as a limiting form of the multiple Muskingum model. In the present paper, the multiple Muskingum model is taken as the starting point of the process and the nature and properties of the limiting form derived before comparing this distributed Muskingum model with the solution of the linearised St. Venant equation. The procedure followed is a more general and more difficult analysis than the special case of demonstrating that the limiting form of a cascade of n equal linear reservoirs each of delay time K is given by a pure translation with a lag equal to nK as shown by Nash (1960). The latter pure delay corresponds to the linear kinematic wave solution which is obtained by neglecting all second terms in the linearised St. Venant equation and is an exact solution for the case of limiting stability e.g. $F_0 = 2$ for a wide rectangular channel with Chezy friction.

If in the multiple Muskingum model the value of n is increased indefinitely, then the value of the delay time K for each of the very short reaches must decrease indefinitely in such a way that nK, which represents the delay time for the total channel length, remains finite. When this is done we have for the first cumulant (i.e. the first moment about the origin):

$$k_1 = nK = \frac{L}{c_k} \tag{14}$$

It is clear from eqn. (13b) that for any given total channel length L the value of a will approach minus infinity as the number of short reaches n approaches infinity. Since the variance of the linear channel response is positive and finite, then we have from the expression for the second cumulant (i.e. the second moment about the centre):

$$k_2 = n[1 - 2a]K^2 \tag{15}$$

a second condition that $(1 - 2a)K$ must remain finite as K approaches zero and a approaches minus infinity.

The system function for the distributed Muskingum model can be obtained by letting n pass to infinity in eqn. (11) while maintaining the conditions of finite nK and finite $(1 - 2a)K$ mentioned in the last paragraph. The main steps in this limiting process are described in Appendix A. The system function for the distributed Muskingum model is found to be:

$$H_\infty(s) = \exp\left[-\frac{k_1 s}{1 + (k_2/2k_1)s} \right] \tag{16}$$

in which k_1 and k_2 are the cumulants of the linear response of the channel being simulated by the model.

Though the system function given by eqn. (16) is relatively simple, the inversion to the time domain is not straightforward. Since the model contains an infinite cascade of reaches, each represented in the time domain by eqn. (5) above, one would expect the infinite cascade to be represented by an infinite series. However, as shown in Appendix B we can write as the response function in the time domain:

$$h(t) = \exp(-2k_1^2/k_2)\delta(t) + \exp[-(t + k_1)2k_1/k_2]I_1[4(tk_1^3/k_2^2)^{1/2}][2(k_1^3/k_2^2 t)^{1/2}] \tag{17}$$

where $I_1[\quad]$ is a modified Bessel function of the first order.

3. PROPERTIES OF DISTRIBUTED MUSKINGUM MODEL

Moments and cumulants of DMM

The use of moments for characterising the scale and shape of response functions was introduced in hydrology by Nash (1959). The application of cumulants to the characterisation of linear channel response was a further development of this approach (Dooge and Harley, 1967a; Dooge, 1973). The first cumulant is identical to the first moment about the origin; the second and third cumulants are identical to the second and third moments about the centre; the fourth cumulant equals the fourth moment about the centre minus three times the square of the second moment about the centre; higher cumulants can be expressed in terms of central moments up to and including the same order (Kendall and Stuart, 1958). For computational purposes, moments are determined and cumulants derived from them for orders above three. In analysis, however, cumulants are more convenient because of the property that any cumulant of the output is equal to the sum of the cumulants of the same order of the input and of the response function.

When a response function is known, the cumulants can be determined analytically since the logarithm of the system function is a generating function for the cumulants (Dooge, 1973). Thus if $H(s)$ is the Laplace transform of the system response $h(t)$, the Rth cumulant of the response will be given by:

$$k_R(h) = (-1)^R \frac{\mathrm{d}^R}{\mathrm{d}s^R}[\log H(s)]_{s=0} \tag{18}$$

and can be obtained by successive differentiation. In the case of the distributed Muskingum model, the exponential form of the system function given in eqn. (16) makes this procedure particularly convenient.

The generating function for the cumulants of the distributed Muskingum model obtained by substituting the system function in eqn. (16) into the general expression in eqn. (18) is therefore given by:

$$G(s) = -\frac{k_1 s}{1 + (k_2/2k_1)s} \tag{19}$$

and the Rth cumulant of the DMM by:

$$k_R(\text{DMM}) = (-1)^R \frac{d^R}{ds^R}\left[-\frac{k_1 s}{1 + (k_2/2k_1)s}\right]_{s=0} \tag{20}$$

The process of successive differentiation of the generating function $G(s)$ and the derivation of the cumulants is described in Appendix C. It is shown there that the Rth cumulant of the distributed Muskingum model is given by:

$$k_R(\text{DMM}) = R!\left(\frac{k_2}{2k_1}\right)^{R-1} k_1 \tag{21}$$

for any value of R. It can be readily verified that the first and second cumulants which were preserved in the limiting process are correctly given by eqn. (21) above.

In using the model to simulate a given channel, the first and second cumulants can be matched to the known first and second moments of the prototype. These two cumulants are the only parameters required for the generation of the DMM response function. It is a common experience in applied hydrology that the performance of a conceptual model is improved by the including of a pure delay into the model. This can be done for the DMM by matching the second and third cumulants of model and prototype and making up the discrepancy in the first cumulant by a pure delay. In this case we would have:

$$k_2(\text{DMM}) = k_2(\text{prototype}) \tag{22a}$$

$$k_1(\text{DMM}) = \frac{3}{2}\frac{[k_2(\text{prototype})]^2}{k_3(\text{prototype})} \tag{22b}$$

$$T\ (\text{lag of DMM}) = k_1(\text{prototype}) - k_1(\text{DMM}) \tag{22c}$$

as the values of the three model parameters.

Shape factor diagrams

The use of dimensionless moments to characterise the shape of response functions was used by Nash (1960) in relation to unit hydrographs. This approach was adapted by Dooge and Harley (1967a) to the use of dimensionless cumulants in relation to the comparison of shapes of linear channel responses. Nash pointed out that in the case of a two-parameter model, the third and higher moments would be fixed once the two parameters of the model had been fixed from the first and second moments. Competing two-parameter models could therefore be compared with one another and with field data by plotting a dimensionless third moment against a dimensionless second moment.

Nash (1959) based his system of dimensionless second and higher moments on the first moment about the origin, i.e.:

$$m_R = \frac{U_R}{(U_1')^R} \tag{23}$$

where m_R is the dimensionless moment of order R, U_R is the Rth moment about the centre, and U_1' the first moment about the origin. In order to include orders above three without undue complication in analysis, it is more convenient to replace these dimensionless moments by dimensionless cumulants defined by:

$$s_R = \frac{k_R}{(k_1)^R} \tag{24}$$

where s_R is a shape factor of order R, k_R the Rth cumulant, and k_1 the first cumulant.

The classical Muskingum model ($n = 1$), the multiple Muskingum model ($n > 1$) and the distributed Muskingum model ($n = \infty$) can be conveniently compared on a shape factor diagram in which s_3 is plotted against s_2. For a cascade of n Muskingum reaches two first cumulants are given by eqns. (14) and (15) while the third one can be derived from eqn. (12):

$$k_3(\text{MMM}) = 2n(1 - 3a + 3a^2)K^3 \tag{25}$$

Substituting these values in eqn. (24) gives:

$$s_2 = \frac{(1 - 2a)}{n} \tag{26a}$$

$$s_3 = \frac{2(1 - 3a + 3a^2)}{n^2} \tag{26b}$$

Elimination of the parameter a between eqns. (26a) and (26b) gives the relationship:

$$s_3(\text{MMM}) = \frac{3}{2}(s_2)^2 + \frac{1}{2n^2} \tag{27}$$

For the particular case of $n = 1$, this expression becomes:

$$s_3(\text{CMM}) = \frac{3}{2}(s_2)^2 + \frac{1}{2} \tag{28}$$

while for $n = \infty$ it becomes:

$$s_3(\text{DMM}) = \frac{3}{2}(s_2)^2 \tag{29}$$

The latter expression can readily be confirmed from eqn. (21) which gives in general the expression for the Rth shape factor as:

$$s_R(\text{DMM}) = R!\left(\frac{k_2}{2k_1^2}\right)^{R-1} \tag{30a}$$

and its relationship with s_2 as:

$$s_R = R!\left(\frac{s_2}{2}\right)^{R-1} \tag{30b}$$

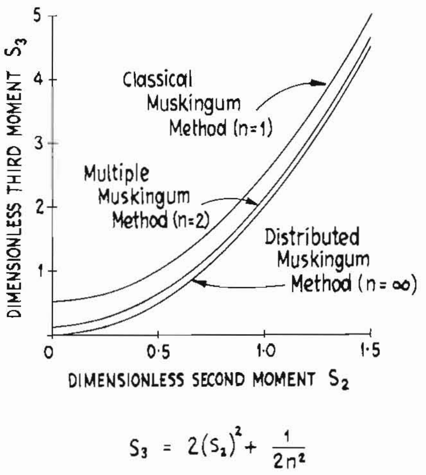

Fig. 1. Shape factors for Muskingum type models.

for all values of R greater than 2.

The relationship between s_3 and s_2 is shown on Fig. 1 for the classical Muskingum model ($n = 1$), the multiple Muskingum model with $n = 2$, and the distributed Muskingum model ($n = \infty$). It should be noted that on this diagram small values of s_2 correspond to long lengths of channel. For very long channels, s_2 for the classical Muskingum model approaches zero while s_3 approaches 0.5 thus leading, as noted in section 2, to a breakdown in the simulation of the channel response. For the multiple Muskingum model this limiting value of s_3 is $1/2n^2$ and thus the problem is rapidly eliminated as the value of n increases.

Amplitude and phase characteristics

In many other branches of geophysics, the properties of systems are discussed in relation to their response to a sinusoidal rather than an impulsive input (e.g. Pedlosky, 1979). The amplitude and phase spectra are used as the basis of comparison of various models and of the comparison of models with data. These spectra are also used to gain insight into the characteristics of the model or of the system being simulated.

The amplitude and phase spectra can be derived from the Fourier transform of the system response (Osiowski, 1972). This transform is readily obtained by taking only the imaginary part ($i\omega$) of the argument ($s = c + i\omega$) of the system function. Hence we can adapt eqn. (16) for the system function and write:

$$H(i\omega) = \exp\left[-\frac{\omega k_1 i}{1 + (k_2/2k_1)\omega i}\right] \tag{31a}$$

as the Fourier transform of the response function of the distributed Muskingum model. In order to separate the effects of attenuation with distance and phase shift along the channel it is necessary to split the term inside the square brackets into its real and imaginary parts. This is done by multiplying top and bottom by the complex conjugate of the denominator i.e. by $(1 - i\omega k_2/2k_1)$. When this is done we obtain:

$$H(i\omega) = \exp\left[-\left\{\frac{(\omega^2 k_2/2)}{1 + \omega^2(k_2/2k_1)^2} + \frac{\omega k_1}{1 + \omega^2(k_1/2k_1)^2}\, i\right\}\right] \tag{31b}$$

which when written as:

$$H(i\omega) = \exp\left[\frac{-(\omega^2 k_2/2)}{1 + \omega^2(k_2/2k_1)^2}\right]\exp\left[-i\,\frac{\omega k_1}{1 + \omega^2(k_2/2k_1)^2}\right] \tag{31c}$$

then is in the standard form:

$$H(i\omega) = A(\omega)\exp(i\phi) \tag{31d}$$

The first part of the right-hand side of eqn. (31c) represents the amplitude of the outflow from the channel reach and the second part represents the sinusoidal variation.

The salient features can be conveniently illustrated by plotting the amplitude and phase spectra in terms of the dimensionless frequency given by the product of the frequency (ω) and the lag (k_1). This gives for the amplitude:

$$A(\omega k_1) = \exp\left[-\frac{\frac{1}{2}(k_2/k_1^2)(\omega k_1)^2}{1 + \frac{1}{4}(k_2/k_1^2)^2(\omega k_1)^2}\right] \tag{32}$$

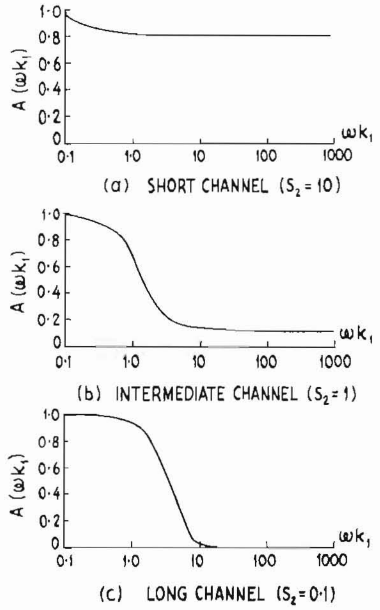

Fig. 2. Amplitude characteristic for DMM.

in which it is seen that the parameter controlling the amplitude attenuation is the shape factor s_2 defined by eqn. (24). The variation of the amplitude reduction with dimensionless frequency is shown in Fig. 2. It will be noted that the attenuation is quite small for low frequencies, and becomes constant for high frequencies at a level appropriate for the parameter s_2.

The phase shift can similarly be written in terms of the dimensionless frequency as:

$$\phi(\omega k_1) = \frac{-\omega k_1}{1 + \frac{1}{4}(k_2/k_1^2)^2(\omega k_1)^2} \tag{33}$$

The phase shift is seen to be vanishing small for both very low and very high frequency. It can be deduced from eqn. (33) that the greatest negative phase shift occurs when:

$$\omega k_1 = \frac{2k_1^2}{k_2} \tag{34}$$

Though frequency analysis has not been widely used in hydrology, the above brief analysis shows that such an analysis can be quite informative.

4. COMPARISON WITH ST. VENANT EQUATION

The linearised St. Venant solution

The linearised St. Venant equation for one-dimensional unsteady flow in uniform channel may be written as (Dooge et al., 1987a):

$$(1 - F_0^2)gy_0 \frac{\partial^2 Q^1}{\partial x^2} - 2u_0 \frac{\partial^2 Q^1}{\partial x \partial t} - \frac{\partial^2 Q^1}{\partial t^2} = gA_0\left(\frac{\partial S_f}{\partial Q}\frac{\partial Q^1}{\partial t} - \frac{\partial S_f}{\partial A}\frac{\partial Q^1}{\partial x}\right) \tag{35}$$

where Q^1 is the perturbation of flow about an initial condition of unsteady uniform flow Q_0, A_0 is the cross-sectional area corresponding to this flow, S_f is the friction slope, y_0 is the hydraulic mean depth, u_0 is the mean velocity, S_0 is the bottom slope, x is the distance from the upstream boundary, t is the elapsed time and derivatives of the friction slope S_f are evaluated at the reference conditions.

The variation of friction slope with discharge at the reference condition for all frictional formula for rough turbulent flow may be expressed as:

$$\frac{\partial S_f}{\partial Q} = 2\frac{S_0}{Q_0} \tag{36}$$

We may for convenience define a parameter m as a ratio of the kinematic wave speed to the average velocity of flow:

$$m = \frac{c_k}{u_0} \tag{37}$$

where c_k is the kinematic wave speed as given by Lighthill and Whitham (1955):

$$c_k = \frac{dQ}{dA} = -\frac{\partial S_f}{\partial A}\Big/\frac{\partial S_f}{\partial Q} \tag{38}$$

The parameter m is a function of the shape of channel and of the friction law used. When eqns. (36), (37) and (38) are substituted in eqn. (35) one gets:

$$(1 - F_0^2)\frac{\partial^2 Q^1}{\partial x^2} - \frac{2F_0^2}{u_0}\frac{\partial^2 Q^1}{\partial x \partial t} - \frac{F_0^2}{u_0^2}\frac{\partial^2 Q^1}{\partial t^2} = \frac{2mS_0}{y_0}\frac{\partial Q^1}{\partial x} + \frac{2S_0}{y_0 u_0}\frac{\partial Q^1}{\partial t} \tag{39}$$

where F_0 is the Froude number of the reference flow.

The case of a downstream wave for a Froude number less than one was solved by Deymie (1935) and independently by Dooge and Harley (1967b) who investigated systematically the properties of the solution for the case of a wide rectangular channel with Chezy friction. This work has since been extended to cover the case of any shape of channel and any friction law (Dooge et al., 1987). The first four cumulants of the solution are:

$$k_1 = \frac{L}{c_k} \tag{40a}$$

$$k_2 = \frac{1}{m}[1 - (m-1)^2 F_0^2]\left(\frac{y_0}{S_0 x}\right)\left(\frac{L}{c_k}\right)^2 \tag{40b}$$

$$k_3 = \frac{3}{m^2}[1 - (m-1)^2 F_0^2][1 + (m-1)^2 F_0^2]\left(\frac{y_0}{S_0 x}\right)^2\left(\frac{L}{c_k}\right)^3 \tag{40c}$$

$$k_4 = \frac{15}{m^3}[1 - (m-1)^2 F_0^2]\left[1 - \left(\frac{m^2 - 10m + 10}{5}\right)^2 F_0^2 + (m-1)^2 F_0^4\right] \times \left(\frac{y_0}{S_0 x}\right)^3\left(\frac{L}{c_k}\right)^4 \tag{40d}$$

The shape factors as defined by eqn. (24) are given by:

$$s_2 = \frac{1}{m}[1 - (m-1)^2 F_0^2]\left(\frac{y_0}{S_0 x}\right) \tag{41a}$$

$$s_3 = \frac{3}{m^2}[1 - (m-1)^2 F_0^2][1 + (m-1) F_0^2]\left(\frac{y_0}{S_0 x}\right)^2 \tag{41b}$$

so that the relationship between the two shape factors is:

$$s_3 = 3\left[\frac{1 + (m-1)F_0^2}{1 - (m-1)^2 F_0^2}\right](s_2)^2 \tag{42}$$

which is applicable for Froude numbers between zero and one i.e. for tranquil flow.

Simplified forms of St. Venant equation

A number of models of simplified forms of the complete St. Venant equation given by eqn. (35) of the last section have been proposed in the hydrological literature. If all three of the second-order terms on the left-hand side of that equation are neglected we obtain:

$$gA_0\left(\frac{\partial S_f}{\partial Q}\frac{\partial Q^1}{\partial t} - \frac{\partial S_f}{\partial A}\frac{\partial Q^1}{\partial x}\right) = 0 \tag{43a}$$

which is equivalent to:

$$\frac{\partial Q^1}{\partial t} + c_k\frac{\partial Q^1}{dx} = 0 \tag{43b}$$

where c_k is kinematic wave speed defined by eqn. (38). The solution of this linear equation is:

$$Q^1(x,t) = f\left(t - \frac{x}{c_k}\right) = \text{constant} \tag{44}$$

which represents a pure translation. The system function of this solution is:

$$H(s) = \exp\left(-\frac{x}{c_k}s\right) \tag{45a}$$

which is also:

$$H(s) = \exp\,(-k_1 s) \tag{45b}$$

The first cumulant:

$$k_1 = \frac{x}{c_k} \tag{46}$$

reproduces exactly the first cumulant of the complete solution as given by eqn. (40a) and all the higher cumulants are zero. Equation (43) therefore represents an adequate first-order approximation and can be used as the basis of a first-order analysis of flood waves (Lighthill and Whitham, 1955).

A new-order approximation can be obtained by using eqn. (44) to approximate two of the terms on the left-hand side of eqn. (35) in terms of the remaining third term. These approximations are:

$$\frac{\partial^2 Q^1}{\partial x^2} = \frac{1}{c_k^2}f''(t - k_1 x) \tag{47a}$$

$$\frac{\partial^2 Q^1}{\partial x \partial t} = -\frac{1}{c_k}f''(t - k_1 x) \tag{47b}$$

$$\frac{\partial^2 Q^1}{\partial t^2} = f''(t - k_1 x) \tag{47c}$$

If the second and third terms are expressed in terms of the first we have:

$$-2u_0\frac{\partial^2 Q^1}{\partial x \partial t} = 2mu_0^2\frac{\partial^2 Q^1}{\partial x^2} \tag{48a}$$

$$\frac{\partial^2 Q^1}{\partial t^2} = -m^2 u_0^2 \frac{\partial^2 Q^1}{\partial x^2} \tag{48b}$$

where m is the ratio of the kinematic wave c_k to the reference velocity u_0. Substitution from eqn. (48) into eqn. (35) gives:

$$g y_0 [1 - (m - 1)^2 F_0^2] \frac{\partial^2 Q^1}{\partial x^2} = \frac{2 g S_0}{u_0} \frac{\partial Q^1}{\partial t} + 2 m g S_0 \frac{\partial Q^1}{\partial x} \tag{49a}$$

which when written in the form:

$$[1 - (m - 1)^2 F_0^2] \frac{Q_0}{2 S_0 T} \frac{\partial^2 Q^1}{\partial x^2} = \frac{\partial Q^1}{\partial t} + m u_0 \frac{\partial Q^1}{\partial x} \tag{49b}$$

is seen to be an advective diffusion equation with the advective parameter:

$$a = m u_0 \tag{50a}$$

and the diffusivity parameter D:

$$D = [1 - (m - 1)^2 F_0^2] \frac{Q_0}{2 S_0 T_0} \tag{50b}$$

where T_0 is the surface width of the channel at reference conditions. The system function of this diffusion analogy approximation is given by:

$$H(s) = \exp\left[- \left(1 - \sqrt{1 + \frac{Ds}{a^2}}\right) \frac{ax}{2D} \right] \tag{51}$$

and the cumulants of the solution by:

$$k_R = \{1,3,5 \ldots (2R - 3)\} \left(\frac{2D}{ax}\right)^{R-1} \left(\frac{x}{a}\right)^R \tag{52}$$

Comparison of eqns. (50) and (52) with eqn. (40) above reveals that the diffusion analogy model gives the correct value for the second cumulant for any value of F_0. Further comparison for higher cumulants reveals that the diffusion analogy model is identical to the complete solution for the special case of $F_0 = 0$.

If the alternative approach is taken of expressing all the second-order terms as cross-derivatives, then we have the relationships:

$$g y_0 (1 - F_0^2) \frac{\partial^2 Q^1}{\partial x^2} = -\frac{g y_0}{c_k} (1 - F_0^2) \frac{\partial^2 Q^1}{\partial x \partial t} \tag{53a}$$

$$-\frac{\partial^2 Q^1}{\partial t^2} = -c_k \frac{\partial^2 Q^1}{\partial x \partial t} \tag{53b}$$

so that the second-order approximation becomes:

$$\frac{g y_0}{m u_0} [1 - (m - 1)^2 F_0^2] \frac{\partial^2 Q^1}{\partial x \partial t} + \frac{2 g S_0}{u_0} \frac{\partial Q^1}{\partial t} + 2 m g S_0 + \frac{\partial Q^1}{\partial x} = 0 \tag{54a}$$

or:

$$\frac{y_0}{2mS_0}\left[1-(m-1)^2F_0^2\right]\frac{\partial^2 Q^1}{\partial x\partial t}+\frac{\partial Q^1}{\partial t}+mu_0\frac{\partial Q^1}{\partial x} = 0 \tag{54b}$$

which can be conveniently written as:

$$\frac{D}{a}\frac{\partial^2 Q^1}{\partial x\partial t}+\frac{\partial Q^1}{\partial t}+a\frac{\partial Q^1}{\partial x} = 0 \tag{54c}$$

where D and a have the same values as defined by eqn. (50). The solution of this equation has the system function:

$$H(s) = \exp\left[-\left(\frac{s}{a+\frac{D}{a}s}\right)x\right] \tag{55}$$

which is identical in form to the system function for the distributed Muskingum model as given by eqn. (16). Comparison of eqns. (16) and (55) gives:

$$k_1 = \frac{x}{a} = \frac{x}{mu_0} \tag{56a}$$

and:

$$k_2 = \frac{2Dx}{a^3} = \frac{2D}{a^2}\frac{x}{a} = \frac{2D}{ax}\left(\frac{x}{a}\right)^2$$

$$= \frac{1}{m}\left[1-(m-1)^2F_0^2\right]\left(\frac{y_0}{S_0x}\right)\left(\frac{x}{mu_0}\right)^2 \tag{56b}$$

which are again the same as for the complete linear solution. From eqn. (21) we have:

$$k_3 = 6\left(\frac{k_2}{2k_1}\right)^2 k_1 \tag{57a}$$

which in terms of a and D is:

$$k_3 = 6\left(\frac{D}{a^2}\right)^2\frac{x}{a} \tag{57b}$$

$$= \frac{3}{2}\left(\frac{2D}{ax}\right)^2\left(\frac{x}{a}\right)^3$$

which is half the value for the diffusion analogy and hence for the complete linear solution with $F_0 = 0$.

For higher orders of cumulant there is also a difference between the cumulant given by eqn. (21) and that given by eqn. (52). The ratio reduces from 0.5 for $R = 3$ to 0.2 for $R = 4$ and 0.07 for $R = 5$.

Comparison of conceptual model with complete equation

In the previous section it was pointed out that since the first and second moments are the same in all models, the efficiency of any conceptual model in representing the linearised St. Venant solution can be judged from the third cumulant.

For the multiple Muskingum model we have from eqns. (14), (15) and (25) that:

$$k_3 = \frac{3}{2}\frac{k_2^2}{k_1} + \frac{k_1^3}{2n^2} \tag{58}$$

The classical Muskingum model, the multiple Muskingum model and the distributed Muskingum model are shown on an $s_2{:}s_3$ shape diagram on Fig. 1.

If the first two cumulants are matched to those given for the complete St. Venant equation by eqns. (40a) and (40b) we have the predicted value of the third cumulant of the classical Muskingum model as:

$$k_3(\text{CMM}) = \frac{3}{2m^2}[1 - (m-1)^2F_0^2]^2\left(\frac{y_0}{S_0x}\right)^2\left(\frac{x}{mu_0}\right)^3 + \frac{1}{2}\left(\frac{x}{mu_0}\right)^3 \tag{59}$$

whereas the true value of this third cumulant is given by eqn. (40c). Consequently the ratio of the two values is:

$$\frac{k_3(\text{CMM})}{k_3(\text{LSV})} = \frac{1}{2}\left[\frac{1-(m-1)^2F_0^2}{1+(m-1)F_0^2}\right] + \frac{m^2}{6} \times \left[\frac{1}{[1-(m-1)^2F_0^2][1+(m-1)F_0^2]}\right]\left(\frac{S_0x}{y_0}\right)^2 \tag{60}$$

Equation (60) clearly shows that for small lengths of channel the classical Muskingum model will underestimate the true value of k_3 but for longer lengths it will overestimate this value to a greater and greater extent as the length of the channel increases. For the particular dimensionless channel length given by:

$$\left(\frac{S_0x}{y_0}\right)^2 = \frac{3}{m^2}[1 + (m^2-1)F_0^2][1-(m-1)^2F_0^2] \tag{61}$$

the two values will be equal. For $m = 3/2$ and $F_0 = 0$ eqn. (60) becomes:

$$\frac{k_3(\text{CMM})}{k_3(\text{LSV})} = \frac{1}{2} + \frac{3}{8}\left(\frac{S_0x}{y_0}\right)^2 \tag{62}$$

so that the dimensionless length for zero error is 1.15 but the error reaches

150% if the channel is double this length. For $m = 3/2$ and $F_0 = 1$, eqn. (60) becomes:

$$\frac{k_3(\text{CMM})}{k_3(\text{LSV})} = \frac{1}{4} + \frac{1}{3}\left(\frac{S_0 x}{y_0}\right)^2 \tag{63}$$

so that dimensionless length for zero error is 1.50 and the error for double this length is 225%.

For the distributed Muskingum model the predicted third cumulant is given by eqn. (57a). If the first and second cumulants are matched to the values for the St. Venant solution given by eqns. (40a) and (40b) the predicted third cumulant will be:

$$\hat{k}_3(\text{DMM}) = \frac{3}{2m^2}\left[1 - (m - 1)^2 F_0^2\right]^2 \left(\frac{y_0}{S_0 x}\right)^2 \left(\frac{x}{mu_0}\right)^3 \tag{64}$$

The ratio of this predicted value to the true value given by equation is:

$$\frac{\hat{k}_3(\text{DMM})}{k_3(\text{LSV})} = \frac{1}{2}\frac{\left[1 - (m - 1)^2 F_0^2\right]}{\left[1 + (m - 1)F_0^2\right]} \tag{65}$$

It is clear that in this case the error is independent of the channel length and depends only on the values of m and F_0. For $m = 3/2$ the ratio in eqn. (65) becomes:

$$\frac{\hat{k}_3(\text{DMM})}{k_3(\text{LSV})} = \frac{1}{2}\left(\frac{1 - F_0^2/4}{1 + F_0^2/2}\right) \tag{66}$$

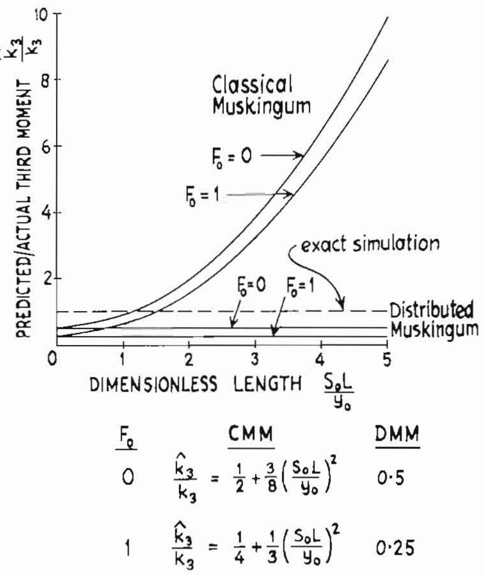

Fig. 3. Error in predicting third moment.

which gives a constant error of 50% for $F_0 = 0$ and a constant error of 75% for $F_0 = 1$. The values of the ratio of predicted to actual third cumulants for $F_0 = 0.0$ and $F_0 = 1.0$ for a range of dimensionless length $(S_0 x/y_0)$ from 0 to 5 is shown on Fig. 3.

Up to here the efficiency of the DMM in representing the LSV solution was assessed in terms of the impulse response. As an alternative it can be judged in terms of the frequency response for cosinusoidal input function. The amplitude and phase characteristics of the DMM are given by eqns. (32) and (33) in terms of cumulants. Using eqns. (8b) and (8c) in Appendix C one can get them in terms of the channel parameters:

$$A_{\mathrm{DMM}}(x, \omega) = \exp\left(-\frac{\alpha\beta\omega^2}{1+\beta^2\omega^2}\right) \tag{67}$$

$$\phi_{\mathrm{DMM}}(x, \omega) = -\frac{\alpha\omega}{1+\beta^2\omega^2} \tag{68}$$

The properties of the transfer function for the Linear Channel Response (LCR), i.e. the solution of the linearised St. Venant equation for a semi-infinite uniform channel, and for $F_0 < 1$ was analysed by Dooge et al. (1987b). The transfer function of the LCR is given by:

$$H_{\mathrm{LCR}}(x, s) = \exp(\mathrm{d}s + \sqrt{c} - \sqrt{as^2 + bs + c}) \tag{69}$$

where the coefficients are related to the parameters of the channel as follows:

$$a = m^2 \frac{F_0^2}{(1-F_0^2)^2}\alpha^2 \tag{70a}$$

$$b = \frac{[1+(m-1)F_0^2][1-(m-1)^2 F_0^2]}{(1-F_0^2)^2}\frac{\alpha^2}{\beta} \tag{70b}$$

$$c = 0.25\frac{[1-(m-1)^2 F_0^2]^2}{(1-F_0^2)}\frac{\alpha^2}{\beta^2} \tag{70c}$$

$$d = m\frac{F_0^2}{1-F_0^2}\alpha \tag{70d}$$

The frequency characteristics of the LCR are (Dooge et al., 1987b):

$$A_{\mathrm{LCR}}(x, \omega) = \exp\left(c^{0.5} - \frac{\{[b^2\omega^2 + (-a\omega^2 + c)^{0.5}]^{0.5} - a\omega^2 + c\}^{0.5}}{\sqrt{2}}\right) \tag{71}$$

$$\phi_{\mathrm{LCR}}(x, \omega) = d\omega - \frac{\{[b^2\omega^2 + (-a\omega^2 + c)^2]^{0.5} + a\omega^2 - c\}^{0.5}}{\sqrt{2}} \tag{72}$$

where the parameters a, b, c and d are defined by eqns. (70).

Figure 4 shows the amplitude spectra of the DMM and the LSV for the case of a wide rectangular channel of unit dimensionless length ($D = 1$) with the Chezy friction ($m = 1.5$) and two Froude numbers ($F = 0.2$ and $F = 0.8$). They are shown as functions of dimensionless frequency $\omega^1 = \omega y_0/S_0 u_0$.

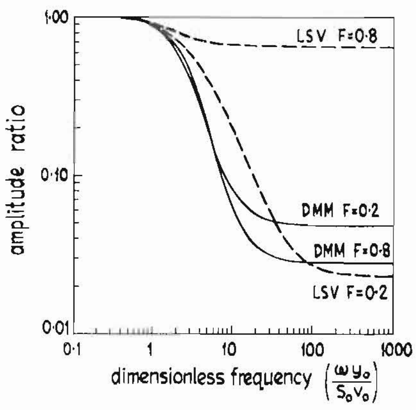

Fig. 4. Amplitude reduction per unit length.

In the case of the attenuation (Fig. 4) the results will differ for various Froude numbers. It can be seen that the DMM gives a good approximation of the LCR for low Froude numbers only. It should be noted that the amplitudes for infinite frequency do not decay to zero thus indicating infinite power. Note, that for other lengths of the channel the logarithm of the amplitude reduction and the phase shift will be proportional to the dimensionless channel length.

5. CONCLUSIONS

The study described in this paper leads to the following conclusions:

(1) The process of dividing a total channel length into shorter reaches each of which is modelled as a Muskingum reach can be carried to the limit of an infinite cascade of such infinitesimal reaches without producing any physical contradiction.

(2) In order to preserve a finite first moment about the origin in the impulse response, the product of the two parameters n and K must remain finite as n goes to infinity and K goes to zero. In order to preserve a finite second moment about the centre, the weighting parameter a must approach minus infinity as K approaches zero in such a way that their product remains finite.

(3) The system function (i.e. the Laplace transform of the impulse response) of the resulting distributed Muskingum model is given by:

$$H_\infty(s) = \exp\left[-\frac{k_1 s}{1 + (k_2 s/2k_1)}\right] \tag{16}$$

where k_1 and k_2 are the first and second cumulants respectively.

(4) The general expression for any cumulant of the distributed Muskingum model is given by:

$$k_R = R!\left(\frac{k_2}{2k_1}\right)^{R-1} k_1 \tag{21}$$

where R is the order of cumulant involved.

(5) The error of the distributed Muskingum model in predicting the third cumulant of the linear St. Venant solution is independent of the length of the channel, thus indicating that the problem of dealing with long channels by the classical Muskingum method is entirely overcome.

(6) The errors of the distributed Muskingum model in predicting the third cumulant vary from 50% for $F_0 = 0$ to 75% for $F_0 = 1$.

APPENDIX

A. System function for distributed Muskingum model

The system function for the multiple Muskingum model is given by eqn. (11) of the main text:

$$H_n(s) = \left[\frac{1 - aKs}{1 + (1 - a)Ks}\right]^n \tag{A1}$$

By combining eqns. (14) and (15) of the main text we can write:

$$a = \frac{1}{2}\left(1 - \frac{k_2}{k_1}\frac{1}{K}\right) \tag{A2}$$

where k_1 and k_2 are the cumulants of the linear channel response being modelled. Substituting in eqn. (A1) the system response is now written as:

$$H_n(s) = \left[\frac{1 - Ks/2 + (k_2/2k_1)s}{1 + Ks/2 + (k_2/2k_1)s}\right]^n \tag{A3}$$

By grouping together the finite and infinitesimal parts and using eqn. (14) of the main text again we can write:

$$H_n(s) = \left[\frac{1 + (k_2/2k_1)s - k_1s/2n}{1 + (k_2/2k_1)s + k_1s/2n}\right]^n \tag{A4}$$

Dividing top and bottom by the common finite terms this becomes:

$$H_n(s) = \left\{\frac{1 - [(k_1s/2)/(1 + k_2s/2k_1)]\ 1/n}{1 + [(k_1s/2)/(1 + k_2s/2k_1)]\ 1/n}\right\}^n \tag{A5}$$

From the standard mathematical theorem (see e.g. Hardy, 1908–1952 for proof):

$$\lim_{n \to \infty}\left[1 \pm \frac{c}{n}\right]^n = \exp(\pm c) \tag{A6}$$

the limiting form of the numerator in eqn. (A5) as n goes to infinity is given by:

$$N = \exp\left[-\frac{k_1s/2}{1 + k_2s/2k_1}\right] \tag{A7a}$$

and the limiting form for the denominator by:

$$D = \exp\left[+\frac{k_1s/2}{1 + k_2s/2k_1}\right] \tag{A7b}$$

which readily combine to give:

$$H_{\infty} = \exp\left[- \frac{k_1 s}{1 + k_2 s/2k_1} \right] \tag{A7c}$$

Since the cumulants k_1 and k_2 are both finite for real channels, the latter expression can be used with confidence.

B. Inversion of system function to time domain

The system function for the distributed Muskingum model given by eqn. (A7c) and by eqn. (16) of the main text can be readily inverted to the time domain. For convenience the system function is written as:

$$H(s) = \exp\left(- \frac{\alpha s}{1 + \beta s} \right) \tag{A8a}$$

where:

$$\alpha = k_1 \tag{A8b}$$

and:

$$\beta = \frac{k_2}{2k_1} \tag{A8c}$$

The first step is to clear the numerator of the transform variable s by writing:

$$H(s) = \exp\left(- \frac{\alpha}{\beta} + \frac{\alpha/\beta}{1 + \beta s} \right) \tag{A9a}$$

which is equivalent to:

$$H(s) = \exp\left(- \frac{\alpha}{\beta} \right) \exp\left(\frac{\alpha/\beta}{1 + \beta s} \right) \tag{A9b}$$

The second step is to expand the second exponential as an infinite series, i.e.:

$$H(s) = \exp\left(- \frac{\alpha}{\beta} \right) \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha/\beta}{1 + \beta s} \right)^n \right] \tag{A10a}$$

The first term of the infinite series is unity and can be separated out to give:

$$H(s) = \exp\left(- \frac{\alpha}{\beta} \right) \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\alpha/\beta}{1 + \beta s} \right)^n \right] \tag{A10b}$$

The third step is to use the standard transform pair:

$$\left(\frac{1}{1 + \beta s} \right)^n \leftrightarrow \frac{\exp(-t/\beta)(t/\beta)^{n-1}}{\beta(n-1)!} \tag{A11}$$

to invert the series in eqn. (A10b) term by term. When this is done the inverse of the system function $H(s)$ in the time domain is found to be:

$$h(t) = \exp\left(- \frac{\alpha}{\beta} \right) \left[\delta(t) + \exp\left(- \frac{t}{\beta} \right) \sum_{k=0}^{\infty} \frac{(\alpha/\beta)^{k+1} (t/\beta)^k}{(k+1)!k!\beta} \right] \tag{A12a}$$

which can be expressed as:

$$h(t) = \exp\left(-\frac{\alpha}{\beta}\right)\left[\delta(t) + \exp(-t/\beta)I_1\left(2\sqrt{\frac{\alpha t}{\beta^2}}\right)\left(\frac{\alpha}{\beta^2 t}\right)^{1/2}\right] \tag{A12b}$$

where $I_1(\ )$ is the modified Bessel function of the first order defined by:

$$I_1(z) = \left(\frac{1}{2}z\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{(k)!(k+1)!} \tag{A12c}$$

The insertion of the parameter values from eqns. (A8b) and (A8c) gives:

$$h(t) = \exp(-2k_1^2/k_2)\delta(t) + \exp[-(t+k_1)\,2k_1/k_2]I_1[4(tk_1^3/k_2^2)^{1/2}][2(k_1^3/k_2^2t)^{1/2}] \tag{A13}$$

as the required inversion to the time domain of eqn. (16) in the main text.

C. Cumulants of distributed Muskingum model

The Rth cumulant of the distributed Muskingum model is given by:

$$k_R(\mathrm{DMM}) = (-1)^R\frac{\mathrm{d}^R}{\mathrm{d}s^R}[G(s)]_{s=0} \tag{A14}$$

where $G(s)$ is the cumulant generating function obtained by substituting from eqn. (16) into eqn. (18) of the main text, thus obtaining:

$$G(s) = \left[-\frac{k_1 s}{1 + (k_2/2k_1)s}\right] \tag{A15}$$

where k_1 is the first cumulant (i.e. the lag) and k_2 the second cumulant (i.e. the variance) of the linear response of the channel.

Differentiating the cumulant generating function once we obtain:

$$\frac{\mathrm{d}G}{\mathrm{d}s} = -\frac{[1 + (k_2/2k_1)s]k_1 - k_1 s\,(k_2/2k_1)}{[1 + (k_2/2k_1)s]^2} \tag{A16a}$$

$$\frac{\mathrm{d}G}{\mathrm{d}s} = -\frac{k_1}{[1 + (k_2/2k_1)s]^2} \tag{A16b}$$

Hence from eqn. (A14) with $R = 1$ we have confirmation that:

$$k_1(\mathrm{DMM}) = k_1 \tag{A17}$$

Differentiating eqn. (A15) a second time we obtain:

$$\frac{\mathrm{d}^2G}{\mathrm{d}s^2} = \frac{2(k_2/2k_1)k_1}{[1 + (k_2/2k_1)s]^3} \tag{A18}$$

which on substitution into eqn. (A8) with $R = 2$ gives:

$$k_2(\mathrm{DMM}) = k_2 \tag{A19}$$

as would be expected since the system function derivation was based on going to the limit in such a way as to preserve the finite second moment about the centre.

Because of the simple form of eqn. (A15) the higher derivatives of the generating function can be written in the general form:

$$\frac{\mathrm{d}^R G}{\mathrm{d}s^R} = \frac{(-1)^R R!\,(k_2/2k_1)^{R-1}k_1}{[1 + (k_2/2k_1)s]^{R+1}} \tag{A20}$$

which is valid for all values of R including $R = 1$ and $R = 2$. Substitution from eqn. (A20) into eqn. (A14) gives the expression:

$$k_R(\mathrm{DMM}) = R!\left(\frac{k_2}{2k_1}\right)^{R-1} k_1 \tag{A21}$$

which is valid for any value of R.

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