BUDAPEST 1988

INTERNATIONAL CONFERENCE ON FLUVIAL HYDRAULICS

SOLUTION OF THE COMPLETE LINEARIZED ST. VENANT EQUATION FOR LIMITING CASE OF THE FROUDE NUMBER EQUAL TO ONE

Witold G. Strupczewski and Jarosław J. Napiórkowski
Institute of Geophysics, Polish Academy of Sciences
Pasteura 3. 00-973 Warsaw

SUMMARY: The solution of the linearized St.Venant equation for Froude number equal to one has simple form and clear conceptual interpretation. It could serve as a good approximation of the complete solution for mountain rivers.

The linearised St.Venant equation for one-dimensional unsteady flow in uniform channel may be written *as* (Dooge et *al.,* 1986):

$$(1 - F_0^2)\frac{\partial^2 Q}{\partial x^2} - \frac{2F_0^2}{v_0}\frac{\partial^2 Q}{\partial x \partial t} - \frac{1}{g\overline{y}_0}\frac{\partial^2 Q}{\partial t^2} = \frac{2mS_0}{\overline{y}_0}\frac{\partial Q}{\partial x} + \frac{2S_0}{v_0\overline{y}_0}\frac{\partial Q}{\partial t} \tag{1}$$

where Q is the perturbation of flow, $\overline{y}_0$ is the hydraulic mean depth, $m = c_k/v_0$ is a ratio of the kinematic wave speed to the average velocity of flow, S_0 is the bottom slop, x is the distance from the upstream boundary, t is the elapsed time, F_o is a Froude number.

The St. Venant equation (1) is *a* hyperbolical one, i.e. has two real characteristics. The direction of these characteristics controls the celerity of both primary and secondary waves. However, for $F_0 = 1$ the first term in eq.(1) is equal to zero (i.e. the celerity of secondary wave is equal to zero) and only upstream boundary condition $Q_u(t)$ is required. This case is discussed in the paper.

Instead of solving the differential eq.(1) for $F_o=1$ with initial and boundary conditions directly, we detour into the image space: using the *L* - transformation the partial differential equation becomes an ordinary differential equation (Doetsch, 1961). When the ordinary differential equation is solved one gets the transfer function:

$$H(x, s) = \exp\left[-\Delta s - \frac{\beta s}{1 + \alpha s}\right] \tag{2}$$

where

$$\Delta = .5mx / c_k; \ \alpha = \bar{y}_0 / S_0 c_k; \ \beta = (1 - .5)x / c_k \tag{3}$$

Solution of the original problem can be obtained the inverting the L - transformation:

$$h(x,t) = P_0(\beta/\alpha)\,\delta(t-\Delta) + \sum_{i=1}^{\infty} P_i(\beta / \alpha)\ h_i[(t-\Delta)/\alpha] \tag{4}$$

where

$$P_i(\beta / \alpha) = \frac{(\beta / \alpha)^i}{i\,!} \exp(-\beta / \alpha) \tag{5}$$

and

$$h_i(t / \alpha) = \frac{(t / \alpha)^{i-1}}{\alpha(i - 1)\,!} \exp(-t / \alpha) \tag{6}$$

The impulse response given by eq.(4) has two distinct parts. One of them contains the Dirac delta function and the other represents the attenuation of the model. It is worth noting that eq.(4) can be considered as a sum of products of Poisson distribution function (5) and impulse response of linear cascade (6) shifted in time. The Poisson distribution defines the part of unit total volume transformed through i-linear reservoirs (with a time constant α), and β / α is the average number of reservoirs in a cascade.

To study the properties of linear responses and to compare the various models proposed to represent the linear channel response h(x,t) the use of cumulants was introduced. The cumulants are generated by the logarithm of the Laplace transform of the impulse response function (Nash, 1959). The cumulants of the transfer function (2) can be expressed in terms of α and β as:

$$k_1 = \beta + \Delta; \qquad k_r = r\,!\,\alpha^{r-1}\beta \ \ r > 1 \tag{7}$$

Note, that if the first N cumulants *of* two different models are equal, these two models give the same response to a polynomial function *of* the N-th degree.

REFERENCES

Doetsch, G., 1961. Guide to Application *of* Laplace Transform. Van Nostrand.

Dooge, J.C.I., Napiórkowski, J.J. and Strupczewski, W.G., 1986. The linear downstream response *of* a generalized uniform channel. Acta Geophysica Polonica. Vol. XXXV. No.4.

Nash, J.E., 1959. Systematic determination *of* unit hydrograph parameters. J. Geoph. Res., 64(1), 111-115.