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Linear flood routing model for rapid flow

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Abstract The linear flood routing model presented has been derived from the linearized St. Venant equation for the case of a uniform open channel with arbitrary cross-sectional shape and friction law. In order to filter out the downstream boundary condition the kinematic wave solution is used to approximate the diffusion term in the St. Venant equation. The hydrodynamic model obtained is called the rapid flow model (RFM). It provides the exact solution for a Froude number equal to one. Such characteristics of the RFM impulse response as cumulants, amplitude and phase spectra are analysed, and then compared with those of the complete linearized St. Venant equations for different reach lengths, values of Froude number and frequencies of flood waves. The RFM can be applied for mountainous rivers that have large Froude numbers and both quick and slow rising waves.

Modèle linéaire de propagation des crues pour l'étude de l'écoulement rapide à surface libre

Résumé Le modèle linéaire de propagation des crues présenté est déduit des équations de Saint Venant linéarisées dans le cas d'un canal prismatique de section transversale quelconque et pour des lois de frottement quelconques. La solution d'une onde cinématique pour l'approximation du terme de diffusion a permis d'éliminer la condition limite aval. Le modèle obtenu est nommé "le modéle d'écoulement rapide". Il donne des résultats exacts pour le nombre de Froude égal à un. Les cumulants, la réponse en amplitude et la réponse en fréquence ont été analysés pour la fonction de transfert de ce modèle. Ensuite ils ont été comparés avec les mêmes caractéristiques des équations de Saint Venant pour différentes valeurs des longueurs des canaux, des nombres de Froude et de fréquences des crues. Le modèle élaboré peut être appliqué aux rivières montagneuses, c'est-à-dire pour les grands nombres de Froude.

INTRODUCTION

The hydraulic formulation for unsteady flow in open channels requires two boundary conditions and in the case of tranquil flow (i.e. Froude number less than unity) one of these is at the downstream end of the channel. In

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practical flood routing the influence of a downstream control is nearly always neglected and the routing takes place only in the downstream direction.

For tranquil flow at low Froude numbers, the small convective terms in the flow equations can be approximated on the basis of the linear kinematic wave approximation. This results in a reduction of the hyperbolic St. Venant equation to a convective diffusion equation (Dooge & Harley, 1967; Dooge & Napiórkowski, 1987) or to a parabolic-like form involving the cross-derivative as the only second order term (Strupczewski & Napiórkowski, 1986; Strupczewski *et al.*, 1988).

The case of rapid flows with high Froude numbers is discussed in this paper. In order to filter out the downstream boundary condition the second derivatives with respect to x are expressed by means of the other second order terms. The impulse response of the hydrodynamic model obtained has a clear conceptual interpretation being the total of the products of a Poisson distribution and the impulse responses of linear reservoirs in series shifted in time.

COMPLETE LINEAR EQUATION (CLE)

The linearized St. Venant equation for one-dimensional unsteady flow in uniform channel may be written as (Dooge *et al.*, 1987a):

$$(1 - F^2) g\bar{y}_0 \frac{\partial^2 Q}{\partial x^2} - 2v_0 \frac{\partial^2 Q}{\partial x \partial t} - \frac{\partial^2 Q}{\partial t^2} = gA_0 \left[\frac{\partial S_f}{\partial Q} \frac{\partial Q}{\partial t} - \frac{\partial S_f}{\partial A} \frac{\partial Q}{\partial x} \right] \quad (1)$$

where Q is the perturbation of flow about an initial condition of steady uniform flow, Q_0; A_0 is the cross-sectional area corresponding to this flow; F is the Froude number; S_f is the friction slope; $\bar{y}_0$ is the hydraulic mean depth; v_0 is the mean velocity; S_0 is the bottom slope; x is the distance from the upstream boundary; t is the elapsed time, and derivatives of the friction slope are evaluated at the reference conditions.

The variation of the friction slope with discharge at the reference condition for all frictional formulae for rough turbulent flow may be expressed as:

$$\frac{\partial S_f}{\partial Q} = 2S_0/Q_0 \quad (2)$$

For convenience a parameter, m, may be defined as the ratio of the kinematic wave speed to the average velocity of flow:

$$m = c_k/(Q_0/A_0) \quad (3)$$

where c_k is the kinematic wave speed as given by Lighthill & Whitham (1955):

$$c_k = \frac{dQ}{dA} = -\frac{\partial S_f/\partial A}{\partial S_f/\partial Q} \quad (4)$$

The parameter, m, is a function of the shape of the channel and of a friction law parameter. Substituting equations (2) to (4) into equation (1) and denoting:

$$D = \frac{S_0}{\bar{y}_0} x \qquad \text{(dimensionless length)} \tag{5}$$

$$z = x/c_k \qquad \text{(passage time of a kinematic wave through the channel)} \tag{6}$$

one obtains:

$$(1 - F^2)\, \frac{c_k}{2m}\, \frac{z}{D}\, \frac{\partial^2 Q}{\partial x^2} - F^2 \frac{z}{D}\, \frac{\partial^2 Q}{\partial x \partial t} - F^2\, \frac{m}{2c_k}\, \frac{z}{D}\, \frac{\partial^2 Q}{\partial t^2} = \frac{\partial Q}{\partial x} + \frac{1}{c_k}\, \frac{\partial Q}{\partial t} \tag{7}$$

DERIVATION OF THE RAPID FLOW MODEL (RFM) FROM THE CLE

The complete linear equation is a hyperbolic one, i.e. it has two real characteristics. The direction of these characteristics gives the celerity of both the primary and secondary waves. For Froude numbers less than unity, the celerity of the secondary wave is in the upstream direction.

In order to filter out the downstream boundary condition for Froude numbers in the neighbourhood of $F = 1$ the small convective term (the first term in equation (7)) can be neglected entirely or represented on the basis of the linear kinematic wave approximation. For the kinematic wave approximation we can write the solution as:

$$Q(x, t) = \mathrm{f}(x - c_k t) \tag{8}$$

This lower order solution can be used to approximate the "diffusion" terms on the left hand side of equation (7) as follows:

for the second term:

$$\frac{\partial^2 Q}{\partial x^2} = -\frac{1}{c_k}\, \frac{\partial^2 Q}{\partial x \partial t} \tag{9}$$

for the third term:

$$\frac{\partial^2 Q}{\partial x^2} = \frac{1}{c_k^2}\, \frac{\partial^2 Q}{\partial t^2} \tag{10}$$

or by a linear combination of the second and third terms:

$$\frac{\partial^2 Q}{\partial x^2} = - C_1 \frac{1}{c_k} \frac{\partial^2 Q}{\partial x \partial t} + C_2 \frac{1}{c_k^2} \frac{\partial^2 Q}{\partial t^2} \tag{11}$$

where C_1 and C_2 are coefficients to be determined. Substitution of equation (11) into equation (7) gives:

$$- \alpha \frac{\partial^2 Q}{\partial x \partial t} - \beta \frac{\partial^2 Q}{\partial t^2} = \frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t} \tag{12}$$

where:

$$\alpha = \left[\frac{1 - F^2}{2m} C_1 + F^2 \right] \frac{z}{D} \tag{13}$$

$$\beta = \left[- \frac{1 - F^2}{2m} C_2 + \frac{mF^2}{2} \right] \frac{z}{D} \frac{1}{c_k} \tag{14}$$

Note that to solve equation (12) only an upstream boundary condition $Q_u(t) = Q(0, t)$ is required. The downstream boundary condition has been filtered out from the complete linear equation.

The linear equation (12) can conveniently be solved by the use of a Laplace transform technique. The initial value of the dependent variable and its derivatives are all zero, so when equation (12) is transformed to the Laplace transform domain one gets:

$$(1 + \alpha s) \frac{\mathrm{d}Q}{\mathrm{d}x} + (s/c_k + \beta s^2)\, Q = 0 \tag{15}$$

The solution of the ordinary differential equation (15) can be written as:

$$Q(x, s) = H(x, s)\, Q_u(s) \tag{16}$$

The transfer function, $H(x, s)$, i.e. the Laplace transform of the impulse response, describes all the transfer properties of the RFM and is given by:

$$H^{RFM}(x, s) = \exp \left[- \Delta s - \lambda + \frac{\lambda}{1 + \alpha s} \right] \tag{17}$$

where:

$$\Delta = x\beta/\alpha \tag{18}$$

$$\lambda = (z - \Delta)/\alpha \tag{19}$$

The term $\exp(-\Delta s)$ is easily interpreted as a time shift, the term $\exp(-\lambda)$ as a damping factor independent of time, and the term $\exp(\lambda/(1 + \alpha s))$ is responsible for the modulatory performance of the model.

To invert equation (17) from the Laplace transform domain to the original time domain, the term responsible for modulatory performance can be expanded into a convergent series and operated on term by term. Adopting standard transform pairs (Doetsch, 1961) and using the translation theorem one gets:

$$h(x, t) = P_0(\lambda)\ \delta(t - \Delta) + \textstyle\sum_{i=1}^{\infty} P_i(\lambda)\ h_i[(t - \Delta)/\alpha] \tag{20}$$

where:

$$P_i(\lambda) = \frac{\lambda^i}{i!} \exp(-\lambda) \tag{21}$$

is a Poisson distribution, and

$$h_i(t/\alpha) = \frac{1}{\alpha(i - 1)!} (t/\alpha)^{i-1} \exp(-t/\alpha) \tag{22}$$

is the impulse response of i linear reservoirs with a time constant α, and Δ, defined in equation (18), is a time delay.

Note that the solution of the physically-based RFM can be represented in terms of basic conceptual elements used in hydrology, namely a cascade of linear reservoirs and a linear channel (Fig. 1). The upstream boundary condition is delayed by a linear channel with time lag, Δ, divided according to a Poisson distribution with mean λ, and then transformed by parallel cascades of equal linear reservoirs (with time constant α) of varying lengths. Note that λ is the average number of reservoirs in a cascade.

PROPERTIES OF THE RFM IMPULSE RESPONSE

Cumulants and frequency characteristics have been widely used to study the properties of linear responses and to compare the various models proposed to represent the Linear Channel Response (LCR), i.e. the solution of the CLE for a semi-infinite uniform channel, and for F_0<1.

The cumulants of the RFM

The use of cumulants to study the properties of impulse responses was introduced by Dooge & Harley (1967). The cumulants are generated by the logarithm of the Laplace transform of the impulse response function:

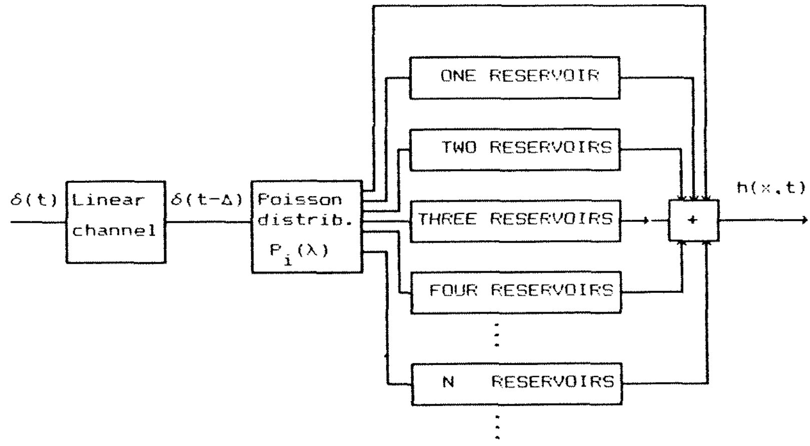

Fig. 1 Distribution of unit volume between paths of linear reservoirs for the RFM.

$$k_i[h(x, t)] = (-1)^r \frac{d^r}{ds^r} \{\ln[H(x, s)]\}_{s=0} \tag{23}$$

Cumulants of the RFM can be obtained from equation (17). Evaluating the rth derivative at $s = 0$ we get the following expression for the rth cumulant:

$$k_1^{RFM} = \Delta + \alpha\lambda = z \tag{24a}$$

$$k_r^{RFM} = r!\,\alpha^r \lambda \qquad \text{for } r>1 \tag{24b}$$

Cumulants of the LCR were obtained by Dooge *et al.* (1987b) and are given in Appendix A. The first cumulant of the LCR is equal to the first cumulant of the RFM because it does not depend on the left hand side of equation (7). Matching the second and third cumulants of the LCR and of the RFM one can determine the parameters of the RFM and express them in terms of channel and flow characteristics:

$$\alpha = \frac{1}{m} [1 + (m - 1)F^2] \frac{z}{D} \tag{25a}$$

$$\lambda = \frac{m}{2} \frac{1 - (m - 1)^2 F^2}{[1 + (m - 1)F^2]^2} D \tag{25b}$$

$$\Delta = 0.5 \frac{1 + (m^2 - 1)F^2}{1 + (m - 1)F^2} z \tag{25c}$$

Since the LCR cumulants are derived for $F < 1$, equations (25) are therefore valid also for $F < 1$.

From equation (25a) one can see that α, the time constant of the reservoirs, does not vary with the length of the river reach. In contrast, equations (25b) and (25c) show that Δ, the parameter of pure time delay, and λ, the parameter of inflow distribution, are both proportional to that length.

As shown in Appendix B, to get equivalence of the second moments of the RFM and CLE the coefficients C_1 and C_2 should fulfil the relation $C_1 + C_2 = 1$, while for the additional equivalence of the third moments $C_1 = 2$.

Therefore the final equation which preserves all three moments is:

$$-\frac{1}{m}\left[1 + (m - 1)F^2\right]\frac{z}{D}\frac{\partial^2 Q}{\partial x \partial t} - \frac{1}{2mc_k}\left[1 + (m^2 - 1)F^2\right]\frac{z}{D} = \frac{\partial Q}{\partial x} + \frac{1}{c_k}\frac{\partial Q}{\partial t} \qquad (26)$$

It is shown in Appendix A that the LCR transfer function (derived for $F < 1$) converges to the RFM transfer function for the limiting case of $F = 1$.

Amplitude and phase spectra of the RFM

In previous sections the RFM was discussed in terms of the impulse response. As an alternative, the RFM can be described in terms of the frequency response (e.g. Osiowski, 1972). In the present section the frequency approach is used and the expressions derived for the amplitude spectrum and frequency spectrum of the RFM.

The transfer function of equation (17) describes all transfer properties of the model for any input function and zero initial conditions. Sometimes it is convenient to employ only a part of the function $H^{RFM}(x, s)$ on the imaginary axis $s = i\omega$, i.e. to replace the Laplace transform by the Fourier transform. The function $H^{RFM}(x, i\omega)$ is called an amplitude-phase characteristic of the system or a frequency transfer function. The quantities:

$$A^{RFM}(x, \omega) = \left|H^{RFM}(x, i\omega)\right| \qquad (27a)$$

$$\psi^{RFM}(x, \omega) = \arg\left[H^{RFM}(x, i\omega)\right] \qquad (27b)$$

are called the amplitude and phase characteristic respectively. From equations (27a) and (27b) one can see that:

$$H^{RFM}(x, i\omega) = A^{RFM}(x, \omega)\ \exp[i\psi^{RFM}(x, \omega)] \qquad (28)$$

Thus the amplitude and phase characteristics determine changes in amplitude and phase caused by the model for a cosinusoidal input function with frequency ω.

The RFM amplitude and phase characteristics are:

$$A^{RFM}(x, \omega) = \exp\left[-\frac{\alpha^2\lambda\omega^2}{1+\alpha^2\omega^2}\right] \tag{29}$$

$$\psi^{RFM}(x, \omega) = -\left[\Delta + \frac{\alpha\lambda}{1+\alpha^2\omega^2}\right]\omega \tag{30}$$

Figures 2 and 3 show the amplitude and phase spectra of the RFM for a wide rectangular channel of unit dimensionless length ($D = 1$) with Manning friction ($m = 5/3$). Both Figures are drawn as a function of dimensionless frequency, $\omega' = \omega\bar{y}_0/S_0y_0$. Note that for other lengths of channel the logarithm of the amplitude reduction and the phase shift will be proportional to the dimensionless channel length.

COMPARISON OF THE RFM IMPULSE RESPONSE AND THE LCR

The next step is to examine the accuracy of the approximation of the LCR by the RFM based on a comparison of cumulants, pure delays and frequency characteristics.

Comparison of cumulants and pure delays

Since the first three cumulants of the RFM and the LCR are equal, all the models discussed give the same response to a polynomial function of the third degree. Hence the differences between the fourth cumulants of the impulse responses can be used as a criterion for comparison.

Consider now the case in which the input signal is a polynomial function of the fourth degree:

$$y(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 \tag{31}$$

The difference between the LCR and the RFM response does not depend on time (Strupczewski & Kundzewicz, 1980) and is given by:

$$E = a_4\,(k_4^{RFM} - k_4^{LCR}) \tag{32}$$

Hence to compare the accuracy of the approximation by the RFM one can use the relative error in the fourth cumulants:

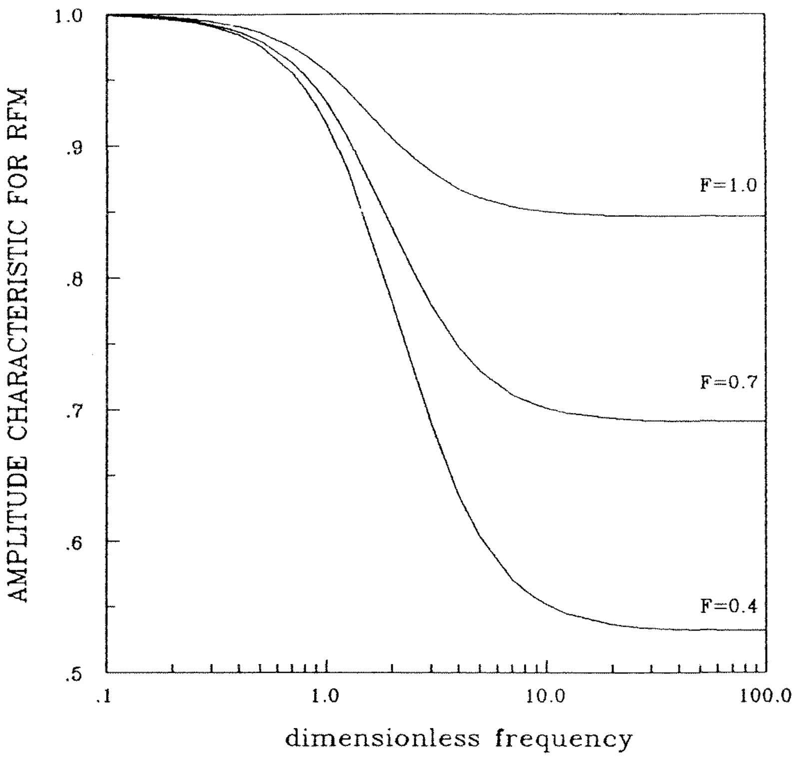

Fig. 2 Amplitude spectrum for the RFM for unit length.

$$r_4 = \frac{k_4^{RFM} - k_4^{LCR}}{k_4^{LCR}} \tag{33}$$

The second criterion which can be used for comparison of the LCR and the RFM impulse responses is the relative error of pure lags defined as:

$$r_\Delta = \frac{\Delta - \Delta^{LCR}}{\Delta^{LCR}} \tag{34}$$

The pure delay of the RFM, Δ, is given by equation (25c). The pure delay of the LCR, which reflects the time of propagation of a perturbation along the positive characteristic, can be expressed by:

$$\Delta^{LCR} = mz \frac{F}{F + 1} \tag{35}$$

The relative errors defined by equations (33) and (34) are plotted in Fig. 4 as a function of the Froude number for $m = 5/3$. Note that the errors discussed are opposite in sign.

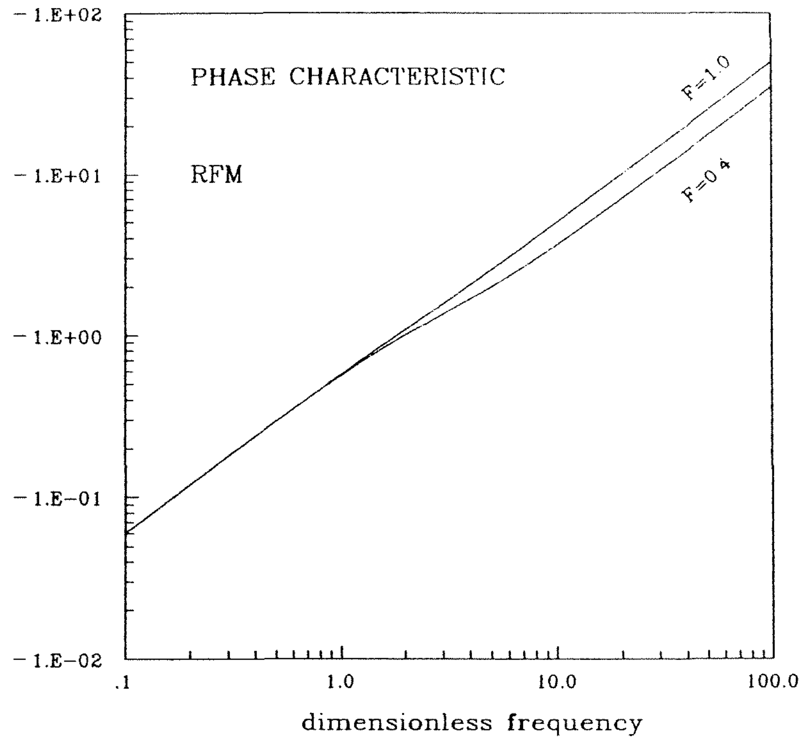

Fig. 3 Phase spectrum for the RFM for unit length.

Comparison by means of frequency characteristics

It is instructive to examine the form of the amplitude and phase spectra for the limiting values of the frequency, ω. For very low frequencies, i.e. very long waves, the amplitude given by equation (29) and the phase given by equation (30) can be approximated by:

$$A^{RFM}(x, \omega) \cong \exp(-0) = 1 \tag{36}$$

$$\psi^{RFM}(x, \omega) \cong -(\Delta + \alpha\lambda)\,\omega = -z\omega \tag{37}$$

An upstream input function in the form of a harmonic oscillation $f(t) = f_0 \cos(\omega t)$ results in a harmonic oscillation at the point x given by:

$$f(x, t) = f_0 \cos(\omega t - z\omega) \tag{38}$$

so there is no attenuation for very long waves and the phase velocity corresponds to the kinematic wave speed.

The response of the LCR to a harmonic oscillation for very low frequencies is also given by equation (38) (Dooge *et al.*, 1987b).

At the other extreme of very high frequencies, i.e. very short waves, the

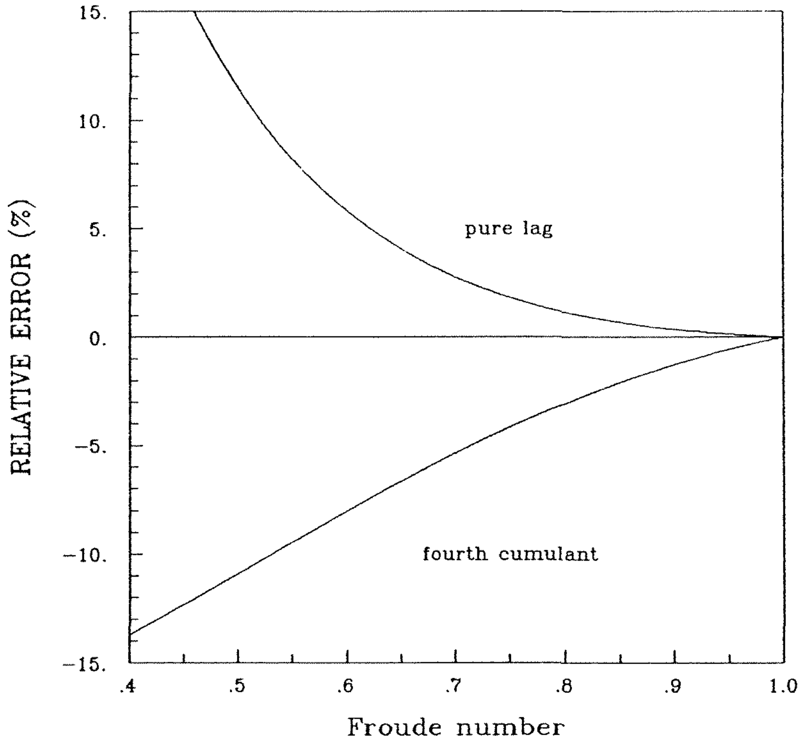

Fig. 4 The relative errors of fourth cumulant and pure lag for the RFM.

amplitude and the phase approach values given by:

$$A^{RFM}(x, \omega) \cong \exp(-\lambda) \tag{39}$$

$$\psi^{RFM}(x, \omega) \cong -\Delta\omega \tag{40}$$

so the resulting harmonic oscillation at the point x takes the form:

$$f^{RFM}(x, t) \cong f_0 \exp(-\lambda) \cos(\omega t - \Delta\omega) \tag{41}$$

which corresponds to the head of the wave travelling with a celerity x/Δ and attenuation $\exp(-\lambda)$.

The LCR response to the harmonic oscillation for very high frequencies takes a different form (Dooge *et al.*, 1987b):

$$f^{LCR}(x, t) = f_0 \exp(-\gamma) \cos(\omega t - \omega x/c_1) \tag{42}$$

which corresponds to the wave travelling with phase velocity $c_1 = v_0 + (g\bar{y}_0)^{1/2}$

and attenuation $\exp(-\gamma)$ where:

$$\gamma = \frac{1 - (m - 1) F}{(1 + F) F} D \tag{43}$$

The relative error in amplitude:

$$r_{A(x,\,\omega)} = \frac{A^{RFM}(x,\omega) - A^{LCR}(x,\omega)}{A^{LCR}(x,\omega)} \tag{44}$$

and in phase:

$$r_{\psi(x,\omega)} = \frac{\psi^{RFM}(x,\omega) - \psi^{LCR}(x,\omega)}{\psi^{LCR}(x,\omega)} \tag{45}$$

for the same conditions as for Figs 2 and 3 are plotted in Figs 5 and 6 respectively. It can be seen that the RFM gives a good approximation of the LCR only for higher Froude numbers.

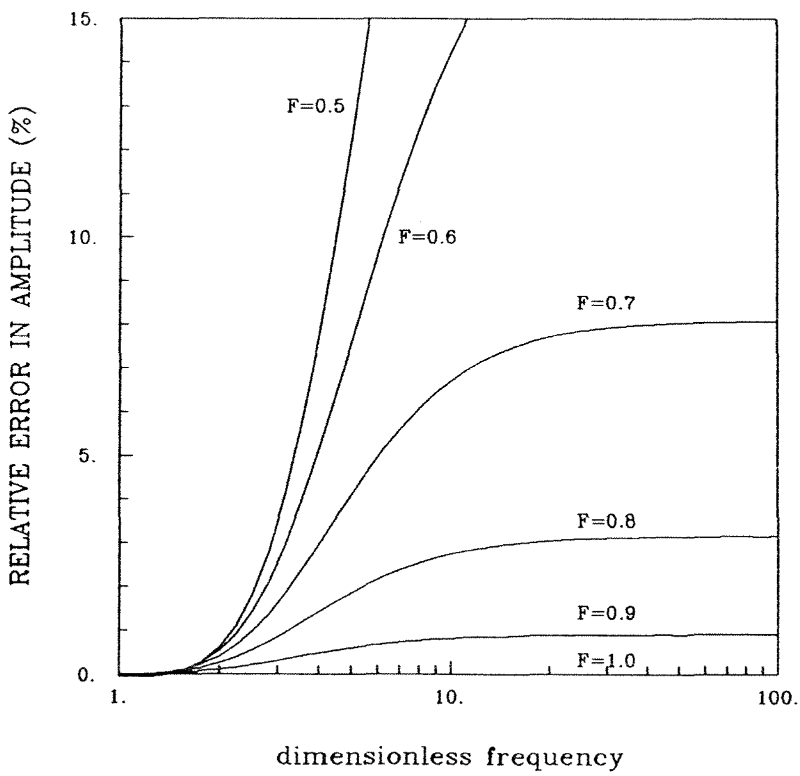

Fig. 5 The relative error in amplitude for the RFM.

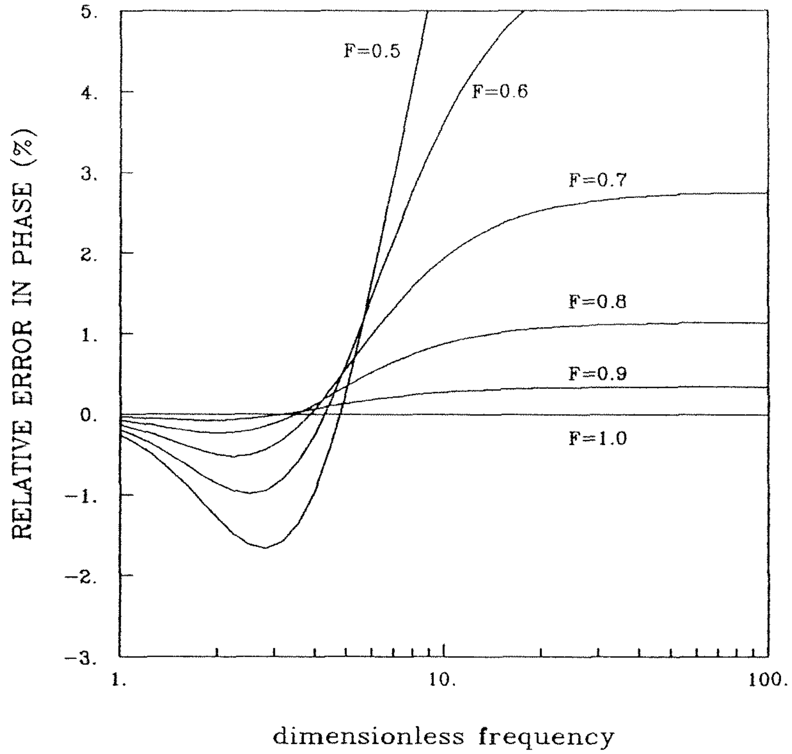

Fig. 6 The relative error in phase for the RFM.

CONCLUSIONS AND RECOMMENDATIONS

The Rapid Flow Model is a model which can be considered as both a conceptual and physical one. On the one hand it is a conceptual model with physically derived parameters; on the other it is a simplification of the linearized St. Venant equations. This simplification results in reducing the number of model parameters and filtering out the downstream boundary condition. Moreover, the RFM, due to its simple structure, can be easily extended to cover lateral inflow.

The accuracy of the LCR approximation by the RFM is assessed by comparing the frequency characteristics and the cumulants of impulse responses. The RFM can be applied for any length of channel reach. However, the quality of the CLE approximation by the RFM depends on the type of motion. It fits the LCR for large Froude numbers and both quick and slow rising waves.

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APPENDIX A. THE LINEAR CHANNEL RESPONSE OF A GENERALIZED UNIFORM CHANNEL

The linearized solution for the downstream movement of a flood wave in a semi-infinite channel and for any friction law was derived by Dooge *et al.*, (1987a).

For the case of a semi-infinite uniform channel with an impulse input at the upstream end the Laplace transform of the linear channel response for Froude number less than unity is given by:

$$H^{LCR}(x, s) = \exp\left[es + f - (as^2 + bs + c)^{1/2}\right] \quad \text{(A1)}$$

where the coefficients are related to the parameters of the channel as follows:

$$a = m^2 \frac{F^2}{(1 - F^2)^2} z^2 \quad \text{(A2)}$$

$$b = 2m \frac{1 + (m - 1)F^2}{(1 - F^2)^2} zD \quad \text{(A3)}$$

$$c = \frac{m^2}{(1 - F^2)^2} D^2 \quad \text{(A4)}$$

$$e = m \frac{F^2}{1 - F^2} z \quad \text{(A5)}$$

$$f = \frac{m}{(1 - F^2)} D \quad \text{(A6)}$$

Equations (3), (5) and (6) define, respectively, the velocity ratio m, the dimensionless length of the channel D, and the time of passage of a kinematic wave through the channel z. F is the Froude number for the reference flow condition.

The first four cumulants of the LCR can be expressed as:

$$k_1^{LCR} = z \tag{A7}$$

$$k_2^{LCR} = \frac{1}{m}\,[1 - (m - 1)^2F^2]z^2/D \tag{A8}$$

$$k_3^{LCR} = \frac{3}{m}\,[1 + (m - 1)\,F^2]k_2^{LCR}\,z/D \tag{A9}$$

$$k_4^{LCR} = \frac{3(1 - F)^2}{m}\,\left[k_2^{LCR}\right]^2/D + \frac{4}{3}\,\left[k_3^{LCR}\right]^2/k_2^{LCR} \tag{A10}$$

For any given shape of channel and friction law the cumulants of the LCR are functions of z, D, m and F.

The transfer function of equation (A1) describes all transfer properties of the model for any input function and zero initial conditions.

The frequency characteristics of the LCR are (Dooge *et al.*, 1987b):

$$A^{LCR}(x,\,\omega) = \exp(c^{½} - \left\{[b^2\omega^2 + (-\,a\omega^2 + c)^{½}]^{½} - a\omega^2 + c\right\}^{½}/\sqrt{2}) \tag{A11}$$

$$\psi^{LCR}(x,\,\omega) = e\omega - \left\{[b^2\omega^2 + (-\,a\omega^2 + c)^2]^{½} + a\omega^2 - c\right\}^{½}/\sqrt{2}) \tag{A12}$$

where the parameters a, b, c, e and f are defined by equations (A2) to (A6).

The limiting case of Froude number equal to unity

Substituting equations (A2) to (A6) into equation (A1) gives:

$$H^{LCR}(x,\,s) = \tag{A13}$$

$$\exp\left[\frac{mzF^2s + mD - \left\{m^2z^2F^2s^2 + 2\,mzD[1 + (m - 1)F^2]s + m^2D^2\right\}^{½}}{1 - F^2}\right]$$

From the l'Hopital theorem one gets the transfer function of the LCR for the limiting case of $F = 1$:

$$H_{F=1}^{LCR}(x,\ s) = \exp\left[-\ 0.5mzs\ -\ 0.5(2\ -\ m)D\ +\ \frac{0.5(2\ -\ m)D}{1\ +\ zs/D}\right] \quad \text{(A14)}$$

It can be seen that the transfer function of the RFM described by equation (17) with the parameters α, λ and Δ defined by equations (25) for a Froude number equal to unity is given by equation (A14) as well, i.e:

$$H_{F=1}^{LCR} = H_{F=1}^{RFM} \quad \text{(A15)}$$

Hence the expressions for the cumulants of the LCR derived for $F < 1$ are valid for $F \leqslant 1$.

APPENDIX B: DERIVATION OF THE RFM PARAMETERS BY CUMULANT MATCHING

The coefficients C_1 and C_2 of equation (12) can be estimated using the condition of equivalence of the RFM cumulants and those of the LCR. Since the first order terms in equations (7) and (12) are identical, the first cumulants of both models are equivalent for any C_1 and C_2.

From equations (24b), (18) and (19) one has:

$$k_2^{RFM} = 2(\alpha z - \beta x) \quad \text{(B1)}$$

By comparing equation (B1) with equation (A8) and taking into account equations (13) and (14) one gets:

$$(1 - F^2)\ (C_1 + C_2) + 2mF^2 - m^2F^2 = 1 - (m\ -\ 1)^2F^2 \quad \text{(B2)}$$

which gives $C_1 + C_2 = 1$ as the criterion of the equivalence of second cumulants.

From equation (24b) we have:

$$k_3^{RFM} = 6\alpha^3\lambda \quad \text{(B4a)}$$

or

$$k_3^{RFM} = 3\alpha\ k_2^{RFM} \quad \text{(B4b)}$$

By comparing this with equation (A9) one gets equation (25a) which on the basis of equation (13) gives $C_1 = 2$ as the criterion for the additional equivalence of the third cumulants.

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