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Transformation of the Processes in the Linear Hydrological Systems

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SYNOPSIS

Aim of the Paper

Mathematical models describing water systems are formed in empirical way. Because of a big choice of the models it seems that some theoretical works concerning the transformation of random processes in hydrologic systems may be useful.

Scope of the Paper

Transformation of the stochastic processes (White noise, Markov noise) in linear systems (serial and parallel two-river systems with common input) will be considered. Successively Nash, Muskingum, Diskin models will be taken as system models. Auto and crosscorrelation functions and power spectrum for the runoff process will be determined theoretically. The stochastic nature of storage is also analyzed. The results will be illustrated with plots. We shall also deal with time averaged processes. The above discussion may be extended onto the other linear and non-linear systems.

1. Introduction

The development of researches on satisfactory (optimal if possible) exploitation of water resources has created the need of extension of mathematical analysis of hydrologic data. In the last decade the field of researches moved from the applications of theory of random variables into the theory of random functions. The knowledge of the space-time stochastic structure of the hydrologic processes is necessary for modelling of water-economical systems. The problem of adequate choice of hypotheses and computing errors becomes more significant comparing to the problems existing in the case of the choice of the type of distribution and estimation of its parameters. Statistics is not a magic tool that enables obtaining satisfactory results from uncertain data, small sample sizes and short time series. Due to this fact the generalization and transfer of experience worked out on different watersheds relating to the structure of the outflow process, types of the distribution or possibility of utilizing the longer time series for the extrapolation and estimation of shorter series parameters are important. These attempts, however, are based on some new hypotheses that can be rejected basing on short random samples. The problem of proof of assumed hypotheses basing on our ideas of physical properties of investigated hydrologic systems is said to exist.

When modelling the hydrologic systems, the first common assumption is linearity. This assumption becomes sufficiently valid for the most of engineering and research works. Although the non-linear system modelling is already extensively developed it seems that due to the need of simplicity linear systems must be used in the stochastic hydrology applications. The choice of combination of inputs and outputs in hydrologic systems is very big and it results from some physical reasons. We shall draw our attention to the investigation of the structure of the river outflow process and of the water resources under the assumption of conceptual watershed models. We shall determine the relation between the structure and parameters of the outflow process. These results would be difficult to obtain experimentally because of the fact that these models approximate real hydrologic system and on the other hand hydrometeorological measurement instruments add some error to the actual signal that contributes to the noise produced by the system itself. The corrolaries resulting from the integration of conceptual models and the hydrologic processes can be helpful in acceptation of the hypotheses about the structure of the outflow process. When the watershed system operators are known, the results of transformation of the process may be applied to the prediction of the structure and parameters of the outflow process.

However, continuous processes are not commonly used for hydrologic applications. The time series are created from the average values for the same equal periods (an hour, a day). In the paragraph 6 we shall consider the relation between the averaging time and the structure of the process.

The prolongation of the observation series by utilizing the longer observation series from another station and the prediction of the volume levels based on water-level relations is of great practical importance. Both these actions are based on utilizing the correlative relation between the corresponding (in the sense of maximum correlation) observations in two space points. In both cases the procedure is the same as in the case of incomplete random sample two-dimensional normal variable. The similarity of inflow processes and watershed operators is the most important factor determining the power of this relation.

For the case of two river cross-section water levels these factors are transformating properties of the river sector and the ratio of an upstream inflow and lateral inflow. We shall analyze the accuracy of such relations basing on the conceptual linear models. To avoid repeating tedious algebraic calculations we do not present procedures leading to the final results. It was done in Ref. (5).

2. Some General Remarks

According to many papers [e. g., Kisiel (2), Yevjevich (3), Dooge (1),...] the action of hydrologic system can be represented by

$$Y(t) = H_d\,[X_d\,(t)]+H_s\,[X_s(t)] \qquad \ldots(2.1)$$

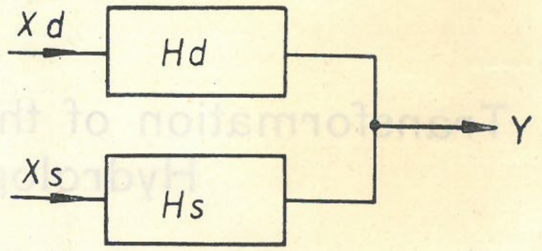

FIGURE 1.

where,

$X_d(t)$ = deterministic input signal

$X_s(t)$ = random noise input signal

H_d = linear model operator transforming the deterministic component of the input signal

H_s = linear model operator transforming the random component of the input signal

$Y(t)$ = output signal.

When analyzing the deterministic component of the input signal, it is most commonly represented by means of a sum of three components:

—linear or quasi-linear trend representing slowly varied tendency observed in the system due to the human activity influence on the physical conditions

—jump component representing short-lasting changes of the input process mean

—periodic component resulting from the periodicity of the precipitation, outflow, etc.

The first two components usually have relatively very small values what makes them very difficult to identify. Gathered empirical material does not allow to determine them quantitatively.

Let us assume the system to be linear, deterministic and time invariant. Then the responses of the model to each deterministic input component can be considered separately according to the principle of superposition. The operator transforms linear trend into output signal, tending asymptotically to a linear function. The nature of a periodic (harmonic) signal is not affected by the transformation by the linear operator (change of the signal amplitude and phase shift occurs).

The separation of the deterministic component of the input signal from the stochastic component enables the assumption of zero mean of the stochastic signal. That is the necessary condition for the weak stationarity. Moreover, we shall assume the stationarity of the dispersion and covariance of the process $X_s(t)$ throughout our paper. As a subject of the analysis we shall consider the structure of the random component transformed by some selected linear watershed models (Nash, Diskin, Muskingum models). Throughout the paper the inflow to the system is assumed to be the input signal and the outflow is considered as the output signal.

The most general hydrologic model investigated by Kulandaiswamy, that covers all the models considered in this paper, has the form

$$\sum_{i=0}^{p} a_i (X, Y, t) \frac{d^i Y(t)}{dt^i} = \sum_{i=0}^{q} b_i (X, Y, t) \frac{d^i x(t-\tau)}{dt^i} \quad \text{...(2.2)}$$

Assuming the coefficients a_i (X, Y, t) and b_i (X, Y, t) constant and selecting adequate p and q, we obtain linear models more suitable for analysis. In this paper we deal with the models being special cases of Kulandaiswamy's general model.

3. Transformation of Stochastic Processes in the Nash's Model

The model that we analyze in this paragraph is the Nash's conceptual model consisting of n equal linear reservoirs. Cascade of n linear reservoirs having different time constants (storage coefficients) will be called generalized Nash's model (Figure 2).

Its transfer function can be evaluated with the help of Laplace transformation techniques as

$$H(s) = \frac{1}{\bigcap_{i=1}^{N} K_i (S+1/K_i)} \qquad K_i \neq K_j,\ i \neq j \quad \text{...(3.1)}$$

where, s = complex variable.

After transformation to the time domain we obtain the kernel function (instantaneous unit hydrograph) :

$$h(t) = \sum_{i=1}^{N} \alpha_i \, e^{-t/K_i} \quad \text{...(3.2)}$$

where,

$$\alpha_i = \frac{1}{\bigcap_{\substack{n=1 \\ n \neq i}}^{N} (1+K_n/K_i)} \quad \text{...(3.3)}$$

For the Nash's model of $N+1$ linear reservoirs with the same storage coefficients $K = K = ... = K = K$ the kernel function is given by:

$$h(t) = \frac{1}{\Gamma(N)\, K^N} \, t^{N-1} \, e^{-t/K} \quad \text{...(3.4)}$$

where, $\Gamma(N)$ denotes the gamma function.

Let us consider a single linear reservoir (Nash's model for $N = 1$). Its outflow is given by

$$y(t) = \int_{-\infty}^{+\infty} x(\alpha)\, h(t-\alpha)\, d\alpha \quad \text{...(3.5)}$$

where,

$x(t), y(t)$ = stochastic processes

$h(t-\alpha)$ = kernel function, $h(t) = 1/K . \exp(-t/K)$ for $t \geqslant 0$

Let us apply White noise input to the model of a single linear reservoir. The autocovariance of the input process is given by

$$R_{xx}(\tau) = C \,.\, \delta(\tau) \quad \text{...(3.6)}$$

$\delta(\tau)$ = Dirac delta function

Stochastic properties of the output are described by means of the following formulae

Crosscovariance

$$R_{yx}(\tau) = \int_{-\infty}^{+\infty} R_x(\tau-\alpha) .\, h(\alpha)\, d\alpha = \frac{C}{K} . \, e^{-\tau/K}, \ \tau \geqslant 0 \quad \text{...(3.7)}$$

Autocovariance

$$R_{yy}(\tau) = \int_{-\infty}^{+\infty} R_{yx}(\tau+\alpha) .\, h(\alpha)\, d\alpha = \frac{C}{2K} . \, e^{-\tau/K} \quad \text{...(3.8)}$$

Obtained results enable us to say, that White noise passed through the single linear reservoir model is transformed into simple Markov noise(2).

In stochastic hydrology it is often convenient to assume that inflows form simple Markov processes. Let us apply now to the model of a single reservoir stationary and normal Markov noise of the autocovariance function given by

$$R_{xx}(\tau) = D^2_x \, e^{-c(\tau)} \quad \text{...(3.9)}$$

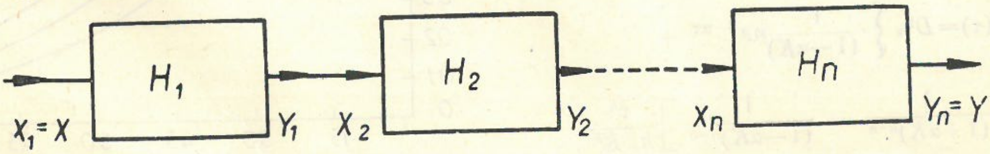

FIGURE 2.

Proceeding similarly as in the case of White noise input we have obtained the following formulae

Crosscovariance

$$R_{yx}(\tau) = \frac{D^2{}_x}{cK-1}\left(\frac{2cK}{cK+1}\; e^{-\tau/K} - e^{-c\tau}\right), \quad \tau \geqslant 0 \qquad \ldots(3.10)$$

Autocovariance

$$R_{yy}(\tau) = \frac{D^2{}_x}{C^2K^2-1}\left(C.K.e^{-\tau/K} - e^{-c\tau}\right) \qquad \ldots(3.11)$$

It will be shown further that the above statement has the form similar to the output autocovariance function obtained for the generalized Nash's model consisting of two reservoirs when applying White noise input. Dividing Equation (3.11) by $R_{yy}(0)$ we get autocorrelation function

$$\rho_{yy}(\tau) = \frac{c.K\,(e^{-\tau/K} - e^{-c\tau})}{c.K-1} \qquad \ldots(3.12)$$

We shall consider now the Nash's model consisting of n similar reservoirs having the same storage coefficients K. The kernel function for this case is given by Equation (3.4).

For the White noise input we have obtained the following relations

Crosscovariance

$$R_{yx}(\tau) = \frac{C}{K^n\Gamma(n)}\; \tau^{n-1}\, e^{-\tau/K} \qquad \ldots(3.13)$$

Autocovariance

$$R_{yy}(\tau) = \frac{C}{K^n\Gamma(n)\,2^n}\, e^{-\tau/K} \sum_{j=0}^{n-1} \frac{\tau^{n-j-1}.K^j.(n+j-1)!}{(n-j-1)!\; j!.2^j} \qquad \ldots(3.14)$$

For $n=1$ the above formulae are consistent with these worked out for a single reservoir

For $n=2$

$$R_{yy}(\tau) = \frac{C}{4K^2}\; e^{-\tau/K}\,[\tau+K] \qquad \ldots(3.15)$$

and so on.

For the simple, stationary and normal Markov process on the input of the above model we have obtained the formulae

Crosscovariance

$$R_{yx}(\tau) = D^2{}_x \left\{ \frac{1}{(1-\alpha K)} n e^{-\alpha\tau} + e^{-\tau/K} \sum_{h=0}^{n-1} \left[\frac{1}{(1+\alpha K)^{n-h}} - \frac{1}{(1-\alpha K)^{n-h}} \right] \frac{\tau^h}{h!\,K^h} \right.$$

$$\text{dla } \tau \geqslant 0 \qquad \ldots(3.16)$$

Autocovariance

$$R_{yy}(\tau) = \frac{D^2{}_x}{\Gamma(n)} \left\{ \frac{\Gamma(n).e^{-\alpha\tau}}{(1-\alpha K)^n(1+\alpha K)^n} + \frac{1}{2^n}\, e^{-\tau/K} \sum_{h=0}^{n-1} \sum_{j=0}^{h} \frac{[(1-\alpha K)^{n-h} - (1+\alpha K)^{n-h}]\,(n+j-1)!\;K^{j-h}}{(1-\alpha K)^{n-h}\,(1+\alpha)^{n-h}\,j!\,(h-j)!\,2^j}\, \tau^{h-j} \right\} \qquad \ldots(3.17)$$

The autocovariance functions described by Equations (3.14—3.17) are plotted in Figures 3 & 4.

We shall analyze now the generalized Nash's model consisting of n linear reservoirs having different storage coefficients. Its kernel function is described by Equations (3.2-3.3).

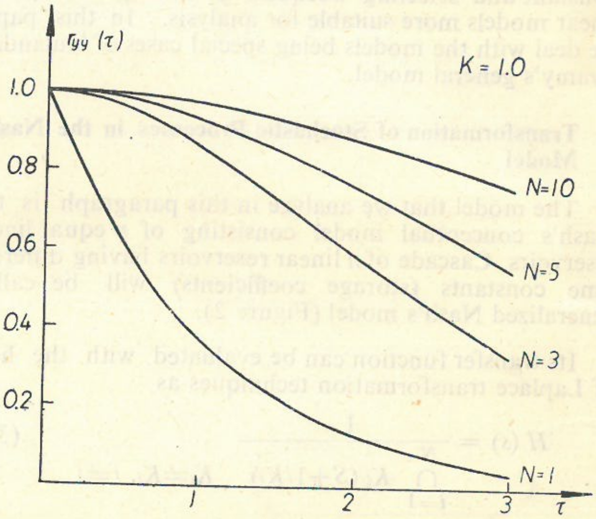

FIGURE 3 : Autocovariance function for Nash model with White noise on the input.

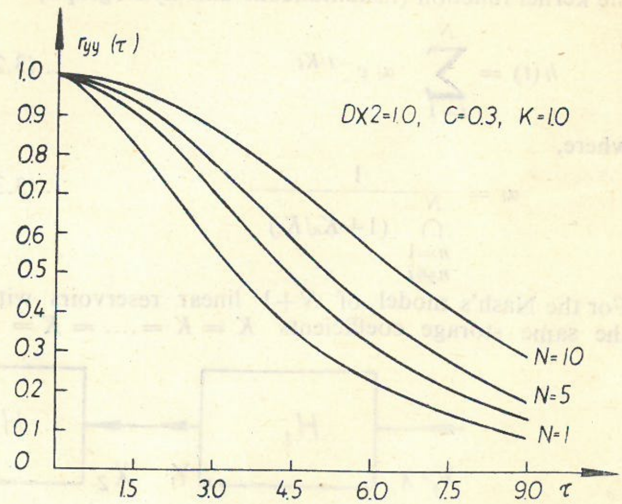

FIGURE 4 : Autocovariance function for Nash model with Markov process on the input.

When applying White noise to the input the following relations can be obtained

Crosscovariance

$$R_{yx}(\tau)=C.\sum_{i=1}^{n}\alpha\, i\, e^{-\tau/Ki} \qquad \tau\geqslant 0 \quad \ldots(3.18)$$

Autocovariance

$$R_{yy}(\tau)=C.\sum_{i=1}^{n}e^{-\tau/Ki},\ \left(\alpha i^2.\frac{Ki^2}{2}+\sum_{j=1}^{N}\frac{-1}{1/Ki+1/K_j}\right) \quad \ldots(3.19)$$

Denoting

$$A_i=\left(\alpha_i\frac{2Ki}{2}+C.\sum_{j=1}^{N}\frac{-1}{1/Ki+1/K_j}\right) \quad \ldots(3.20)$$

we get

$$R_{yy}(\tau)=\sum_{i=1}^{n}e^{-\tau/Ki}.A_i \quad \ldots(3.21)$$

where, A_i depends on α_i and K_i, $i=1,\ldots,n$

The above equation can be considered as the autocovariance function for n-th order autoregressive process. It is easy to see that transformation of White noise in a Nash's model of n reservoirs is qualitatively equivalent to transformation of $(n-1)$th order autoregression scheme by a single storage reservoir. Both situations yield n-th order autoregression scheme of the output process. The above result means that the order of autoregression scheme of an input stochastic process increases by 1 after transformation in a single linear reservoir.

4. Crosscorrelation of the Processes on the Outputs of two Nash's Models

We shall consider two linear systems described by means of the kernel functions $h_1(t)$, $h_2(t)$ (see Figure 5).

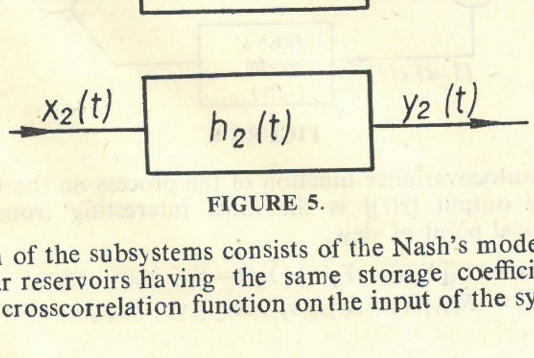

FIGURE 5.

Each of the subsystems consists of the Nash's model of linear reservoirs having the same storage coefficients. The crosscorrelation function on the input of the system is assumed to be known. The power spectrum of the input signal is given by

$$S_{x_1x_2}(\omega)=\int_{-\infty}^{+\infty}R_{x_1x_2}(\tau)e^{-j\omega\tau}\,d\tau=S_{x_2x_1}{}^*(\omega) \ldots(4.1)$$

The crosscovariance function of the output processes from the both subsystems is most interesting from the practical point of view. This function can be obtained by integrating the cross power spectrum of the output signal

$$R_{y_1y_2}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_{y_1y_2}(\omega)\,e^{j\omega\tau}\,d\omega \quad \ldots(4.2)$$

If the frequency intervals where the transfer functions have non-zero values do not overlap, the value of output power spectrum $S_{y_1y_2}(\omega)=0$ for all frequencies, what means that for all $x_1(t)$, $x_2(t)$, are orthogonal. The output cross power spectrum is generally expressed in the form

$$S_{y_1y_2}(\omega)=S_{x_1x_2}(\omega)\,H_1(j\omega)H_2{}^*(j\omega) \quad \ldots(4.3)$$

Let us assume that $S_{x_1x_2}=A$ (White noise input) After integrating we have obtained

$$R_{y_1y_2}(\tau)=\frac{A\tau^{2n-1}}{KL^n(n-1)!}\,e^{-\tau/K}\sum_{p=0}^{n-1}\frac{(n+p-1)!}{p!(n-p-1)!\,Z^{n+p}} \quad \ldots(4.4)$$

For zero time-lag the above expression has the form :

$$R_{y_1y_2}(0)=A\,\frac{(2n-2)!}{[(n-1)!]^2}\,\frac{(KL)^{n-1}}{(K+L)^{2n-1}} \quad \ldots(4.5)$$

When assuming the linear dependence of the storage coefficients of both cascades the following autocorrelation function can be obtained

$$r_{y_1y_2}\left(\frac{\tau}{K}\right)=\frac{(n-1)!}{(2n-2)!}\,\frac{2^{2n-1}}{e^{\tau/K}}$$

$$\sum_{p=0}^{n-1}\frac{(n+p-1)!}{p!(n-p-1)!}\,\frac{m^{p+1/2}}{(m+1)^{n+p}}\left(\frac{\tau}{K}\right)^{n-p} \qquad \text{for } \tau\geqslant 0 \quad \ldots(4.6)$$

This function has its maximum value for zero time-lag

$$\max_{\tau/K}\ r_{y_1y_2}\left(\frac{\tau}{K}\right)=r_{y_1y_2}(0)=2^{2n-1}\left(\frac{\sqrt{m}}{1+m}\right)^{2n-1} \quad \ldots(4.7)$$

The specific case of the above relations, obtained for the common input $x_1=x_2$ is of vast interest for hydrologic applications. With the exception of high frequencies in the power spectrum, not transferred by the system, all the processes contributing to the inflow are strongly spatially correlated. Thus usually it can be assumed that two adjacent watersheds have common input processes. Then the structure and parameters of the outflow process of one watershed can be determined basing on the inflow process and on the knowledge of both system operators. It is worth considering because the realizations of outflow process can be obtained in much easier way and more accurately than of an input stochastic process.

Let us assume common inflow process and consider two forms of these processes. When the inflow signal has the form of White noise, the crosscovariance function of the output signal is given by

$$R_{y_1y_2}(\tau)=\frac{A}{K^nL^m\Gamma(n)}\,e^{-\tau/K}\times$$

$$\sum_{j=0}^{n-1}\frac{(m+j-i)!}{(n-j-1)!\,j!}\,\tau^{n-j-1}\cdot\left(\frac{K+L}{KL}\right)^{-m-j} \quad \ldots(4.8)$$

where, K, n and l, m represent the storage coefficient and the number of reservoirs respectively for both subsystems (see Figure 6).

The simple Markov noise applied to the common input yields the formula:

$$R_{y_1y_2}(\tau)=\frac{D^2{}_x.e^{-C\tau}}{(1+CL)^m(1-CK)^n}-\frac{D^2{}_x.e^{-\tau/K}}{(1+CL)^m(1-CK)^n}$$

$$\sum_{h=0}^{n-1}\frac{(1-CK)^h}{h!}\cdot\frac{\tau^h}{K^h}+\frac{D^2{}_x.e^{-\tau/K}}{(1-CL)^m(1+CK)^n}\cdot$$

$$\sum_{h=0}^{n-1}\frac{(1+CK)^h}{h!}\cdot\frac{\tau^h}{K^h}+\frac{D^2{}_x.e^{-\tau/K}}{\Gamma(n).K^n}\times$$

$$\sum_{h=0}^{n-1}\sum_{j=0}^{h}\sum_{i=0}^{n+j-1}\frac{[(1-CL)^{m-h}-(1+CL)^{m-h}]}{(1+CL)^{m-h}(1-CL)^{m-h}}\times$$

$$\frac{(n+j-1)!\,\tau^{h-j+i}\,(-1)^{h-j}}{L^h(h-j)!\,j!\,i!\left(\frac{K+L}{K.L}\right)^{n+j-i}} \quad \ldots(4.9)$$

The above expression is illustrated in Figure 7.

The other model close to these previously analyzed is Diskin linear model (Figure 8).

The crosscorrelation of the output processes of both subsystems is given by

$$R_{y_1y_2}(\tau)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}R_{x_1x_2}(\tau-\alpha+\beta)\,h_1(\alpha)h_2(\beta)\,d\alpha d\beta \quad \ldots(4.10)$$

Let us apply to the input the normal, stationary Markov process. Then

$$R_{x_1x_2}(\tau)=E\{\alpha X(t)-\beta X(t-\tau)\}=\alpha\beta R_x(\tau)=\alpha\beta D^2{}_xe^{-C|\tau|} \quad \ldots(4.11)$$

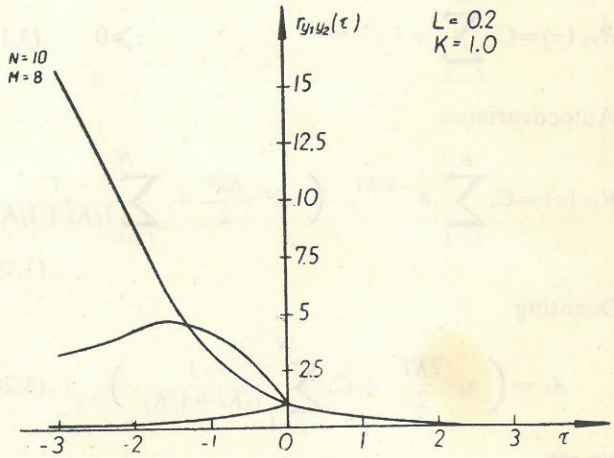

FIGURE 6.

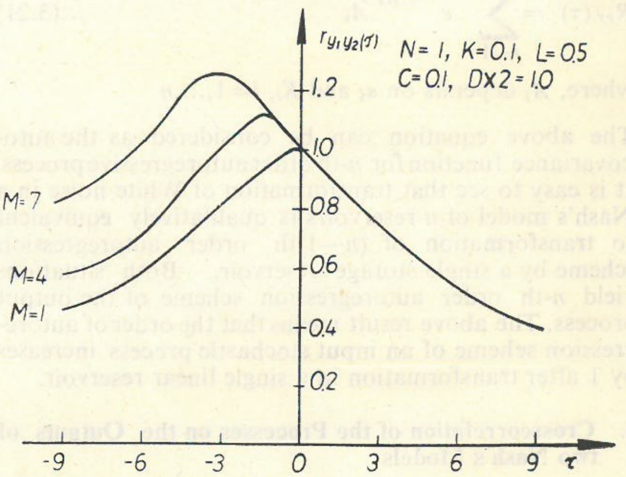

FIGURE 7.

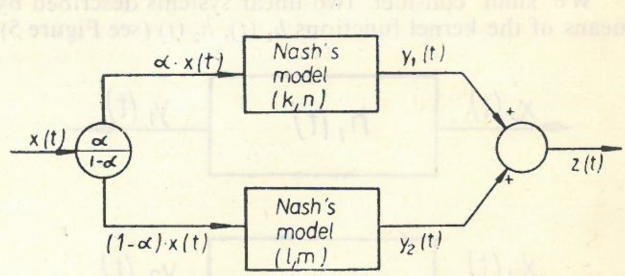

FIGURE 8.

The autocovariance function of the process on the combined output $[z(t)]$ is the most interesting from the practical point of view.

$$\begin{aligned}R_z(\tau)&=E\{[Y_1(t)+Y_2(t)]\,[Y_1(t-\tau)+Y_2(t-\tau)]\\&=R_{y_1y_1}(\tau)+R_{y_1y_2}(\tau)+R_{y_2y_1}(\tau)+R_{y_2y_2}(\tau)\end{aligned} \quad \ldots(4.12)$$

where,

$$R_{x_1y_2}(\tau) = \int_{-\infty}^{+\infty} R_{x_1x_2}(\tau+\beta)\, h_2(\beta)\, d\beta \qquad \dots(4.13)$$

$$R_{y_1y_2}(\tau) = \int_{-\infty}^{+\infty} R_{x_1y_2}(\tau-\alpha)\, h_1(\alpha)\, d\alpha$$

The above relations are illustrated in Figures 9 & 10.

5. Transformation of Stochastic Processes in Muskingum Model

Muskingum model is a special case of general

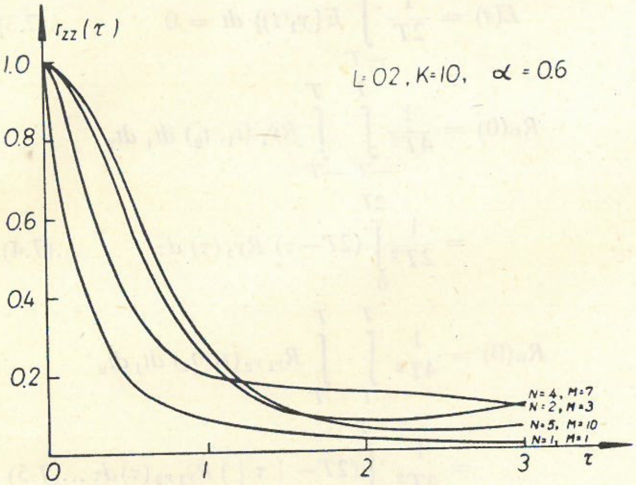

FIGURE 9 : Autocovariance function. White noise on the input.

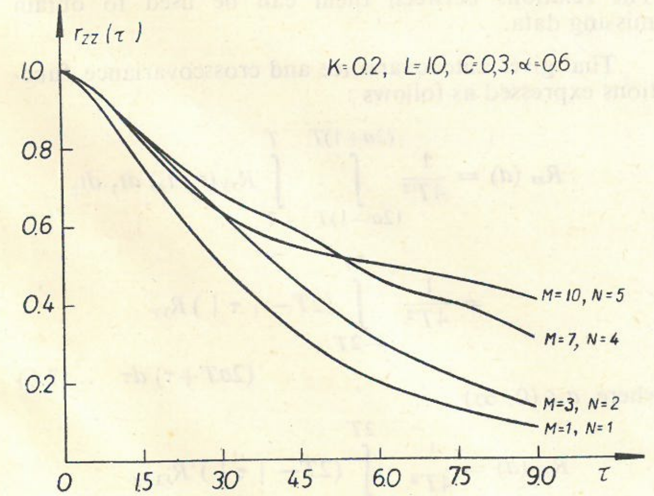

FIGURE 10 : Autocovariance function. Markov process on the input.

conceptual hydrologic model (2.2), described by :

$$X(t) = y(t) + \frac{d\,W(t)}{dt} \qquad \dots(5.1)$$

where,

$W(t) = Kx(t) + Ly(t)$ —storage

$x(t)$ —input to the system (inflow)

$y(t)$ —output (discharge)

Transfer function for that model, using Laplace transformation is expressed by

$$H(s) = \frac{1-Ks}{Ls+1} \qquad \dots(5.2)$$

When returning to the time domain we get the expression for the kernel function of the model :

$$h(t) = \frac{L+K}{L^2} \cdot e^{-t/L} - \frac{K}{L}\,\delta(t) \qquad \dots(5.3)$$

where, $\delta(t)$ denotes Dirac delta function.

With White noise on the input (Equation 3.6) we obtain the following autocovariance function

$$R_{yy}(\tau) = D^2{}_x \left[\frac{(L+K)^2 e^{-\tau/L}}{2L^3} - \frac{(L+K)K}{L^3} 2ch\left(\frac{\tau}{L}\right) + \frac{K^2}{L^2}\,\delta(\tau) \right] \qquad \dots(5.4)$$

For the simple Markov noise input crosscovariance function can be expressed as

$$R_{yx}(\tau) = \left\{ D^2{}_x \left\{ \frac{L+K}{L(\alpha L-1)} \left[\frac{2\alpha L}{\alpha L+1} e^{-\tau/L} - e^{-\alpha\tau} \right] - \frac{K}{L} e^{-\alpha\tau} \right\} \tau \geqslant 0 \right. \qquad \dots(5.5)$$

Autocovariance function of the output has the form

$$R_{yy}(\tau) = \frac{D^2{}_x}{\alpha^2 L^2 - 1} \left[\frac{\alpha(L^2-K^2)}{L} e^{-\tau/L} + (\alpha^2 K^2 - 1)\, e^{-\alpha\tau} \right] \qquad \dots(5.6)$$

6. Process of Storage

In the preceding paragraphs we have considered outflow as the system output. Here we shall briefly analyze some relations for storage in linear reservoirs. It should be noted that by terms "outflow" and "storage" we mean the stochastic component of outflow or storage.

The storage equation for each included in the Nash model linear reservoir is given by

$$S_i = k_i\, y_i \qquad i = 1, 2, \dots, n \qquad \dots(6.1)$$

The total system storage for the Nash model having the same storage coefficients may be described as

$$S = \sum_{i=1}^{n} S_i = k \sum_{i=1}^{n} y_i \qquad \dots(6.2)$$

Let $y : (t)$ be a stationary process in the wide sense, then $S(t)$ is also a stationary process in the wide sense with the autocovariance function

$$R_{ss}(\tau) = E[S(t+\tau)\,S(t)] = k^2 \sum_{i,j=1}^{n} Ry_i\, y_j(\tau) \quad \ldots(6.3)$$

and from the other side

$$\dot{S} = \frac{d\,S(t)}{dt} = x(t) - y(t) \quad \ldots(6.4)$$

$$\dot{S}(t+\tau)\cdot\dot{S}(t) = x(t+\tau)\,x(t) - x(t+\tau)\,y(t) - y(t+\tau)\,x(t) + y(t+\tau)\,y(t) \quad \ldots(6.5)$$

It yields

$$R_{\dot{s}\dot{s}}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) - R_{xy}(\tau) - R_{yx}(\tau) \quad \ldots(6.6)$$

Thus the autocovariance function can be calculated by double integrating of the above formula.

If the input process has the form of White noise, we obtain the following formula for the output process autocovariance function

$$R_{ss}(\tau) = \frac{A}{K^n\,\Gamma(n)}\, e^{-\tau/K} \sum_{p=0}^{n-1} a_p\, \tau^{n-p-1} \quad \ldots(6.7)$$

where,

$$a_p = \begin{cases} +[d_p + (n-p)\,a_{p-1}]k & p \neq 0 \\ +d_o K & p = 0 \end{cases} \quad \ldots(6.7a)$$

$$d_p = \begin{cases} -[C_p - (n-p)\,d_{p-1}]k & p \neq 0 \\ -C_o K & p = 0 \end{cases} \quad \ldots(6.7b)$$

For $n = 1$ the White noise input yields the storage in the form of simple Markov process. Hence for the case of a single linear reservoir retention is proportional to the outflow.

So its autoregression model has the same order as that of the outflow from the reservoir. Following the reasoning from the end of paragraph 3 it can be proved that the input stochastic process of n-th order of autoregression scheme gives the storage stochastic process of $(n+1)$ th order for a single linear reservoir.

7. Analysis of Time averaged Processes

Continuous time processes are rarely used in hydrologic system analysis. Discrete time sequences are usually formed by means of time averaging of continuous real processes.

In this part of the paper properties of the stochastic process transferred through the linear model of the part of a watershed (Nash model consisting of n linear reservoirs with the same time constants k) and time averaged will be analyzed.

The influence of averaging period length on the properties of the processes will be also described.

Let us consider two stationary stochastic processes $y_1(t)$ and $y_2(t)$ having zero means.

We shall form the following integrals :

$$S = \frac{1}{2T}\int_{-T}^{T} y_1(t)\,dt \quad \ldots(7.1)$$

and

$$V = \frac{1}{2T}\int_{-T}^{T} y_2(t)\,dt \quad \ldots(7.2)$$

Assuming their existence in Riemann sense for each process realization, s and v become random variables with the following properties [see Ref. (3)] :

$$E(s) = \frac{1}{2T}\int_{-T}^{T} E\{y_1(t)\}\,dt = 0 \quad \ldots(7.3)$$

$$R_{ss}(0) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} Ry_1(t_1, t_2)\,dt_1\,dt_2 = \frac{1}{2T^2}\int_{0}^{2T}(2T-\tau)\,Ry_1(\tau)\,d\tau \quad \ldots(7.4)$$

$$R_{sv}(0) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} R_{y_1 y_2}(t_1, t_2)\,dt_1\,dt_2 = \frac{1}{4T^2}\int_{-T}^{T}(2T - |\,\tau\,|\,)\,R_{y_1 y_2}(\tau)\,d\tau \quad \ldots(7.5)$$

In the hydrologic practice time sequences usually describe the runoff phenomena on different watersheds. The relations between them can be used to obtain missing data.

That gives autocovariance and crosscovariance functions expressed as follows :

$$R_{ss}(a) = \frac{1}{4T^2}\int_{(2a-1)T}^{(2a+1)T}\int_{-T}^{T} R_{yy}(t_1, t_2)\,dt_1\,dt_2 = \frac{1}{4T^2}\int_{-2T}^{2T}(2T - |\,\tau\,|\,)\,R_{yy}(2aT+\tau)\,d\tau \quad \ldots(7.6)$$

where, $a \in (0, \infty)$

$$R_{sv}(a) = \frac{1}{4T^2}\int_{-2T}^{2T}(2T - |\,\tau\,|\,)\,R_{y_1 y_2}(2aT+\tau)\,d\tau \quad \ldots(7.7)$$

We shall illustrate now the properties of stochastic process transferred through the linear model of the part of a watershed and time averaged.

Let us consider the input in the form of White noise, process being very simple for the analysis and of some importance for the hydrologic applications. The mean value of the time averaged output process equals zero, as it results from Equation (7.3).

The variance of the averaged output process can be evaluated from Equation (7.4), substituting Equation (3.14) as the autocovariance function of the continuous output process. After some rearranging we get :

$$E\{s^2\} = \frac{2A}{(n-1)!\,Kb^2} \sum_{p=0}^{n-1} \frac{(n+1-p)!}{p!\,2^{n+p}} \left\{(b-n+p)\left[1-e^{-b} \sum_{g=0}^{n-p-1} b^g \frac{1}{g!}\right] + e^{-b} \frac{b^{n-p}}{(n-p-1)!}\right\} \quad ...(7.8)$$

where, $b = 2T/K$

Crosscorrelation function of the random synchronic sequences under the same averaging period is the most interesting from the practical point of view. Assuming, that both systems consist of Nash's model of the same number of reservoirs and applying common input in the form of White noise the crosscorrelation function for $\tau = 0$ is given by :

$$r_{sv}(b, c) = \frac{\frac{A}{2T}\left\{\left(\frac{b}{c}+1\right)^{-n} \sum_{p=0}^{n-1} \left(\frac{c}{b}+1\right)^{-p} . f(b,n,p) + \left(\frac{c}{b}+1\right)^{-n} \sum_{p=0}^{n-1} \left(\frac{b}{c}+1\right)^{-p} . f(c,n,p)\right\}}{\sqrt{\left[\frac{A}{T}\sum_{p=0}^{n-1} \frac{1}{2^{n+p}} f(b,n,p)\right]} \sqrt{\left[\frac{A}{T}\sum_{p=0}^{n-1} \frac{1}{2^{n+p}} f(c,n,p)\right]}} \quad ...(7.9)$$

where,

$$f(x,n,p) = \frac{1}{x(n-1)!} \frac{(n+p-1)!}{p!} \left\{(x-n+p)\left[1-e^{-x} \sum_{g=0}^{n-p-1} \frac{x^g}{g!}\right] + \frac{x^{n-p} e^{-x}}{(n-p-1)!}\right\} \quad ...(7.10)$$

and b, c have the same meaning as b in Equation (7.8), for the both systems. Results obtained for the case of one reservoir in each system are easy for the interpretation.

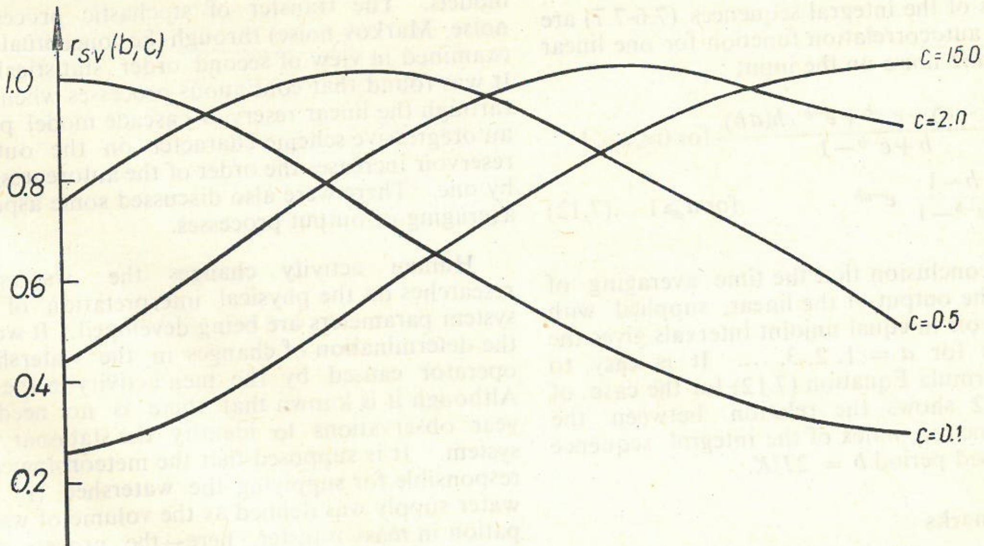

FIGURE 11.

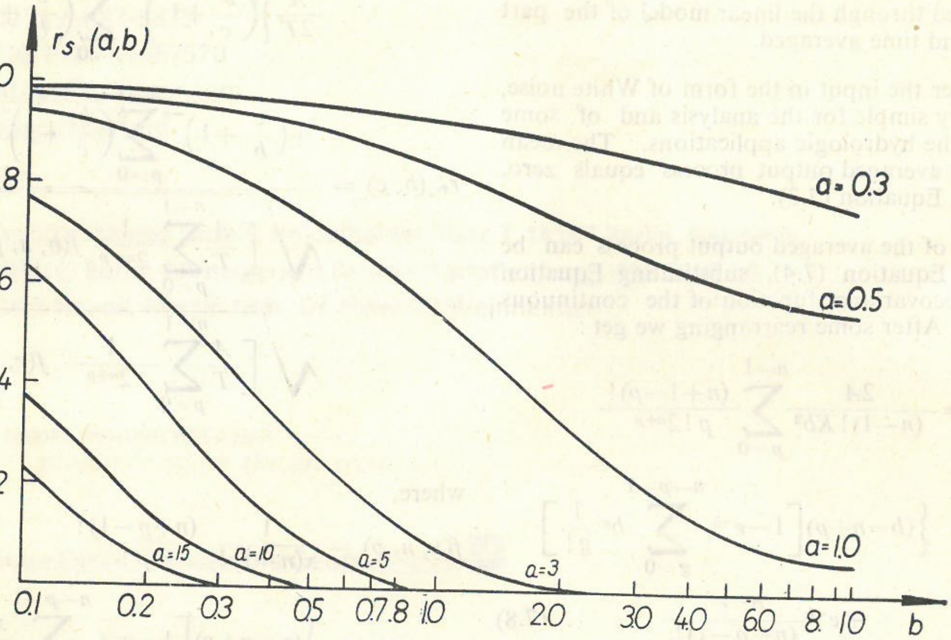

FIGURE 12.

The crosscorrelation function for $\tau = 0$ is given by :

$$r_{sv}(0) = \frac{\frac{bc\sqrt{bc}}{b+c}\left(\frac{b+e^{-b}-1}{b^2}+\frac{c+e^{-c}+1}{c^2}\right)}{\sqrt{(b+e^{-b}-1)(c+e^{-c}-1)}} \quad \ldots(7.11)$$

This is illustrated in Figure 11.

The properties of the integral sequences (7.6-7.7) are illustrated by the autocorrelation function for one linear reservoir with White noise on the input :

$$r_s(a, b) = \begin{cases} \dfrac{b(1-a)-e^{-ab}+e^{-b}\,ch(ab)}{b+e^{-b}-1} & \text{for } 0 \leqslant a \leqslant 1 \\ \dfrac{ch\,b-1}{b+e^{-b}-1}\,e^{-ab} & \text{for } a \geqslant 1 \end{cases} \quad \ldots(7.12)$$

This leads to the conclusion that the time averaging of the processes on the output of the linear, supplied with White noise reservoir in equal unjoint intervals gives the Markov sequence for $a = 1, 2, 3, \ldots$. It is easy to notice from the formula Equation (7.12) for the case of $a \geqslant 1$. Figure 12 shows the relation between the autocorrelation function, index of the integral sequence a and the normalised period $b = 2T/K$.

8. Concluding Remarks

The usage of information concerning water resource stochastic properties in water-economic projects needs the developing of the methods enabling the utilization of longer informational sequences in short series. In hydrologic model systems there exists noise caused by the lack of the information concerning all input signals. Considering of random properties of watershed operator complicates the analytic determination of transformed stochastic process. Thus in this case and in the case of non-linear watershed model it is better to simulate the work of the system supplied with artificially generated process. In the paper there were found the relations between conceptual watershed models and stochastic models. The transfer of stochastic processes (White noise, Markov noise) through the conceptual models was examined in view of second order statistical moments. It was found that continuous processes when transferred through the linear reservoir cascade model preserve the autoregressive scheme character on the output. Each reservoir increases the order of the autoregressive scheme by one. There were also discussed some aspects of time averaging of output processes.

Human activity changes the watershed. The researches on the physical interpretation of watershed system parameters are being developed. It would enable the determination of changes in the watershed system operator caused by the men activity in the landscape. Although it is known that there is no need of many year observations to identify the stationary watershed system. It is supposed that the meteorological process, responsible for supplying the watershed (in the paper water supply was defined as the volume of water participation in mass transfer, here—the process responsible for the supplying that is precipitation, temperature, moisture) is not very sensitive on the human activity changes. Hence the random process transformation in the hydrologic systems will give the possibility of obtaining the runoff process characteristics under the new conditions.

9. References

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