PROPERTIES OF THE DISTRIBUTED MUSKINGUM MODEL

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Abstract

The presented linear flood routing model can be considered both as a hydrodynamic and conceptual one. It has been derived for the case of a uniform open channel with arbitrary cross-section shape and friction law as the asymptotic case of physically based multiple Muskingum model. It has been shown that the discussed model is equivalent to the linear kinematic diffusion model. Because of the Muskingum origin of the model and its description by partial differential equation it is called Distributed Muskingum Model (DMM). The DMM impulse response retains clear conceptual interpretation being the total of the products of the Poisson distribution and the impulse response of linear reservoirs in series.

Such characteristics of the DMM impulse response as cumulants, amplitude and phase spectra are analyzed, and then compared with those of the complete linearized St. Venant equations for different reach lengths, values of Froude number and frequencies of flood waves. The DMM can be applied for small Froude numbers and slow rising waves. Having the same number of parameters it is more accurate than the Muskingum model except of very short lengths of channel reach.

1. INTRODUCTION

To describe the unsteady flow of water in an open channel by means of the St. Venant equations it is necessary to know with sufficient accuracy the geometrical and hydraulic characteristics of the channel reach as well as the initial and boundary conditions. The difficulties of meeting these requirements led to the replacement of distributed characteristics by their representative values for the whole river reach and to the development of simple, yet accurate models. In the approximate flood routing models there is an important class of linear stationary models which covers hydrodynamic, lumped conceptual and black box models. The complete linearized St. Venant equations provide a basis for comparison for all other linear models and permits a physical evaluation of their parameters.

The conceptual Muskingum model which seemed to be purely empirical was shown to be linked with models based on convective diffusion equations. Cunge (1969) found similarity in the difference schemes of both models and obtained relationships between their parameters. A more direct connection of the Muskingum model with the physical situation was shown by Strupczewski and Kundzewicz (1980) and Napiórkowski et al. (1981). The nonlinear convective diffusion model has been lumped under assumption of linear changes of water level along the river reach and linearized around the steady state.
Dooge et al. (1981) using the method of inverse order obtained the Muskingum model as
the first term of Volterra series and got results applicable to any shape of cross-section and
to any type of friction law.

The Muskingum model fails if the length of the flood wave is small compared to the
length of the channel reach. In such cases a series of the Muskingum models (multiple
Muskingum model) can be apply as was first proposed by Cunge (1969). Strupczewski
and Napiórkowski (1986) showed that for a finite river reach the infinite multiply Musk-
ingum model with physically based parameters is equivalent to the kinematic diffusion
model. Therefore the best analogue of the Muskingum lumped model in the distributed
domain is the kinematic diffusion model, not convective diffusion model. The aim of
the present paper is to describe some properties of the impulse response of the infinite multiple
Muskingum model and to compare those with these obtained as solution of the complete
linearized St. Venant equations and with the Muskingum model impulse response. It is
the first model being both hydrodynamic and conceptual. This feature is also reflected
by its name "distributed Muskingum model".

2. COMPLETE LINEAR EQUATION

The linearized St. Venant equation for one-dimensional unsteady flow in uniform chan-
nel may be written as (Dooge et al., 1987a):

\[(1-F_0^2)g\bar{y}_0 \frac{\partial^2 Q}{\partial x^2} - 2v_0 \frac{\partial^2 Q}{\partial x \partial t} - \frac{\partial^2 Q}{\partial t^2} = gA_0 \left( \frac{\partial S_f}{\partial Q} \frac{\partial Q}{\partial t} - \frac{\partial S_f}{\partial A} \frac{\partial Q}{\partial x} \right), \tag{1}\]

where \(Q\) is the perturbation of flow about an initial condition of steady uniform flow \(Q_0\),
\(A_0\) is the cross-sectional area corresponding to this flow, \(S_f\) is the friction slope, \(\bar{y}_0\) is the
hydraulic mean depth, \(v_0\) is the mean velocity, \(x\) is the distance from the upstream boundary,
\(t\) is the elapsed time, and derivatives of the friction slope \(S_f\) are evaluated at the reference
conditions.

The variation of friction slope with discharge at the reference condition for all frictional
formula for rough turbulent flow may be expressed as:

\[\frac{\partial S_f}{\partial Q} = \frac{2S_0}{Q_0}, \tag{2}\]

where \(S_0\) is the bottom slope. We may for convenience define a parameter \(m\) as a ratio of the
kinematic wave speed to the average velocity of flow

\[m = \frac{c_k}{Q_0}, \tag{3}\]

where \(c_k\) is the kinematic wave speed as given by Lighthill and Whitham (1955)

\[c_k = \frac{dQ}{dA} = \frac{\partial S_f}{\partial A} \frac{\partial S_f}{\partial Q}. \tag{4}\]
The parameter $m$ is a function of the shape of channel and of a friction law parameter. When equations (2), (3), and (4) are substituted in equation (1) one gets:

$$(1 - F_0^2) \frac{\partial^2 Q}{\partial x^2} = \frac{2F_0^2}{v_0} \frac{\partial^2 Q}{\partial x \partial t} - \frac{1}{g} \frac{\partial^2 Q}{\partial y_0 \partial t^2} = \frac{2mS_0}{v_0} \frac{\partial Q}{\partial x} + \frac{2S_0}{v_0 y_0} \frac{\partial Q}{\partial t},$$  

(5)

where $F_0$ is a Froude number. Denoting

$$D = \frac{S_0}{y_0} x,$$

(6)

dimensional length,

and

$$\beta = \frac{x}{c_k}$$

(7)

passage time of a kinematic wave through the channel

one obtains

$$(1 - F_0^2) \frac{c_k}{2mD} \frac{\partial^2 Q}{\partial x^2} - F_0^2 \frac{\beta}{D} \frac{\partial^2 Q}{\partial x \partial t} - F_0^2 \frac{m \beta}{2 c_k D} \frac{\partial^2 Q}{\partial t^2} = \frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t}.$$

(8)

3. MUSKINGUM MODEL AND ITS GENERALIZATION

In this section three approaches used for modelling the effect of a channel reach of finite length are discussed, viz., the classical Muskingum model, the multiple Muskingum model with physically based parameters and the distributed Muskingum model, obtained as a limiting case of the multiple Muskingum model.

3.1. Muskingum model (MM). One of the most popular approaches to the mathematical description of open channel flow is the Muskingum method, which was first proposed by McCarthy (1939). The Muskingum method is a set of continuity and dynamic equations

$$\dot{S}(t) = Q_1(t) - Q_2(t),$$  

(9)

$$S(t) = K \left[ e Q_1(t) + (1 - e) Q_2(t) \right],$$  

(10)

where $Q_1$ is the inflow to the reach, $Q_2$ is the outflow from the reach, $S$ is the storage in the reach and $K$ and $e$ are model parameters.

As with all types of models, it is necessary to find the best values of the parameters for use in a particular case. Dooge et al. (1982) established the relationship between the model parameters and hydraulic characteristics of the uniform channel, applicable to any shape of cross-section and any friction law using state trajectory variation method. This results in the physically based values

$$K = \beta,$$

(11)

$$e = 0.5 - \frac{\alpha}{\beta},$$

(12)
where

\[
\alpha = \frac{1}{2m} \frac{\bar{y}_0}{S_0 c_k} \left[ 1 - (m - 1)^2 F_0^2 \right]. \tag{13}
\]

The transfer function of the Muskingum model in a convenient form for calculating the inverse Laplace transformation is given by:

\[
H(s) = \frac{\frac{1}{2} \alpha - \frac{1}{2} \beta}{\frac{1}{2} \alpha + \frac{1}{2} \beta} \left[ 1 + \frac{\beta}{2} \left( \frac{1}{\alpha - \frac{\beta}{2}} \left( \frac{\alpha + \beta}{2} \right) s + \frac{1}{\alpha + \frac{\beta}{2}} \right) \right]. \tag{14}
\]

On the basis of equation (14) one can see that the impulse response of the Muskingum model reads:

\[
h(t) = \frac{\frac{1}{2} \alpha - \frac{1}{2} \beta}{\frac{1}{2} \alpha + \frac{1}{2} \beta} \left[ \delta(t) + \frac{\beta}{2} \left( \frac{1}{\alpha - \frac{\beta}{2}} \left( \frac{\alpha + \beta}{2} \right) e^{\left( -\frac{t}{\alpha + \frac{\beta}{2}} \right)} \right) \right]. \tag{15}
\]

The impulse response of the MM is found to have two distinct parts. First of them contains the Dirac delta function. This term provides direct transformation of the damped input signal. The other is responsible for the modulatory system performance.

3.2. Multiple Muskingum model (MMM). It is well known, that the Muskingum method is not applicable to longer channels. The effects of such channels may however be expressed by a multiple Muskingum model obtained by dividing the total reach into \( n \) equal subreaches. In such a case the value of \( \beta \) is dependent on the subreach length and is given by:

\[
\beta' = \frac{\beta}{n}. \tag{16}
\]

Accordingly, the transfer function of a cascade of Muskingum models is:

\[
H_n(x, s) = \left( \frac{\frac{1}{2} \alpha - \frac{1}{2} \beta}{\frac{1}{2} \alpha + \frac{1}{2} \beta} \right)^n \left[ 1 + \frac{\beta}{n} \left( \frac{1}{\alpha - \frac{\beta}{2n}} \left( \frac{\alpha + \beta}{2n} \right) s + \frac{1}{\alpha + \frac{\beta}{2n}} \right) \right]^n. \tag{17}
\]

Using Newton binomial formula we can expand the second term into a finite series and ope-
rate on it term by term. Hence, the MMM impulse response is:

\[
h_n(x, t) = \left( \frac{\alpha - \beta}{2n} \right)^n \left( \delta(t) + \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \frac{\beta}{2n} \left( \frac{\beta}{2n} \right)^i \frac{\beta}{\alpha + \beta} \frac{t^{i-1}}{(i-1)!} \exp \left( \frac{-t}{\alpha + \beta} \right) \right)
\]

(18)

Similarly to the MM, the impulse response of the MMM has two distinct parts. One of them contains the Dirac delta function and the other represents the attenuation of the model. To obtain insight into the model structure it is worth noting that equation (18) can be considered as a sum of products of some distribution function:

\[
P_i(n, \frac{\beta}{\alpha}) = \left( \begin{array}{c} n \\ i \end{array} \right) \left( \frac{\beta}{\alpha + 2n} \right)^{n-i} \left( 1 - \frac{\beta}{\alpha + 2n} \right)^i,
\]

\[i = 0, 1, \ldots, n,
\]

(19)

and the impulse response of linear \(i\)-reservoirs with a time constant \(\alpha(1 + \beta/\alpha 2n)\):

\[
h_i \left[ \frac{t}{\alpha \left( 1 + \frac{\beta}{\alpha 2n} \right)} \right] = \left[ \frac{t}{\alpha \left( 1 + \frac{\beta}{\alpha 2n} \right)} \right]^{i-1} \exp \left[ \frac{-t}{\alpha \left( 1 + \frac{\beta}{\alpha 2n} \right)} \right],
\]

\[i = 1, 2, \ldots, n.
\]

(20)

So, the MMM impulse response can be rewritten as:

\[
h_n(x, t) = P_0(n, \frac{\beta}{\alpha}) \delta(t) + \sum_{i=1}^{n} P_i(n, \frac{\beta}{\alpha}) h_i \left[ \frac{t}{\alpha \left( 1 + \frac{\beta}{\alpha 2n} \right)} \right].
\]

(21)

The distribution function given by equation (19) can be considered as the binomial distribution probability if \(p = (1 - \beta/\alpha 2n)/(1 + \beta/\alpha 2n)\) fulfils the condition \(0 \leq p \leq 1\). Both \(\alpha\) and \(\beta\) are greater than zero, hence \(p\) is less than one. On the other hand, \(p\) is greater than zero if \(\alpha - \beta/\alpha 2n \geq 0\) or if (from equations (12) and (16)) \(\alpha/\beta' - 0.5 > 0\), that is if the parameter \(\alpha'\) of the MMM is less than zero.

If the MMM is activated by an infinite number of water drops of the unit total volume entering the model instantaneously, then the binomial distribution, equation (19), defines that part of unit total volume which is transformed by a cascade of \(i\)-linear reservoirs, equation (20), with a time constant \(\alpha(1 + \beta/\alpha 2n)\).

3.3. Distributed Muskingum model (DMM). In this section the limiting case of the MMM model is discussed. Note, that the model implies modelling of finite length reach by an infinite number of physically based Muskingum models.
The transfer function of the DMM can be calculated as the limiting case of equation (17) when $n$ tends to infinity (Strupczewski and Napiórkowski, 1986):

$$H_{o}(x, s) = \lim_{n \to \infty} H_{n}(x, s) = \exp \left( -\frac{\beta s}{1 + \alpha s} \right). \tag{22}$$

The impulse response of the DMM can be calculated as the limiting case of equations (19), (20), and (21).

Equation (19) can be rewritten as:

$$P_{i}(n, \beta, \alpha) = \frac{\left( \frac{\beta}{\alpha} \right)^{i} \left( 1 - \frac{\beta}{\alpha n} \right)^{n}}{i!} \frac{1}{\left( 1 + \frac{\beta}{\alpha 2n} \right)^{n}} \frac{1}{(n-i)!} \frac{n!}{n!} =$$

$$= \frac{\left( \frac{\beta}{\alpha} \right)^{i} \left( 1 - \frac{\beta}{\alpha n} \right)^{n}}{i!} \frac{1}{\left( 1 + \frac{\beta}{\alpha 2n} \right)^{n}} \frac{1}{(n-i)!} \frac{n!}{n!} \left( \frac{1}{n} \right) \left( \frac{1}{n} \right) \cdots \left( \frac{1}{n} \right). \tag{23}$$

For limiting case when $n$ tends to infinity the binomial distribution described by equation (19) converges to the Poisson distribution and is given by:

$$\lim_{n \to \infty} P_{i}(n, \beta, \alpha) = \frac{\left( \frac{\beta}{\alpha} \right)^{i}}{i!} \exp \left( -\frac{\beta}{\alpha} \right) = P_{i}(\beta, \alpha). \tag{24}$$

The rate of convergence of the binomial distribution to the Poisson distribution for a wide rectangular channel with the Chezy friction law ($m = 1.5, F = 0.2$ and $D = 1$, that is for

$$\frac{\beta}{\alpha} = \frac{2mD}{1-(m-1)^{2}F_{0}^{2}} = 3.03 \tag{25}$$

and for various numbers of subreaches $n$, is shown in Fig. 1. One can see, that the convergence is stronger when $\beta/\alpha$ is small (small Froude number and short channel reach).

When $n$ tends to infinity the impulse response given by equation (20) converges to the impulse response of $n$ equal linear reservoirs with a time constant equal to $\alpha$:

$$\lim_{n \to \infty} h_{i} \left[ \frac{t}{\alpha \left( 1 + \frac{\beta}{\alpha 2n} \right)} \right] = h_{i} \left( \frac{t}{\alpha} \right)^{i-1} \frac{1}{(i-1)!} \exp \left( -\frac{t}{\alpha} \right). \tag{26}$$

The relative difference in the time constant of the linear reservoirs in the MMM and the DMM is $\beta/\alpha 2n$, i.e. it changes linearly with $1/n$ and with $\beta/\alpha$. 
So, the explicit formulation of the DMM impulse consists of products of the Poisson distribution probability (equation (24)) and the impulse response of linear reservoirs (equation (26)) and is given by:

\[ h_\alpha(x, t) = P_0 \left( \frac{\beta}{\alpha} \right) \delta(t) + \sum_{i=1}^{\infty} P_i \left( \frac{\beta}{\alpha} \right) h_i \left( \frac{t}{\alpha} \right). \]  \hspace{1cm} (27)

Note, that equation (27) is the limiting case of equation (21). It can be interpreted in a similar way as the MMM impulse response. The Poisson distribution defines the part of the unit total volume transformed through \( i \)-th linear reservoirs (with a time constant \( \alpha \)), and \( \beta/\alpha \) is the average number of reservoirs in a cascade. Note, that the length of a river reach controls a way of water distribution between infinite number of paths, but not a time constant of reservoirs.

**Fig. 1. Distribution of unit volume between paths of linear reservoirs for the MMM and the DMM**

\( n \) — number of Muskingum reaches, \( i \) — path with \( i \)-th reservoirs only, \( P_i \) — part of unit volume entering \( i \)-th path

**4. PHYSICAL INTERPRETATION OF THE DMM**

The question arises, what physically based model is described by the DMM, which transfer function given by equation (22) can be considered as the solution of the ordinary differential equation in the image space.

The application of the Laplace transformation to partial differential equation can be summed in the following scheme (Doe et al., 1961). Instead of solving the differential equation with initial and boundary condition directly, we detour into the image space: using the \( \mathcal{L} \)-transformation the partial differential equation becomes an ordinary differential equation. When the ordinary differential equation is solved the solution of the original problem can be obtained by inverting the \( \mathcal{L} \)-transformation.

The problem to be solved is to find the partial differential equation knowing the solution in the image space given by equation (22). The image equation can be written as:

\[ Q(x, s) = \exp \left[ \lambda(s) x \right] Q_1(s), \]  \hspace{1cm} (28)

where \( Q_1(s) \) is the upstream boundary condition and \( \lambda(s) \) is the root of so called charac-
teristic equation. On the basis of equations (22) and (7) the characteristic equation is:

$$c_k(1+\alpha s) \dot{\lambda} + s = 0.$$  \hspace{1cm} (29)

Hence, the ordinary differential equation in image space takes form:

$$c_k(1+\alpha s) \frac{dQ}{ds} + sQ = 0.$$  \hspace{1cm} (30)

The original partial differential equation can be found by inverting equation (30) to the time domain:

$$\frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t} = -\alpha \frac{\partial^2 Q}{\partial x \partial t}.$$  \hspace{1cm} (31)

Equation (31) is typical of the equations representing the diffusion of kinematic waves (Lighthill and Whitham, 1955) with a diffusion coefficient $\alpha/c_k$. I.e., the multiple Muskingum model for limiting case of an infinite number of physically based Muskingum elementary models used for representation of a finite river reach is equivalent to the diffusion-like equation (31) described by partial differential equation, it can be called the Distributed Muskingum Model (DMM).

The DMM can be derived directly from the hyperbolic linear St. Venant equation. The direction of two real characteristics of St. Venant equation gives the celerity of both the primary and secondary waves. In the case of tranquil flow, i.e. Froude number less than 1, the celerity of the secondary wave is in an upstream direction. Accordingly, to derive the DMM from the linearized St. Venant equation, the hyperbolic equation (8) is modified in order to filter out upstream waves. In order to accomplish this it is necessary to reduce the hyperbolic equation to a parabolic-like form.

Instead of neglecting small convective terms entirely, they can be represented on the basis of the linear kinematic wave approximation (Dooge and Harley, 1967). For the kinematic wave approximation we can write the solution as:

$$Q(x,t) = f(x - c_k t).$$  \hspace{1cm} (32)

This lower order solution can be used to approximate the “hyperbolic” terms on the left hand side of equation (8)

$$\frac{\partial^2 Q}{\partial t^2} = c_k^2 f''(x - c_k t) = -c_k \frac{\partial^2 Q}{\partial x \partial t},$$  \hspace{1cm} (33)

$$\frac{\partial^2 Q}{\partial x^2} = f''(x - c_k t) = -\frac{1}{c_k} \frac{\partial^2 Q}{\partial x \partial t}.$$  \hspace{1cm} (34)

Substitution of these approximations in equation (8) gives equation (31). Note, that to solve equation (31) only upstream boundary conditions are required. The downstream boundary condition was filtered out from the complete linear equation.
5. COMPARISON OF THE DMM AND THE LINEARIZED ST. VENANT EQUATION

It is instructive to ask after the relation between the DMM and the linearized St. Venant equation and the range of applicability of the DMM. The answer to these questions can provide valuable insight into the underlying physics of the problem.

5.1. Amplitude and phase spectra. In previous sections the DMM was discussed in terms of the impulse response. As an alternative the DMM can be described in terms of the frequency response (e.g. Osiofski, 1972). In the present section the frequency approach is used and expressions derived for the amplitude spectrum and frequency spectrum of the DMM.

The transfer function (22) describes all transfer properties of the model for any input function and zero initial conditions. Sometimes it is convenient to employ only a part of the function $H_\infty(x, s)$ on the imaginary axis $s = i\omega$ i.e. to replace the Laplace transform by the Fourier transform. The function $H_\infty(x, i\omega)$ is called an amplitude-phase characteristic of the system or a frequency transfer function. The quantities

$$A_{DMM}(x, \omega) = |H_\infty(x, i\omega)| \quad (35)$$

and

$$\varphi_{DMM}(x, \omega) = \text{arg}[H_\infty(x, i\omega)] \quad (36)$$

are called the amplitude and phase characteristic, respectively. From equations (35) and (36) one can see that:

$$H_\infty(x, i\omega) = A_{DMM}(x, \omega) \exp[i\varphi_{DMM}(x, \omega)]. \quad (37)$$

So, the amplitude and phase characteristics determine changes in amplitude and phase caused by the model for cosinusoidal input function with frequency $\omega$.

The frequency transfer function can be obtained from the Laplace transform (22) by taking only the imaginary part of the complex variable:

$$H_\infty(x, i\omega) = \exp \left( -i\beta \frac{\omega}{1 + i\omega} \right). \quad (38)$$

Hence, the amplitude and phase characteristics are:

$$A_{DMM}(x, \omega) = \exp \left( -\alpha \beta \frac{\omega^2}{1 + \alpha^2 \omega^2} \right), \quad (39)$$

$$\varphi_{DMM}(x, \omega) = -\frac{\beta \omega}{1 + \alpha^2 \omega^2}. \quad (40)$$

The properties of the transfer function for the Linear Channel Response (LCR), i.e. the solution of the linearized St. Venant equation for a semi-infinite uniform channel, and for $F_0 < 1$ was analysed by Doooge et al. (1987b). The transfer function of the LCR is given by:

$$H_{LCR}(x, s) = \exp(ds + \sqrt{c - \sqrt{a^2s^2 + bs + c}}), \quad (41)$$

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where the coefficients are related to the parameters of the channel as follows:

\[ a = m^2 \frac{F_0^2}{(1 - F_0^2)^2} \beta^2, \]  
\[ b = \frac{[1 + (m-1)F_0^2][1 - (m-1)^2F_0^2] \beta^2}{(1 - F_0^2)^2} \alpha', \]  
\[ c = 0.25 \frac{[1 - (m-1)^2F_0^2]^2 \beta^2}{(1 - F_0^2)^2} \alpha^2, \]  
\[ d = m \frac{F_0^2}{1 - F_0^2} \beta. \]  

(42a)  
(42b)  
(42c)  
(42d)

The frequency characteristics of the LCR are (Dooge et al., 1987b):

\[ A_{LCR}(x, \omega) = \exp\left( c^{0.5} - \frac{[b^2 \omega^2 + (-a \omega^2 + c)^{0.5}]^{0.5} - a \omega^2 + c^{0.5}}{\sqrt{2}} \right), \]

\[ \phi_{LCR}(x, \omega) = d \omega - \frac{[b^2 \omega^2 + (-a \omega^2 + c)^{0.5} + a \omega^2 - c^{0.5}]}{\sqrt{2}}, \]

(43)  
(44)

where the parameters \( a, b, c, \) and \( d \) are defined by equations (42).

It is interesting to examine the form of the amplitude and phase spectra for the limiting values of the frequency \( \omega \). For very low frequencies i.e. very long waves, the amplitude given by equation (39) can be approximated by:

\[ A_{DMM}(x, \omega) \approx \exp(-0) = 1 \]

(45)

so there is no attenuation for very long waves. For the same condition of very low frequency the phase given by equation (40) can be approximated by:

\[ \phi_{DMM}(x, \omega) \approx -\beta \omega = -\frac{x}{c_k} \omega. \]

(46)

The upstream input function in the form of a harmonic oscillation \( f(t) = f_0 \cos(\omega t) \) results in a harmonic oscillation at the point \( x \)

\[ f(x, t) = f_0 \cos(\omega t - \frac{\omega x}{c_k}). \]

(47)

The phase velocity of the above wave corresponds to the kinematic wave speed.

The response of the LCR on a harmonic oscillation for very low frequencies is given by equation (47) as well.

At the other extreme of very high frequencies, i.e. very short waves, the amplitude approaches the value given by:

\[ A_{DMM}(x, \omega) = \exp\left( -\frac{\beta}{\alpha} \right). \]

(48)
The phase for very short waves is

$$\varphi_{\text{DMM}}(x, \omega) = -\frac{\beta}{\alpha \omega} \to 0.$$  (49)

For the case of a very short wave as an input the resulting harmonic oscillation at point \(x\) takes the form:

$$f_{\text{DMM}}(x, t) = f_0 \exp\left(-\frac{\beta}{\alpha}\right) \cos(\omega t)$$  (50)

which corresponds to the head of the wave travelling with the infinite celerity and the attenuation \(\exp(-\beta/\alpha)\).

The LCR response to the harmonic oscillation for very high frequencies takes different form (Dooge et al., 1987b):

$$f_{\text{LCR}}(x, t) = f_0 \exp(-\gamma) \cos\left(\omega t - \frac{\omega x}{c_1}\right)$$  (51)

which corresponds to the wave travelling with the phase velocity \(c_1 = v_0 + \sqrt{g y_0}\) and attenuation \(\exp(-\gamma)\) where:

$$\gamma = \frac{\left[1-(m-1)^2 F^2\right] \left[1-(m-1) F_0\right]}{2 m (1+F_0) F_0} \frac{\beta}{\alpha}.$$  (52)

Figs. 2 and 3 show the amplitude and phase spectra of the DMM and the LCR for the case discussed in previous sections, i.e., for wide rectangular channel of unit dimensionless length \((D=1)\) with the Chezy friction \((m=1.5)\) and two Froude numbers \((F=0.2 \text{ and } F=0.8)\). In both figures amplitudes and phases are drawn in function of dimensionless frequency \(\omega' = \omega y_0 / S_0 v_0\).
In the case of the attenuation (Fig. 2) the results will differ for various Froude numbers. It can be seen that the DMM gives a good approximation of the LCR for low Froude numbers only. It should be noted that the amplitudes for infinite frequency do not decay to zero thus indicating infinite power.

In the case of the phase shift the DMM breaks down for all Froude numbers when the dimensionless frequency $\omega'$ is great than 8.

Note, that for other lengths of the channel the logarithm of the amplitude reduction and the phase shift will be proportional to the dimensionless channel length.

5.2. Cumulants of the impulse responses. The next step in the analysis is to examine accuracy of the approximation of the LCR by the DMM and the single Muskingum models. From many possible criteria a simple one is used, namely that based on comparison of cumulants.

The use of moments to characterize a distribution so widely applied in statistics was introduced into hydrology by Nash (1959). Since then it has been widely used to study the properties of linear responses and to compare the various models proposed to represent the linear channel response $h(x, t)$.

In establishing theoretical relationships, it is more convenient to replace the moments by the cumulants which are related to them. While the moments are generated by the Laplace transform, the cumulants are generated by the logarithm of the Laplace transform of the impulse response function

$$k_r[h(x, t)] = (-1)^r \frac{d^r}{ds^r} \{\ln [H(x, s)]\}_{s=0}. \quad (53)$$

The first five cumulants of the LCR were obtained by Dooge et al. (1987b) and can be expressed in terms of $\alpha$ and $\beta$ as:

$$k_1^{LCR} = \beta, \quad (54a)$$

$$k_2^{LCR} = 2\alpha\beta, \quad (54b)$$

$$k_3^{LCR} = 12\alpha^2\beta \frac{1+(m-1)F_0^2}{1-(m-1)^2F_0^2}, \quad (54c)$$

$$k_4^{LCR} = 120\alpha^3\beta \frac{1-(0.2m^2-2m+2)F_0^2+(m-1)^2F_0^4}{[1-(m-1)^2F_0^2]^2}, \quad (54d)$$

$$k_5^{LCR} = 1680\alpha^4\beta \frac{[1+(m-1)F_0^2][1-(\frac{3}{2}m^2-2m+2)F_0^2+(m-1)^2F_0^4]}{[1-(m-1)^2F_0^2]^3}. \quad (54e)$$

For any given shape of channel and friction law the cumulants of the LCR are functions of the time of passage of a kinematic wave through the channel $\beta$, the dimensionless length of the channel $D$, the Froude number for the reference flow condition $F_0$ and the parameter $m$.

Cumulants of the DMM can be obtained from equation (22). Evaluating the $r$-th
derivative at \( s=0 \) we get the following expression for \( r \)-th cumulant

\[
k_r^{\text{DMM}} = (-1)^r \left\{ \frac{d^r [\ln H_n(x, s)]}{ds^r} \right\}_{s=0} = r! \alpha^{r-1} \beta. \tag{55}
\]

Similarly, the cumulants of the MMM are obtained. Evaluating the \( r \)-th derivative of equation (17) at \( s=0 \) one gets

\[
k_r^{\text{MMM}} = (-1)^r \left\{ \frac{d^r [\ln H_n(x, s)]}{ds^r} \right\}_{s=0} = 2n (r-1)! \alpha^r \sum_{k=1}^{(r+1)/2} \left( \frac{r}{2k-1} \right) \left( \frac{\beta}{\alpha 2n} \right)^{2k-1}, \tag{56}
\]

where \([\cdot]\) denotes the entier function.

Equations (55) and (56) can be used to analyse the convergence of the MMM to the DMM in terms of cumulants. Since the first two cumulants of the MM, DMM and the LCR are equal, all discussed models give the same response to a polynomial function of the second degree. Hence, the differences between third cumulants of the impulse responses can be used as a criterion for comparison. Note, that in general an impulse response of any model obtained by linear kinematic wave approximation of one or two left hand side terms of equation (8) preserves two first cumulants of the LCR. Furthermore, the first cumulant does not depend on the left hand side of equation (8) at all.

![Fig. 4. Comparison of LCR approximation accuracy by the DMM and the MM using third cumulants](image)

Consider now the case in which signal feeding is a polynomial function of the third degree

\[
y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3. \tag{57}
\]

The differences between the LCR and the responses of the MM and DMM do not depend on time (Strupczewski and Kundzewicz, 1980) and they are, respectively,

\[
E_{\text{MM}} = a_3 (k_3^{\text{LCR}} - k_3^{\text{MMM}}), \tag{58}
\]

\[
E_{\text{DMM}} = a_3 (k_3^{\text{LCR}} - k_3^{\text{DMM}}). \tag{59}
\]
Hence, to compare the accuracy of approximation by the MM and the DMM one can use the relative error in third cumulants

\[ r_{MM} = \left\| \frac{k_3^{LCR} - k_3^{MM}}{k_3^{LCR}} \right\|, \]  

(60)

\[ r_{DMM} = \left\| \frac{k_3^{LCR} - k_3^{DMM}}{k_3^{LCR}} \right\|. \]  

(61)

The relative errors defined by equations (60) and (61) are plotted in Fig. 4 in function of dimensionless length for \( m = 1.5 \) and for three Froude numbers \( (F=0, F=0.5 \) and \( F=1) \). One may notice, that the single Muskingum model is worse than the distributed Muskingum model for long channel reaches as it has been expected.

<table>
<thead>
<tr>
<th>( r/F_0 )</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
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<tr>
<td>3</td>
<td>0.5</td>
<td>0.583</td>
<td>0.75</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>0.848</td>
<td>0.938</td>
</tr>
<tr>
<td>5</td>
<td>0.928</td>
<td>0.949</td>
<td>0.984</td>
</tr>
</tbody>
</table>

Table 1

The relative approximation error of higher order cumulants

The comparison of equation (55) with equations (54) shows similarity of structures. Hence, the DMM relative error of any order of cumulant for given friction law depends on Froude number only. Its value increases with the order of cumulant as it is shown in Table 1.

6. CONCLUSIONS

The Distributed Muskingum Model is the first model which can be considered as both, conceptual and physical one. On one hand it is a conceptual model with physically derived parameters, on the other is simplification of the linearised St. Venant equations. It can be easily extended to cover lateral inflow due to its simple structure.

The accuracy of the LCR approximation by the DMM for a Froude number between zero and one is assessed by comparing the frequency characteristics and the cumulants of impulse responses. The DMM can be applied to longer channel reaches than the single Muskingum model. However, the quality of the LCR approximation by the DMM depends on the type of motion. It fits the LCR for small Froude numbers and slow rising waves.

The model performance as the LCR approximation can be highly improved by introducing a pure lag as its third parameter.
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WŁASNOŚCI ROZŁOŻONEGO MODELU MUSKINGUM

Streszczenie

Rozważany liniowy model propagacji fali wezbraniowej może być traktowany zarówno jako model hydrodynamiczny, jak i konceptualny. Wypracowany on został dla koryta pryzmatycznego o dowolnym przekroju poprzecznym i dowolnego prawa tarcia, jako przypadek graniczny wielokrotnego modelu Muskingum o parametrach mających interpretację fizyczną. Wykazano, że dyskutowany model jest równoważny modelowi liniowej dyfuzji kinematycznej. Model ten nazwano Rozłożonym Modelem
Muskingum ze względu na to, że wywodzi się z modelu Muskingum, lecz opisywany jest cząstkowymi równaniami różniczkowymi. Jego odpowiedź impulsowa zachowuje klarowną interpretację modelu konceptualnego, jest mianowicie sumą iloczynów prawdopodobieństw rozkładu Poissona i odpowiedzi impulsowej kaskady zbiorników liniowych.