Stochastic properties of the processes transformed in linear hydrological systems

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Abstract. The transformation of white noise and the Markov process through the linear, time invariant river system is examined. The systems presented correspond to the linear reservoir model, the Nash model, the Muskingum model and two linear river system models with common input.

Normalized functions of auto- and cross-correlation are presented. The time series obtained by integration of the output process are also considered.

Propriétés stochastiques des processus transformés par des systèmes hydrologiques linéaires

Résumé. La communication traite de la transformation d'un bruit blanc et d'un processus markovien par un système fluvial linéaire et invariant dans le temps. Les systèmes présentés correspondent à un modèle de réservoirs linéaires (modèle de Nash et modèle de Muskingum) et a un modèle comportant deux systèmes fluviaux linéaires avec données d'entrée communes.

Les auteurs présentent les fonctions normalisées d'auto-corrélation et de corrélation croisée. On considère aussi les séries chronologiques obtenues par intégration du processus gouvernant les données de sortie.

INTRODUCTION

The development of water resources research has created the need for an extension of the mathematical analysis of hydrological data. In recent years, the centre of gravity of such research has moved away from the applications of the theory of random variables towards the theory of random functions. An awareness of the stochastic structure of the hydrological processes is necessary for modelling water resource systems. The generalization and transfer of experience gained from different basins and relating to the structure of inflow-outflow processes and types of distribution is of great importance. It allows the utilization of longer time series for extrapolation and estimation of shorter series parameters.

The first fundamental assumption usually made in hydrological system modelling is linearity. This assumption is sufficiently valid for most engineering and scientific purposes. It is the evident simplification of the actual nonlinear problems that results from the compromise between simplicity and accuracy. Although nonlinear system modelling is already extensively developed, due to inherent complexity it does not seem useful for stochastic hydrology applications.

The generalization of data obtained from different basins is difficult because of differences in the structure of precipitation-runoff operators, observation and measurement errors and the input processes' structure of space variability. The relations between runoff processes and system operators are examined here, based on the assumption of identical stochastic structure of input processes. The correlation function formulae have been obtained which allow the utilization of information from similar basins for the estimation of stochastic processes or extension and supplementation of observational sequences of runoff.

TRANSFORMATION OF THE PROCESSES IN LINEAR BASIN MODELS (SINGLE RESERVOIR, NASH CASCADE AND MUSKINGUM MODEL)

According to many papers, for example, Kisiel (1969), Yevjevich (1972) and Dooge (1972), a hydrological system can be represented by

$$y(t) = H_d[x_d(t)] + H_s[x_s(t)] \tag{1}$$

where

x_d = deterministic input signal;
x_s = random noise input signal;
H_d = linear model operator transforming the deterministic component of the input signal;
H_s = linear model operator transforming the random component of the input signal;
$y(t)$ = output signal.

When analysing the deterministic component of the input signal, it is most commonly represented as a sum of three components:

(1) the linear (or quasilinear) trend representing the slowly varying tendency observed in the system due to the influence of human activity on the physical conditions;
(2) the jump component representing brief changes in the process input mean;
(3) the periodic component resulting from the periodicity of precipitation, outflow, etc.

The first two components have usually relatively small values which make them difficult to identify. The empirical material gathered does not allow quantitive determination.

Let us assume the system to be linear, deterministic and time invariant. The responses of the model to each input component can then be considered separately according to the principle of superposition.

The separation of the deterministic component of the input signal from the stochastic component enables the assumption of constant mean of the stochastic signal analysed. This is the necessary condition for weak stationarity. Moreover, we shall assume the stationarity of the second moments of the processes $x_s(t)$ throughout the paper.

As part of the analysis, we shall consider the structure of the random component transformed by some linear basin models (single reservoir, Nash cascade and Muskingum model).

The input stochastic processes analysed here are white noise and Markovian noise. These processes are commonly used in stochastic hydrology due to their simplicity and existing relationship to real processes.

White noise is the process having zero mean and autocovariance given by

$$R_{xx}(\tau) = C\,\delta(\tau) \tag{2}$$

where $\delta(\tau)$ is the Dirac delta function.

The stationary and normal simple Markovian noise has the autocovariance function given by

$$R_{xx}(\tau) = D_x^2\,\mathrm{e}^{-c|\tau|} \tag{3}$$

where D_x^2 is the variance of the input process and c is a constant.

The structure of processes obtained in the output of our models with the above two types of input process is given in Table 1. The table is based on the following assumptions.

(1) The input is assumed to be the inflow of water to the system and the output is the mean outflow of water from the system. Thus the general formulae for cross-covariance and autocovariance are

$$R_{yx}(\tau) = \int_{-\infty}^{+\infty} R_x(\tau-\alpha)\,h(\alpha)\,\mathrm{d}\alpha \tag{4}$$

$$R_{yy}(\tau) = \int_{-\infty}^{+\infty} R_{yx}(\tau+\alpha)\,h(\alpha)\,\mathrm{d}\alpha \tag{5}$$

(2) The autocovariance functions of input processes are described by equations (2) and (3) for white noise and simple, normal Markovian noise, respectively.

The subject of great practical importance is the derivation of the correlative relations between the corresponding (in the sense of maximum correlation) observations at two points. Let us consider the system composed of two linear models in parallel. For the case of white noise on the inputs of both subsystems, the output autocovariance function has the form

$$R_{y_1 y_2}(\tau) = \frac{A\,\tau^{2n-1}}{(KL)^n (n-1)!}\,\mathrm{e}^{-\tau/K} \sum_{p=0}^{n-1} \frac{(n+p-1)!}{p!(n-p-1)!\,z^{n+p}} \tag{6}$$

where n is the number of linear reservoirs forming a Nash cascade in every subsystem (equal for both subsystems), $z = \tau(1/K+1/L)$.

For zero time lag the above expression obtains its maximum.

$$R_{y_1 y_2}(0) = A\,\frac{(2n-2)!}{[(n-1)!]^2}\,\frac{(KL)^{n-1}}{(K+L)^{2n-1}} \tag{7}$$

The special case of the above relation obtained for common input signals $x_1 = x_2$ is of great interest for hydrological applications. It represents the situation of two adjacent basins having common input processes. The structure and parameters of the output process of the basin can be then determined, based on the known inflow process and parameters of both systems. It is worth considering however, that realizations of the outflow process can often be measured more easily and more accurately than by the input stochastic process. Thus, correlative functions of the outputs for both subsystems may be utilized in practice. The

TABLE 1. Cross-covariance and autocovariance formulae (valid for $\tau \geqslant 0$)

	Cross-covariance	Autocovariance
White noise input		
Linear reservoir	$\frac{c}{K}\,\mathrm{e}^{-\tau/K}$	$\frac{c}{2K}\,\mathrm{e}^{-\tau/K}$
Nash's cascade of n reservoirs	$\frac{c}{K^n\,\Gamma(n)}\,\tau^{n-1}\,\mathrm{e}^{-\tau/K}$	$\frac{c}{K^n\,\Gamma(n)\,2^n}\,\mathrm{e}^{-\tau/K}\sum_{j=0}^{n-1}\frac{\tau^{n-j-1}\,K^j(n+j-1)!}{(n-j-1)!\,j!\,2^j}$
Muskingum model		$D_x^2\left[\frac{(L+K)^2\,\mathrm{e}^{-\tau/L}}{2L^3}-\frac{(L+K)\,K}{L^3}\,2\cosh(\tau/L)+\frac{K^2}{L^2}\,\delta(\tau)\right]$
Simple, normal Markovian input		
Linear reservoir	$\frac{D_x^2}{cK-1}\left(2\,\frac{cK}{cK+1}\,\mathrm{e}^{-\tau/K}-\mathrm{e}^{-c\tau}\right)$	$\frac{D_x^2}{c^2K^2-1}(c\,K\,\mathrm{e}^{-\tau/K}-\mathrm{e}^{-c\tau})$
Nash's cascade of n reservoirs	$D_x^2\left\{\frac{1}{(1-\alpha K)^n}\,\mathrm{e}^{-\alpha\tau}+\mathrm{e}^{-\tau/K}\sum_{h=0}^{n-1}\left[\frac{1}{(1+\alpha K)^{n-h}}-\frac{1}{(1-\alpha K)^{n-h}}\right]\frac{\tau^h}{h!\,K^h}\right\}$	$\frac{D_x^2}{\Gamma(n)}\left\{\frac{\Gamma(n)\,\mathrm{e}^{-\alpha\tau}}{(1-\alpha K)^n\,(1+\alpha K)^n}+\frac{1}{2^n}\,\mathrm{e}^{-\tau/K}\sum_{h=0}^{n-1}\sum_{j=0}^{h}\frac{[(1-\alpha K)^{n-h}-(1+\alpha K)^{n-h}]\,(n+j-1)!\,K^{j-h}}{(1-\alpha K)^{n-h}\,(1+\alpha)^{n-h}\,j!\,(h-j)!\,2^j}\,\tau^{h-j}\right\}$
Muskingum model		$\frac{D_x^2}{\alpha^2L^2-1}\left[\frac{\alpha(L^2-K^2)}{L}\,\mathrm{e}^{-\tau/L}+(\alpha^2K^2-1)\,\mathrm{e}^{-\alpha\tau}\right]$

α, c = exponent coefficients of Markovian noise,
K = storage coefficient for linear reservoir and Nash's cascade,
K, L = storage coefficients for Muskingum model,
τ = time lag.

common input in the form of white noise and simple, normal Markovian noise yield the following autocovariance formulae, respectively:

$$R_{y_1 y_2}(\tau) = \frac{A}{K^n L^m \Gamma(m)} e^{-\tau/K} \sum_{j=0}^{n-1} \frac{(m+j-1)!}{(n-j-1)!\,j!} \tau^{n-j-1} \left(\frac{K+L}{KL}\right)^{-M-j} \tag{8}$$

$$R_{y_1 y_2}(\tau) = \frac{D_x^2 e^{-c\tau}}{(1+cL)^m (1-cK)^n} - \frac{D_x^2 e^{-\tau/K}}{(1+cL)^m (1-cK)^n} \sum_{h=0}^{n-1} \frac{(1-cK)^h}{h!} \frac{\tau^n}{K^n} +$$

$$+ \frac{D_x^2 e^{-\tau/K}}{(1-cL)^m (1+cK)^n} \sum_{h=0}^{n-1} \frac{(1+cK)^h}{h!} \frac{\tau^h}{K^h} + \frac{D_x^2 e^{-\tau/K}}{\Gamma(n) K^n} \sum_{h=0}^{n-1} \sum_{j=0}^{h} \sum_{i=0}^{n+j-1} \times$$

$$\times \frac{(1+cL)^{m-h} - (1+cL)^{m-h}}{(1+cL)^{m-h} \quad (1-cL)^{m-h}} \frac{(n+j-1)!\, \tau^{h-j+1} (-1)^{h-j}}{L^h (h-j)!\, j!\, i! \left(\frac{K+L}{KL}\right)^{n+j-i}} \tag{9}$$

For n and m, $n \neq m$, it was observed that the maximum of (9) moves away from $\tau = 0$.

ANALYSIS OF TIME AVERAGED PROCESSES ON LINEAR SYSTEM OUTPUT

Continuous time processes are rarely used in hydrological system analysis. Discrete time sequences are usually formed by means of time averaging of continuous real processes. The time series are formed by the average values for equal periods (an hour, a day).

We shall now analyse relations between the averaging time and the structure of the process.

Let us consider two stationary processes $y_1(t)$ and $y_2(t)$ having zero means. We shall form the integrals

$$s = \frac{1}{2T} \int_{-T}^{T} y_1(t)\,dt \tag{10}$$

and

$$v = \frac{1}{2T} \int_{-T}^{T} y_2(t)\,dt \tag{11}$$

Assuming their existence in the Riemann sense for each realization process, s and v become random variables with the properties (see (3))

$$E(s) = \frac{1}{2T} \int_{-T}^{T} E\{y_1(t)\}\,dt = 0 \tag{12}$$

$$R_{ss}(0) = \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R_{y_1}(t_1, t_2)\,dt_1\,dt_2 = \frac{1}{4T^2} \int_{0}^{2T} (2T - \tau) R_{y_1}(\tau)\,d\tau \tag{13}$$

$$R_{sv}(0)=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}R_{y_1 y_2}(t_1,\,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2=\frac{1}{4T^2}\int_{-T}^{T}(2T-|\tau|)\,R_{y_1 y_2}(\tau)\,\mathrm{d}\tau \tag{14}$$

This gives autocovariance and cross-covariance functions for different time lags:

$$R_{ss}(a)=\frac{1}{4T^2}\int_{(2a-1)T}^{(2a+1)T}\int_{-T}^{T}R_{yy}(t_1,\,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$

$$=\frac{1}{4T^2}\int_{-2T}^{2T}(2T-|\tau|)\,R_{yy}(2aT+\tau)\,\mathrm{d}\tau \tag{15}$$

$$R_{sv}(a)=\frac{1}{4T^2}\int_{-2T}^{2T}(2T-|\tau|)\,R_{y_1 y_2}(2aT+\tau)\,\mathrm{d}\tau \tag{16}$$

Let us consider the Nash model with the white noise input (the choice of input has been already explained). The variance of the averaged output process calculated according to (12) and (13) has the form

$$E(s^2)=\frac{2A}{(n-1)!\,Kb^2}\sum_{p=0}^{n-1}\frac{(n+1-p)!}{p!\,2^{n+p}}\times$$

$$\times\left[(b-n+p)\left(1-\mathrm{e}^{-b}\sum_{g=0}^{n-p-1}b^g\frac{1}{g!}\right)+\mathrm{e}^{-b}\frac{b^{n-p}}{(n-p-1)!}\right] \tag{17}$$

where $b=2T/K$ is an averaging time coefficient.

The cross-correlation function of the random synchronic sequences under the same averaging period is the most interesting relation from the practical point of view. Assuming that both systems consist of Nash's cascade of the same number of linear reservoirs having identical storage coefficients and applying common input in the form of white noise, we obtain the following cross-correlation function for $\tau=0$:

$$r_{sv}(b,\,c)=\frac{A}{2T}\left[\left(\frac{b}{c}+1\right)^{-n}\sum_{p=0}^{n-1}\left(\frac{c}{b}+1\right)^{-p}f(b,\,n,\,p)+\right.$$

$$\left.+\left(\frac{c}{b}+1\right)^{-n}\sum_{p=0}^{n-1}\left(\frac{b}{c}+1\right)^{-p}f(c,\,n,\,p)\right]\times$$

$$\times\left\{\left[\frac{A}{T}\sum_{p=0}^{n-1}\frac{1}{2^{n+p}}f(b,\,n,\,p)\right]\left[\frac{A}{T}\sum_{p=0}^{n-1}\frac{1}{2^{n+p}}f(c,\,n,\,p)\right]\right\}^{\frac{1}{2}} \tag{18}$$

where

$$f(x,\,n,\,p)=\frac{1}{x(n-1)!}\,\frac{(n+p-1)!}{p!}\left[(x-n+p)\left(1-\mathrm{e}^{-x}\sum_{g=0}^{n-p-1}\frac{x^g}{g!}\right)+\frac{x^{n-p}\,\mathrm{e}^{-x}}{(n-p-1)!}\right] \tag{19}$$

r_{sv} is the normalized cross correlation function, and b, c have the same meaning as b in (17) for both systems.

The above equation is rather difficult to deal with. A much clearer formula is obtained for the case of one reservoir in each cascade. Then the cross-correlation function for $\tau = 0$ is given by:

$$r_{sv}(0) = \frac{bc\sqrt{bc}}{b+c}\left(\frac{b+e^{-b}-1}{b^2}+\frac{c+e^{-c}+1}{c^2}\right)[(b+e^{-b}-1)(c+e^{-c}-1)]^{-\frac{1}{2}} \tag{20}$$

The properties of integral sequences (15), (16) are illustrated by the autocorrelation function for one linear reservoir with white noise input:

$$r_s(a, b) = \begin{cases} \dfrac{b(1-a)-e^{-ab}+e^{-b}\cosh(ab)}{b+e^{-b}-1} & \text{for } 0 \leqslant a \leqslant 1 \\[2ex] \dfrac{\cosh b - 1}{b+e^{-b}-1}\; e^{-ab} & \text{for } a \geqslant 1 \end{cases} \tag{21}$$

CONCLUSIONS

The results obtained in the paper would be difficult to obtain experimentally because of the fact that these models approximate actual nonlinear hydrological systems and hydrometeorological measurement devices add some error to the actual signal in the form of additional noise. Since the conceptual linear hydrological models are widely used, the analysis of some stochastic properties may seem useful for practical applications. To avoid repeating tedious algebraic transformations we do not present the procedures leading to the final formulae obtained.

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