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Multivariate Analysis of Hydrologic Processes

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ASYMPTOTIC BEHAVIOUR OF PHYSICALLY BASED MULTIPLE MUSKINGUM MODEL

By

W. G. Strupczewski, Head of Division
Institute of Geophysics, Polish Academy of Sciences
Warsaw, Poland

J. J. Napiórkowski, Research Hydrologist
Institute of Geophysics, Polish Academy of Sciences
Warsaw, Poland

ABSTRACT

The St. Vénant equations for unsteady flow in open channels are simplified in order to filter out the downstream boundary condition. It is achieved by approximating some terms by means of kinematic wave solution. It is shown that the resulting model is equivalent to the multiple Muskingum model.

INTRODUCTION

The set of St. Vénant equations of gradually varied river flow gave rise to a number of models that are developed and widely used in practice. However, if flow routing in channels must be repeated many times for different scenarios, the St. Vénant based models are likely to be too costly and time consuming. Hence, simpler models of flood wave movement have been developed. Recently, interest in the Muskingum type models (multiple, nonlinear, with variable parameters) has significantly increased.

The Muskingum flood routing method which had seemed to be purely empirical was shown to be linked with models based on convective diffusion equations. By comparison of both models relationships between their parameters have been found. Cunge (1969) compared the difference schemes and Dooge (1973) compared the impulse responses using moment matching technique. Koussis's method (1978) leads from the Muskingum equation to the linear convective diffusion equation. He transformed the lumped Muskingum model into a distributed model by expressing outflow as a function of inflow and its length derivatives and using the relation valid for kinematic wave only. There exists a more direct possibility of deriving the Muskingum equations from St. Vénant equations. One approach initiated by Strupczewski and Kundze-

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Hsieh Wen Shen, Jayantha T.B. Obeysekera
Vujica Yevjevich, Donald G. DeCoursey

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Colorado State University, Fort Collins, Colorado 80523
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wicz (1980) and developed by Napiórkowski et al. (1981) is the lumping of nonlinear convective diffusion model under assumption of linear changes of water level along the river reach and then linearising it around the steady state. Dooge et al. (1982) using the method of inverse order obtained results applicable to any shape of cross-section and to any type of friction law. The present paper gives the answer to the question what physically based model is best approximated by the multiple Muskingum method.

COMPLETE LINEAR EQUATION AND ITS SIMPLIFICATION

The linearised St. Vénant equation for one-dimensional unsteady flow in a broad rectangular channel with Chezy friction may be written as (e.g. Dooge and Harley, 1967a)

$$(gy_o - v_o^2)\frac{\partial^2 Q}{\partial x^2} - 2v_o\frac{\partial^2 Q}{\partial x \partial t} - \frac{\partial^2 Q}{\partial t^2} = 3gS_o\frac{\partial Q}{\partial x} + \frac{2gS_o}{v_o}\frac{\partial Q}{\partial t} \qquad (1)$$

where v_o is reference velócity, y_o is reference depth, S_o is bottom slope, t is elapsed time, x is distance along the channel, Q is the perturbation from the reference flow.

Eq.(1) is a hyperbolic one, i.e. it has two real characteristics. The direction of these characteristics in the (x,t) plane is given by

$$\frac{dx}{dt} = v_o \pm \sqrt{gy_o} \qquad (2)$$

which gives the celerity of both the primary and secondary waves. In the case of tranquil flow, i.e. Froude number

$$F = \frac{v_o}{\sqrt{gy_o}} \qquad (3)$$

less than 1, the celerity of secondary wave will be negative, that is the wave will travel in an upstream direction. In practical flood routing the influence of downstream controls is nearly always neglected and the routing takes part only in a downstream direction. Accordingly, the hyperbolic Eq.(1) is modified in order to filter out these upstream waves. In order to accomplish this it is necessary to reduce the hyperbolic equation to a parabolic-like form.

In the case of many river channels, the terms on the left hand side of Eq.(1) are of an order of magnitude smaller than the terms on the right hand side (Henderson, 1966). Instead of neglecting small "hyperbolic" terms entirely, they can be represented on the basis of the linear kinematic wave approximation (Dooge and Harley, 1967b).

For the kinematic wave approximation we can write the solution for the perturbation as

$$Q = f(x - c_k t) \tag{4}$$

where

$$c_k = 1.5\ v_o \tag{5}$$

is a kinematic wave speed. This lower order solution can be used to approximate the "hyperbolic" terms on the left hand side of Eq.(1)

$$\frac{\partial^2 Q}{\partial t^2} = c_k^2\ f''(x - c_k t) = -\ c_k \frac{\partial^2 Q}{\partial x \partial t} \tag{6}$$

$$\frac{\partial^2 Q}{\partial x^2} = \quad f''(x - c_k t) = -\ \frac{1}{c_k} \frac{\partial^2 Q}{\partial x \partial t} \tag{7}$$

Substitution of these approximations in Eq.(1) gives

$$-\ \frac{D}{c_k}\ \frac{\partial^2 Q}{\partial x \partial t} = c_k \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} \tag{8}$$

where

$$D = \frac{v_o y_o}{2 S_o}\ (1 - 0.25\ F^2) \tag{9}$$

is a constant diffusion coefficient.

Eq.(8) is typical of the equations representing the diffusion of kinematic waves (Lighthill and Whitham, 1955). Note, that to solve Eq.(8) only upstream boundary condition $Q_u(t)$ is required. The downstream boundary condition was filtered out from the St. Vénant Eq.(1).

The linear Eq.(8) can conveniently be solved by the use of the Laplace transform technique. Since Eq.(8) represents perturbation from an initial steady condition, the initial value of the dependent variable $Q(x,t)$ and its derivatives will all be zero. Hence, Eq.(8) when transformed to the Laplace transformed domain becomes

$$\left(c_k + \frac{D}{c_k}\, s\right) \frac{dQ}{dx} + Q\, s = 0 \tag{10}$$

The above equation is a first-oder homogeneous ordinary differential equation for the Laplace transform $Q(x,s)$ as a function of x. The solution of Eq.(10) can be written in the general form

$$Q(x,s) = H(x,s)\, Q_u(s) \tag{11}$$

where

$$H(x,s) = \exp\left(\frac{-\dfrac{x}{c_k}\, s}{1 + \dfrac{D}{c_k^2}\, s}\right) \tag{12}$$

is the transfer function, i.e. the Laplace transform of the impulse response. The impulse response (12) describes all transfer properties of the simplified St. Vénant Eq.(8) for any input function.

It will be shown in the next section, that the transfer function for multiple Muskingum model is given by Eq.(12) as well.

THE MULTIPLE MUSKINGUM MODEL

One of the most popular approaches to the mathematical description of open channel flow is the Muskingum method, which was first proposed by McCarthy (1939). Similarly to other lumped conceptual models, the Muskingum method is a set of continuity and dynamic equations

$$\dot{S}(t) = Q_1(t) - Q_2(t) \tag{13}$$

$$S(t) = K\left[aQ_1(t) + (1-a)Q_2(t)\right] \tag{14}$$

where Q_1 is the inflow to the river reach, Q_2 is the outflow from the reach, S is the storage in the reach and a, K are model parameters.

The transfer function of the Muskingum model reads

$$H(s) = \frac{1 - aK\cdot s}{1 + (1-a)K\cdot s} \tag{15}$$

As with all types of models, it is necessary to find the optimal values of the parameters of the model given by Eqs.(13,14). The parameters K and a can be determined by equating the first and second cummulants of the complete St. Vénant Eq.(1) and the first and second cummulants of the Muskingum model (Dooge, 1973). This results in the physically based values

$$K = \frac{x}{c_k} \tag{16}$$

$$a = 0.5 - \frac{1}{3}\frac{y_o}{S_o x}(1 - 0.25F^2) \tag{17}$$

where x is the length of the river reach.

The Muskingum method completely fails for long lengths of a channel. A straighforward generalization of the model described by Eqs.(13,14) is a multiple Muskingum model obtained by dividing the total reach into n equal subreaches (Laureson, 1959; Kundzewicz and Strupczewski, 1982). In such a case the values of K and a are dependent on the subreach length and are given by

$$K' = \frac{x}{n\ c_k} \tag{18}$$

$$a' = 0.5 - \frac{1}{3}\frac{y_o}{S_o}\frac{n}{x}(1 - 0.25\ F^2) \tag{19}$$

Thus, the transfer function of a cascade of Muskingum models reads

$$H_n(x,s) = \left[\frac{1 - a'K'\ s}{1 + (1-a')K's}\right]^n \tag{20}$$

Substitution of Eqs.(18,19) into Eq.(20) gives

$$H_n(x,s) = \left[\frac{1 - \frac{x \cdot s}{2c_k n} + \frac{y_o \cdot s}{3c_k S_o}\,(1-0.25\,F^2)}{1 + \frac{x \cdot s}{2c_k n} + \frac{y_o \cdot s}{3c_k S_o}\,(1-0.25\,F^2)} \right]^n \tag{21}$$

Let us consider the limiting case of the above transfer function when n tends to infinity. It means modelling a river reach of a finite length x by an infinite number of physically based Muskingum models. Using the de l'Hopital theorem one gets

$$\lim_{n \to \infty} H_n(x,s) = \exp\left(\frac{-\frac{x}{c_k} \cdot s}{1 + \frac{D}{c_k^2} \cdot s} \right) \tag{22}$$

Note, that the transfer function (22) is exactly the same as the transfer function (12). Hence, the main conclusion from the above consideration is:

The diffusion-like partial differential equation (8) is best approximated by the multiple Muskingum method.

SOLUTION IN THE TIME DOMAIN

It remains to invert Eq.(22) from the Laplace transform domain to the original time domain. Eq.(22) can be rewritten

$$H_\infty(x,s) = \exp\left(-\frac{c_k}{D}x\right) \exp\left(\frac{c_k}{D}x \, \frac{1}{1 + \frac{D}{c_k^2}\, s} \right) \tag{23}$$

We can expand the second term of Eq.(23) into a convergent series and operate on it term by term

$$H_\infty(x,s) = \exp\left(-\frac{c_k}{D}x\right) \sum_{i=0}^{\infty} \left(\frac{c_k}{D}x\right)^i \frac{1}{i!} \frac{1}{\left(1 + \frac{D}{c_k^2}s\right)^i} \tag{24}$$

The explicit formulation of the transfer function in the time domain is obtained by adopting the standard transform pairs given by Doetsch (1961).

$$h_\infty(x,t) = \exp(-c_k x/D)\Big[\,\delta(t) + \sum_{i=1}^{\infty}(c_k x/D)^i \frac{1}{i!(i-1)!}\frac{t^{i-1}}{(D/c_k^2)^i}\exp(-\,c_k^2 t/D)\Big] \quad (25)$$

The solution is found to have two distinct parts. One of them contains the Dirac δ-function. This term provides direct transformation of the damped input signal. The other is responsible for the modulatory system performance.

The second part of the system response is shown in Fig.1 for dimensionless variables

$$x' = x/(y_o/S_o) \quad (26)$$

$$t' = t/(y_o/S_o v_o) \quad (27)$$

and for various values of the dimensionless length factor x'.

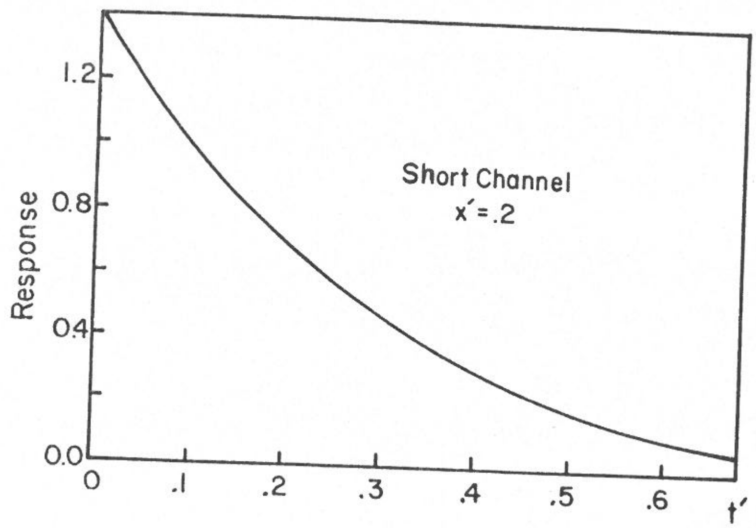

Figure 1. Shape of impulse response for F=.3

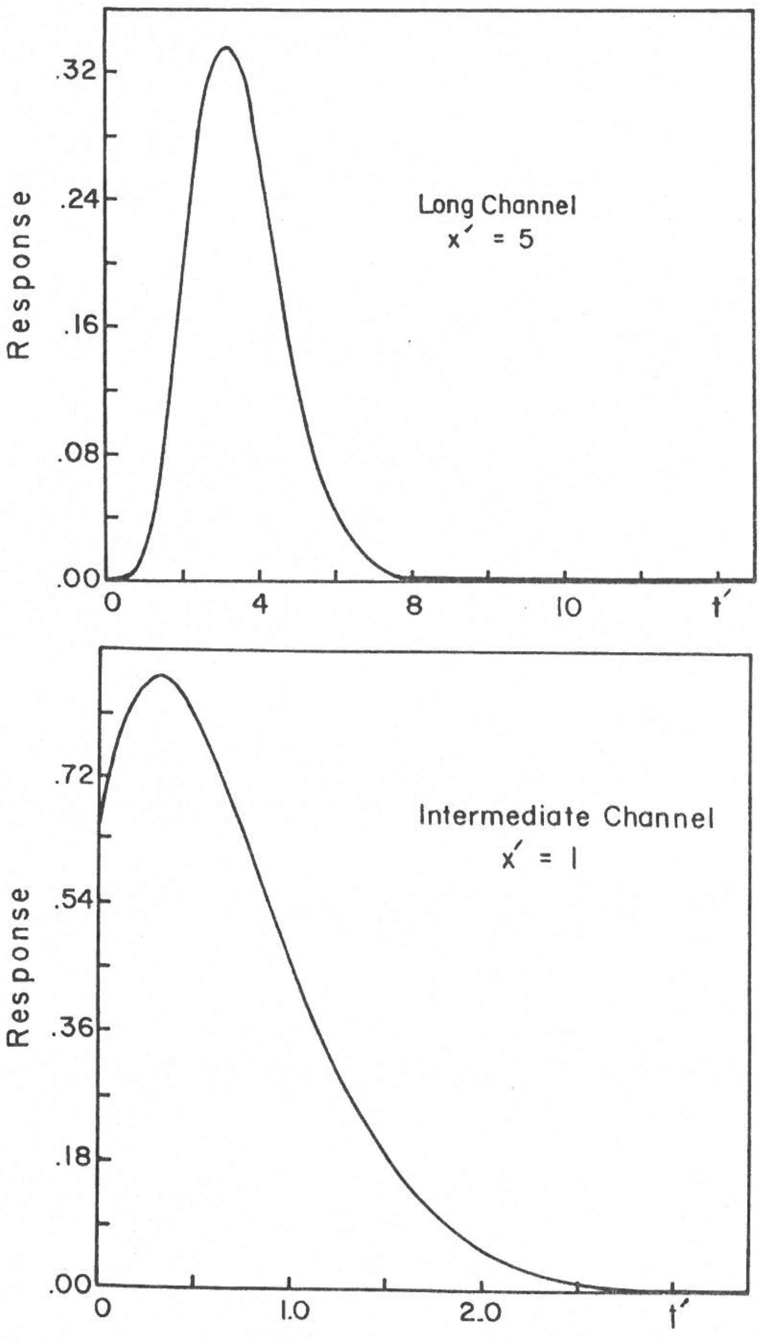

Figure 1. Shape of impulse response for F=.3 (Cont.)

For short lengths, the impulse response declines monotonically; for intermediate lengths, the impulse response is a unimodal curve with an appreciable initial ordinate. For long channels the unimodal shape of response rises from an initial ordinate which is practically zero and declines again to zero. Similar three shapes were obtained for linear downstream response of the the complete St. Vénant equations by Dooge (1973; p.249).

CONCLUSIONS

The present paper gives the answer to the question what physically based distributed model is best approximated by the multiple Muskingum method. Using the transfer function approach it is proved that multiple Muskingum model for limiting case when the river reach of a finite length is modelled by an infinite number of physically based Muskingum reaches is equivalent to the diffusion-like equation (8). Hence, Eq.(8) can be called the distributed Muskingum model.

It has been shown that the impulse response in time domain of the distributed Muskingum model is similar in shape to transfer function of the linearized St. Vénant equations.

REFERENCES

Cunge, J. A., 1969. On the subject of a flood propagation computation method (Muskingum method). J. Hydraul. Res. 7, 205-230.

Doetsch, G., 1961. Guide to the applications of Laplace transforms. Van Nostrand.

Dooge, J. C. I., 1973. Linear theory of hydrologic systems. USDA Agricultural Research Service, Tech. Bull. no. 1468, Washington, DC.

Dooge, J. C. I. and Harley, B. M., 1967a. Linear routing in uniform open channels. Proc. Int. Hydrol. Symp., Fort Collins, Colorado, Pap. 8, 1:57-63.

Dooge, J. C. I. and Harley, B. M., 1967b. Linear theory of open channel flow. Dep. Civ. Eng., Univ. Coll. Cork, unpublished memorandum.

Dooge, J. C. I., Strupczewski, W. G. and Napiórkowski, J. J., 1982. Hydrodynamic derivation of storage parameters of the Muskingum model. J. Hydrol., 54: 371-387.

Henderson, F. M., 1966. Open channel flow. Macmillan.

Koussis, A. D., 1978. Theoretical estimation of flood routing parameters. J. Hydraulics Division, ASCE, vol. 104, no. HY1, Jan., pp 109-115.

Kundzewicz, Z. W. and Strupczewski, W. G., 1982. Approximate translation in the Muskingum model. J. Hydrological Sciences, 27, 1, 3, pp 19-27.

Laurenson, E. M., 1959. Storage analysis and flood routing in long river reaches. J. Geophys. Res. 64, 2423-2431.

Lighthill, M. J. and Whitham, G. P., 1955. On kinematic waves I, flood movements in long rivers. Royal Soc. London Proc., A 229, pp 281-316.

McCarthy, G. T., 1939. The unit hydrograph and flood routing. U.S. Corps Eng., Providence, R.I.

Napiórkowski, J. J., Strupczewski, W. G. and Dooge, J. C. I., 1981. Lumped nonlinear flood routing model and its simplification to the Muskingum model. In: Proc. Int. Symp. on Rainfall-Runoff Modelling, Mississippi State Univ., 543-552.

Strupczewski, W. G. and Kundzewicz, Z. W., 1980. Muskingum method revisited. J. Hydrol., 48: 327-342.