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COPOSITIVITY CONDITION OF A VOLTERRA SERIES MODEL

1. Introduction

The derivation of the physically based equations of hydrologic systems and the determination of the parameters of these equations (usually non-linear partial differential equations) is a very difficult problem, quite often impossible to solve. Usually, the description of hydrologic systems is sought in terms of simplified models (implicit-differential or explicit-integral [1]) that describe input-output relationships fairly accuratelly.

One of the explicit methods of the description of the non-linear conservative systems is the description by means of an integral Volterra series

$$y(t) = \int_0^t h_1(r)x(t-r)dr + \int_0^t\int_0^t h_2(r_1,r_2)x(t-r_1)x(t-r_2)dr_1dr_2 + \dots \qquad (1)$$

where $y(t)$ is the output, $x(t)$ is the input and the functions $\{h_i\}$ $i=1,2,\dots$ are the kernels of the operator.

The essential problem to be solved is the determination of a range of applicability of the infinite (or finte) Volterra model (1) to the input-output modelling. Any infinite Volterra series has a range of convergence. Inside this range the error of approximation of the dynamics of the system grows with decresase of the number of terms. On the other hand in order to identify the dynamics of the system one has to truncate the infinite series. In that case one requires copositivity of the model - a positive output response to a positive input. This physically based criterion was proposed by Boneh and Golan [2].

The aim of the paper is to analyse the copositivity and convergence properties of the Volterra series. To the author' knowledge this problem has not been addressed in the professional literature. The problem expressed in general terms is very complex, nevertheless it must be considered.

The difference between the copositivity condition and convergence condition can be explained for the case of the two-term Volterra model based on a single non-linear reservoir. This simple example allows analytical results to be obtained. In this case, the solution of the resulting ordinary differential equation can be easily compared with the solution given by the Volterra series (1). Moreover, the analytic structure of the kernels $\{h_i\}$ has been determined [3]. The results obtained can be easily generalized for any number of reservoirs and any number of the terms of the Volterra series. However, in this case numerical calculations would be necessary.

2. Convergence condition vs. copositivity condition

Consider the reservoir model of the conservative system described by an ordinary differential equation of continuity and an algebraic equation of outflow (storage)

$$\begin{aligned} &\dot{S}(t) = -a\,S(t) - b\,S^2(t) + x(t) \\ &S(0) = 0 \\ &y(t) = a\,S(t) + b\,S^2(t) \qquad a,b > 0 \end{aligned} \tag{2}$$

where $x(t)$ is the input, $y(t)$ is the output, $S(t)$ is the storage and a, b are the parameters

and the input signal

$$x(t) = \begin{cases} X & \text{for} \quad t \leq T \\ 0 & \text{for} \quad t > T \end{cases} \tag{3}$$

It can easily be seen that Eq.(2) has a finite solution for any finite X, that can be obtained by calculating of several elementary integrals. This solution can be expressed as an infinite Volterra series provided that the convergence condition is fulfilled [3]

$$X \leq 0.25\ a^2/b \tag{4}$$

In order to identify the kernels of the Volterra series one has to truncate the infinite series. As an applicability criterion for such a model one can assume a copositivity requirement.

Let us derive the copositivity condition of the model given by Eq.(2) for the case of the two-term Volterra series and the input given by Eq.(3). The first and second order kernels are [3]

$$\begin{aligned} h_1(t) &= a\ e^{-at} \\ h_2(t_1,t_2) &= 2b\ e^{-a(t_1+t_2)} - b\ e^{-\max(t_1,t_2)} \end{aligned} \tag{5}$$

Let us denote by $\delta y(t)$ the first term and by $\delta^2 y(t)$ the second term of the Volterra series. Then the model response is

$$y(t) = \delta y(t) + \delta^2 y(t) \tag{6}$$

where

$$\delta y(t) = \int_0^t h_1(r)x(t-r)dr = \begin{cases} X\ (1-e^{-at}) & t \leq T \\ X\ (e^{aT}-1)\ e^{-at} & t > T \end{cases} \tag{7}$$

$$\delta^2 y(t) = \int_0^t\int_0^t h_2(r_1,r_2)x(t-r_1)x(t-r_2)dr_1 dr_2 =$$

$$= \begin{cases} \dfrac{2bX^2}{a^2}\left(e^{-2at} + at\ e^{-at} - e^{-at}\right) & \text{for} \quad t \leq T \\ \dfrac{2bX^2}{a^2}\left[e^{-2a(t-T)} - e^{-a(t-T)} + e^{-2at} + aTe^{-at} + e^{-at} - 2e^{-a(2t-T)}\right] & t > T \end{cases} \tag{8}$$

Substituting $v=e^{at}$, $z=e^{-a(t-T)}$, $r=e^{aT}$, $p=2bX/a^2$ in Eqs.(6,7,8) one gets

$$y(t)=\begin{cases} y_1(v)=\dfrac{X}{v^2}\,(v^2+p+pv\ \ln v-pv-v) & \text{for } t\le T \qquad (9a)\\[2ex] y_2(z)=X\left\{z^2p(1-r)^2+z\left[1-\dfrac{1}{r}+\dfrac{p}{r}(1+\ln r-r)\right]\right\} & \text{for } t>T \qquad (9b)\end{cases}$$

where $v\in(1,\ e^{aT})$ and $z\in(0,1)$

One can check that for any $v\geq 1$ the function on the right hand side of Eq.(9a) is greater than zero. Hence, the copositivity requirement is met when $y_2(z)\geq 0$ for $z\in(0,1)$. This is fulfilled if

$$\left.\frac{dy_2}{dz}\right|_{z=0}\geq 0 \qquad (10)$$

that is if

$$p\leq\frac{r-1}{r-\ln r-1} \qquad (11)$$

The above condition is illustrated in Fig.1

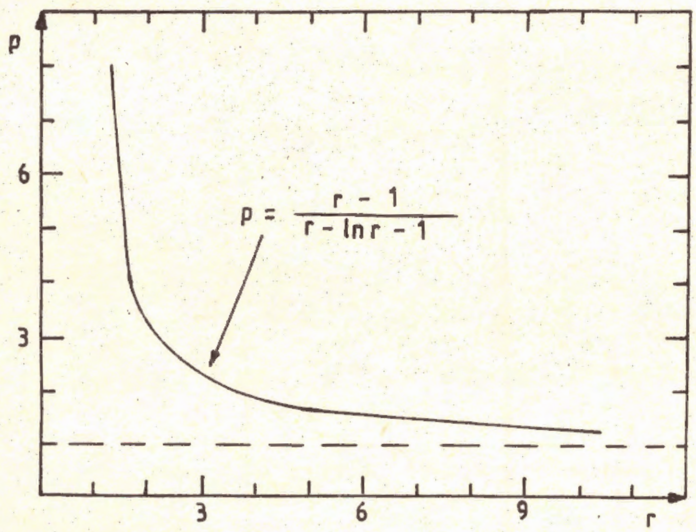

Figure 1. The copositivity condition for a single reservoir

Positive outflows correspond to points below the curve $p=p(r)$ defined by inequality (11). Note, that $p(r)$ is a monotonocally decreasing function and $\lim_{r\to\infty} p(r)=1$. Hence, for $p\leq 1$, that is for

$$X \leq 0.5\, a^2/b \qquad (12)$$

the outflow from the model is positive for any value of T (a sufficient condition). When $p > 1$ the outflow is positive for short time interval T only.

3. Conclusions

By comparing Eqs.(4,12) one can see that the convergence condition is twice stronger than the copositivity condition. Accordingly, the truncated Volterra model with the kernels identified on the basis of the input-output data is in fact applied outside the region of the convergence.

The copositivity condition for the series with more than two terms is very difficult to derive, It is the recommendation of the present authors that the input magnitude used for simulation should not be greater than the input magnitudes used for identification for the copositivity to be established.

4. References

[1] Schetzen M. - The Volterra and Wiener Theories of Nonlinear Systems, John Wiley and Sons, 1980.

[2] Boneh A. and Golan A. - Optimal identification of nonlinear surface runoff systems with copositivity threshold constraints, Advances in Water Res., 1(3), 121-129, 1978.

[3] Napiórkowski J.J. and Strupczewski W.G. - The analytical determination of the kernels of the Volterra series describing the cascade of nonlinear reservoirs, J. Hydr. Sc., 6(3-4), 121-142, 1979.

WARUNEK DODATNIOŚCI DLA MODELU SZEREGU VOLTERRY

Streszczenie

W pracy dyskutowane są warunki stosowalności opisu dynamiki systemów hydrologicznych za pomocą skończonego szeregu Volterry. Rozważane jest kryterium, którego spełnienie zapewnia dodatniość wypływu przy dodatnim dopływie. Wyniki analityczne podane są dla szeregu dwuelementowego.

УСЛОВИЕ ПОЛОЖИТЕЛЬНОСТИ ДЛЯ РЯДА ВОЛЬТЕРРА

Резюме

В статьи дискутированые условия применимости описания гидрологического система при помощи ряда Вольтерра. Рассматриванный признак, которого удовлетворение гарантирует положительность выхода при положительном входе. Аналитические результаты даннуе для ряда состаящего из двох элементов.