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"LUMPED NONLINEAR FLOOD ROUTING MODEL AND ITS SIMPLIFICATION TO THE MUSKINGUM MODEL"

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ABSTRACT

A lumped nonlinear state model (LNLSM) is derived from hydrodynamics. To obtain the model the diffusion wave model was lumped using the mean slope of water level throughout the length of uniform river reach. The linear approximation of the resulting model is equivalent to Muskingum model and gives theoretical evaluation of Muskingum model parameters. The results of linear approximation of the LNLSM around unsteady state was used to generate a discrete scheme of solution. The results of transformation of flow given by the above model were compared with the Discrete Muskingum Method.

1. INTRODUCTION

In recent years the relationships between various types of models used in hydrology have been studied carefully. In particular the linkage between physically based models and the systems approach seems to be a promising one for future development. The Muskingum flood routing method which had seemed to be purely empirical was shown to be linked with models based on convective diffusion equations. By comparison of both models relationships between their parameters were found. Cunge (1969) compared the difference schemes and Dooge (1973) the impulse responses using moment matching technique. Dooge's results are more general as he used the complete linear solution of Saint Venant equations, which for Froude number equals zero reduces to linear diffusion model. Koussis's method (1978) leads from the Muskingum equation to the linear convective diffusion equation. He transformed the lumped Muskingum model into a distributed model by expressing outflow as a function of inflow and its length derivatives and using the relation valid for kinematic wave only. There exists a more direct possibility of deriving the Muskingum equations from Saint -Venant equations. One approach initiated by Strupczewski and Kundzewicz (1980), applied by them to the case of a wide rectangular channel and developed further in the present paper, is the lumping of the nonlinear convective diffusion model under the assumption of linear changes of water level alson the river reach and then linearising it around the steady state. However Dooge, Strupczewski and Napiorkowski (1980) using the method of inverse order obtained results applicable to any shape of cross section and to any type of friction law.

The dependence of the Muskingum parameters on reference values was used by Koussis (1978) and Ponce and Yevjevich (1978) in the discrete Muskingum method with variation in space and time parameters. The results obtained were compared with the solution of a numerical analog of the convective diffusion equation. The present paper attempts to give the answer to the questions

(i) what physically based model is best approximated by the Muskingum method with variable parameters?

(ii) whether it is possible to increase the accuracy of this approximation?

2. DERIVATION OF LUMPED NONLINEAR STATE MODEL (LNLSM)

Unsteady flow in an open channel is described by means of the continuity equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{1}$$

and the dynamic equation

$$\frac{\partial y}{\partial x} + \frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{g}\frac{\partial u}{\partial t} = S_o - S_f \tag{2}$$

Where Q = discharge, A = cross-sectional area, x = distance along the channel, t = time, u = average velocity of the cross-section, y = depth, S_o = bottom slope S_f = friction slope. Both equations have prognostic form as they contain time derivatives.

The dynamic equation for low values of the Froude number can be approximated by the convective diffusion equation

$$e\,\frac{\partial y}{\partial x} = S_o(1 - b) \tag{3}$$

where $b = \frac{S_f}{So}$. The task of the coefficient e is to approximate the omitted acceleration terms. For the purposes of this paper the coefficient e will be assumed constant.

By integrating Eq. 1 along the river reach, its lumped form is obtained as

$$Q_1(t) - Q_2(t) = \frac{d}{dt}V(t) \tag{4}$$

where Q_1, Q_2 V are inflow, outflow and storage for the river reach

Similar integration of Eq. 3, under the assumption of constant values of e and b (ie constant friction slope S_f) along the river reach at any instant and uniform bed slope S_o gives Eq.3 in lumped form as

$$y_2(t) - y_1(t) = S_o L(1 - b(t)) / e \tag{5}$$

where L is a length of the river reach.

In general, the friction slope can be expressed as

$$S_f = f\ (Q,y,\ \text{roughness}) \tag{6}$$

which allows us to express y on the left hand side of Eq.5 in terms of Q(t) and b(t) when the roughnes parameter is assumed as fixed. Eq.5 for a prismatic channel can now be rewritten as

$$\Psi(Q_2,b) - \Psi(Q_1,b) = S_o L\ (1 - b)\ /e \tag{7}$$

Because there are two equation and four variables for evaluation of downstream discharge Q_2 (t)corresponding to a given upstream $Q_1(t)$, one further equation is required

From Eq. 3 one can see that as a consequence of the constant value of b at any instant throughout the reach there is also constant water level slope. This enables us to express the storage in the reach in terms of upstream and downstream depths as

$$V = \S(y_1,\ y_2) \tag{8}$$

or using Eq. 6

$$V = \S(Q_1,Q_2,\ Q\ ,\ b) \tag{9}$$

The linear aproximation of water level should be good enough for flood waves having lengths greater than the channel reach. It obviously cannot handle such input signals as a delta function or a unit step function.

Equations (4), (7), (9) from the physically based lumped nonlinear state model (LNLSM) for open channel flow. As a state variable the storage V is chosen, so the initial condition for the LNLSM is V(o) = Vo. The second variable that is b(t) is an auxiliary one, and it can be eliminated between two diagnostic equation (7) and (9). They contain no time derivatives and hence can only be used to relate dependent variables at the same instant of time.

Even though the only prognostic equation is the linear continuity equation (14), the model remains a nonlinear one since both diagnostic equations (7) and (9) are always nonlinear as the friction slope is a nonlinear function of Q and y.

3. LNLSM FOR RECTANGULAR CHANNEL

For a uniform rectangular wide open channel with the width B, the friction slope may be written (at least locally) form

$$S_f = a\ Q^2\ /\ y^{2m} \tag{6a}$$

For Chezy friction $a = C_2^{-2}\ B_2^{-2}$ and $m = 1.5$, while for Manning friction $a = n^2\ B^{-2}$ and $m = 5/3$. Furthermore for this case Eq. 8 becomes linear since

$$V = B\ L\ (y_1 + y_2)\ /2 \tag{8a}$$

The LNLSM for this simplified case can be expressed by means of following equations. The continuity equation is

$$\frac{d}{dt}\ V(t) = Q_1\ (t) - Q_2\ (t) \tag{4a}$$

dynamic equation is

$$(Q_2^{1/m} - Q_1^{1/m})\ b^{-1/2m} = \beta(1 - b) \tag{7a}$$

and the storage equation is :

$$V = \gamma\ (Q_1^{1/m} + Q_2^{1/m})\ b^{-1/2m} \tag{9a}$$

where $$\gamma = 0.5LBS_o^{-1/2m}\ a^{1/2m} \tag{10}$$

and $$\beta = S_o^{1 + 1/2m}\ a^{-1/2m}\ L/e \tag{11}$$

Even for the case of a linear rating curve, that is for $m = 1$, the model is still non-linear because of the parameter $b(t)$ which reflects the hysteresis loop in the storage relationship.

4. LINEARISATION OF THE LNLSM AROUND THE TRANSIENT STATE

The system of equations (4a), (7a), (9a) describing the LNLSM for a wide rectangular open channel will be now approximated by a linear model for increments around transient state (Napiórkowski 1978). A change of inflow from Q_{10} (t) to Q_1 (t) by an amount δQ_1 (t) will be accompanied by linear perturbations of the remaining variables around their original values, that is

$$Q_2\ (t) = Q_{20}(t) + \delta Q_2\ (t)$$

$$b(t) = b_o(t) + \delta b(t)$$

$$V(t) = V_o(t) + \delta V(t)$$

Expanding the system of equations (4a), (7a), (9a), into a Taylor series of a function of the four variables and retaining only the first-order increments, we obtain the linear system

$$\frac{d}{dt}\ \delta V(t) = \delta Q_1(t) - \delta Q_2(t) \tag{4b}$$

$$\frac{1}{m}\left(Q_{20}^{1/m-1}\,\delta Q_2 - Q_{10}^{1/m-1}\,\delta Q_1\right) b_o^{-1/2m} - \frac{1}{2m}\left(Q_{20}^{1/m} - Q_{10}^{1/m}\right)$$

$$\cdot\, b_o^{-1/2m-1}\,\delta b = -\beta\,\delta b \tag{7b}$$

$$\delta V = \frac{\gamma}{m}\, b_o^{-1/2m}\left(Q_{10}^{1/m-1}\,\delta Q_1 + Q_{20}^{1/m-1}\,\delta Q_2\right) - \frac{\gamma}{m}\left(Q_{10}^{1/m} + Q_{20}^{1/m}\right) b_o^{-1/2m-1}\,\delta b \tag{9b}$$

in which the coefficients are evaluated for the initial transient state conditions.

From Eqs.(7b) and using (7a) for the initial condition

$$\delta b = \frac{2\left(Q_{20}^{1/m-1}\,\delta Q_2 - Q_{10}^{1/m-1}\,\delta Q_1\right)}{\beta b_o^{1/2m}\left(b_o^{-1} - 1 - 2m\right)} \tag{12}$$

Substituting this expression for b(t) into the storage equation (9b) we get the storage increment in terms of inflow and outflow increments

$$\delta V = D_1\,\delta Q_1 + D_2\,\delta Q_2 \tag{13}$$

where

$$D_1 = A(1 + B)\, Q_{10}^{1/m-1} \tag{13a}$$

$$D_2 = A(1 - B)\, Q_{20}^{1/m-1} \tag{13b}$$

and

$$A = \gamma\, b_o^{-1/2m} \tag{13c}$$

$$B = \frac{Q_{10}^{1/m} + Q_{20}^{1/m}}{\beta\, b_o^{1/2m}\left(b_o^{-1} - 1 - 2m\right)} \tag{13d}$$

One can recognize (13) as storage equation of Muskingum Model valid for increments around the transient state. The Muskingum parameters

$$K = D_1 + D_2 \tag{14}$$

$$\alpha = D_1/K \tag{15}$$

are functions of hydraulic characteristics of channel reach (that is γ, β, m) and of both inflow and outflow.

5. LNLSM LINEARISATION AROUND THE STEADY STATE

The system of equations (4a), (7a), (9a), will now be approximated by a linear model around the steady state in which $Q_{10} = Q_{20} = Q_o$ and $b_o = 1$. In this case Eqs(13a)-(13d) take the form

$$A = \gamma /m \qquad (16c)$$

$$B = \frac{1}{\beta m} Q_o^{1/m} \qquad (16d)$$

$$D_1 = \gamma(1 - \frac{1}{\beta m} Q_o^{1/m}) Q_o^{1/m-1} /m \qquad (16a)$$

$$D_2 = \gamma(1 + \frac{1}{\beta m} Q_o^{1/m}) Q_o^{1/m-1} /m \qquad (16b)$$

$$K = 2 \gamma Q_o^{1/m-1} /m \qquad (14a)$$

$$\alpha = 0.5 (1 - \frac{1}{\beta m} Q_o^{1/m}) \qquad (15a)$$

that is the linear approximation of the LNLSM for perturbation from the steady state turns out to be the classical linear Muskingum Model with parameters given by equations (14a) and (15a). Substitution in Dq. 10 for γ and Eq. 11 for β gives

$$K = \frac{L}{m u_o} \qquad (14b)$$

$$\alpha = 0.5 (1 - \frac{e}{m} \frac{y_o}{S_o L}) \qquad (15b)$$

These results are conformable to the results obtained by Dooge (1973) using the moment matching technique between complete linear solution of St. Venant equation and Muskingum Model. However, if acceleration terms in Eq. 2 are neglected, that is if e is equal one, then the equation for the Muskingum Model parameters are exactly the same as were used in the Discrete Muskingum Method by obviously approximates the solution of LNLSM.

6. ON LINE APPROXIMATION OF LNLSM SOLUTION

(Fig.1) If the problem is posed as the discrete approximation of the LNLSM the question arises whether it is possible to improve significantly the accuracy of the Muskingum method with variable parameters. At almost every instant, except the initial one, there is unsteady state in the river reach during the flood. This is why use of Eqs 13 derived for transient state instead of Eqs. 16 should improve accuracy of LNLSM solution.

writing

$$Q_1(t) \sim 0.5 \; (\{Q_1\}^{n+1} + \{Q_1\}^{n})$$

$$Q_2(t) \sim 0.5 \; (\{Q_2\}^{n+1} + \{Q_2\}^{n})$$

$$dV/dt \sim (\{V\}^{n+1} - \{V\}^{n})/\Delta t$$

the continuity equation (4a) takes form

$$(\{V\}^{n+1} - \{V\}^{n})/\Delta t = 0.5\,(\{Q_1\}^{n+1} + \{Q_1\}^{n}) - (\{Q_2\}^{n+1} + \{Q_2\}^{n})$$

where from equation (9a)

$$\{V\}^{n+1} = (\{Q_1{}^{1/m}\}^{n+1} + \{Q_2{}^{1/m}\}^{n+1} \; \{b^{-1/2m}\}^{n+1} \tag{16a}$$

$$\{V\}^{n} = (\{Q_1{}^{1/m}\}^{n} + \{Q_2{}^{1/m}\}^{n} \; \{b^{-1/2m}\}^{n} \tag{16b}$$

and from equation (7a)

$$\{Q_2{}^{1/m}\}^{n+1} - \{Q_1{}^{1/m}\}^{n+1} = \beta\{b^{1/2m}\}^{n+1} \; (1 - \{b\}^{n+1}) \tag{17a}$$

$$\{Q_2{}^{1/m}\}^{n} - \{Q_1{}^{1/m}\}^{n} = \beta\{b^{1/2m}\}^{n} \; (1 - \{b\}^{n}) \tag{17b}$$

Linearisating the set of equations (16), (17) for increments

$$\delta Q_1 = \{Q_1\}^{n+1} - \{Q_1\}^{n}, \; \delta Q_2 = \{Q_2\}^{n+1} - \{Q_2\}^{n}, \delta b = \{b\}^{n+1} - \{b\}^{n}$$

as in section 4, one finds that (see Eq. 13)

$$\{V\}^{n+1} - \{V\}^{n} = D_1\,(\{Q_1\}^{n+1} - \{Q_1\}^{n}) + D_2(\{Q_2\}^{n+1} - \{Q_2\}^{n})$$

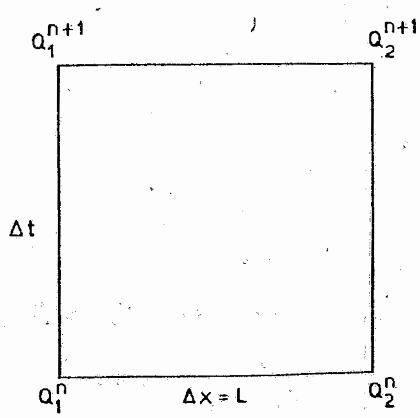

Figure 1. Space-time discretisation of LNLSM.

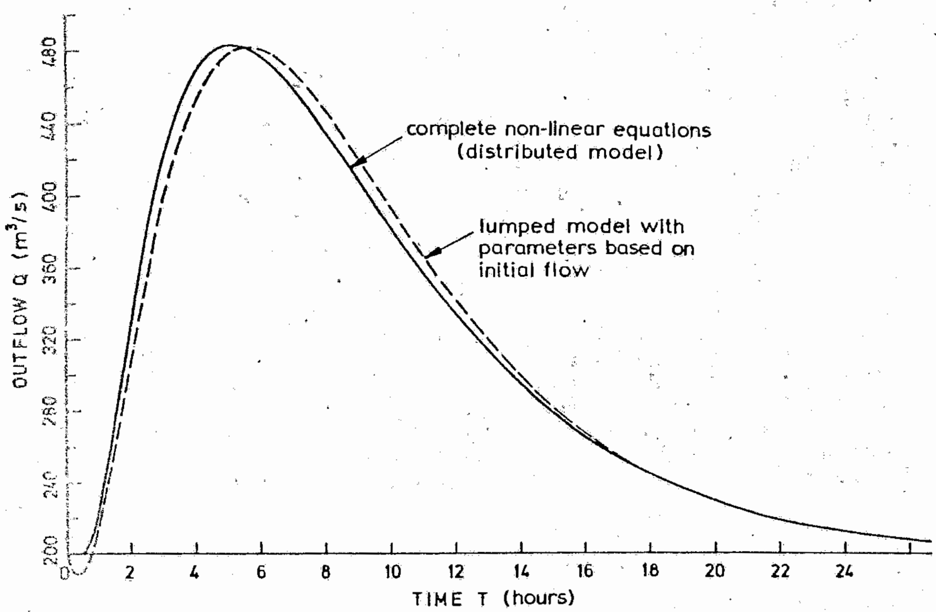

Figure 2. Predicted Outflow for constant parameters.

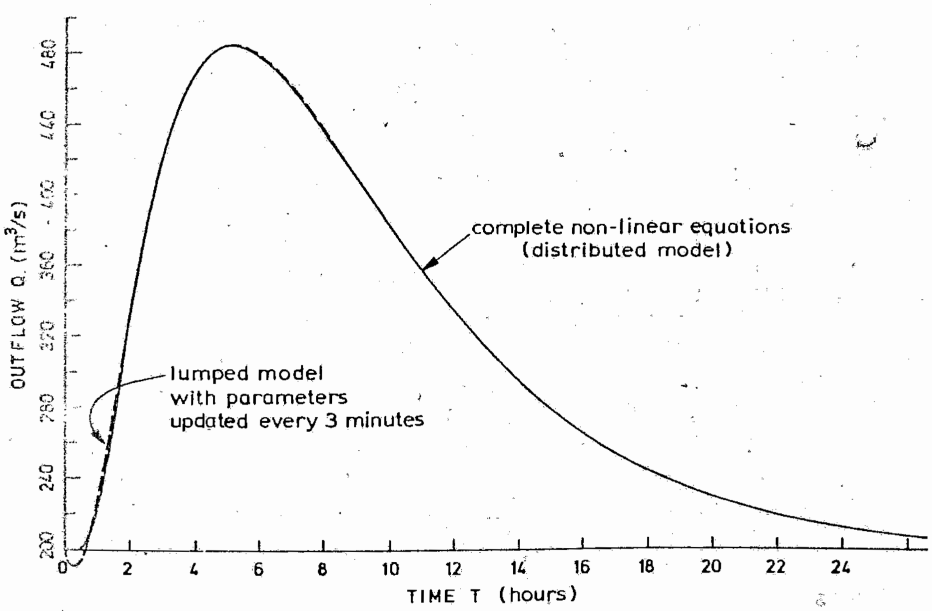

Figure 3. Predicted outflow for updated parameters.

and

$$\{Q_2\}^{n+1} = \frac{-D_1\ \Delta t2}{D_2 + \Delta t/2}\{Q_1\}^{n+1} + \frac{D_1 + \Delta t/2}{D_2 + \Delta t/2}\{Q_1\}^{n} + \frac{D_2 - \Delta t/2}{D_2 + \Delta t/2}\{Q_2\}^{n} \tag{18}$$

where D_1 and D_2 are given by Eqs. 13 (a) and 13 (b)

Above discrete scheme should be applied as follows:

First step

For initial conditions Q_{10} and Q_{20}, variable b is determined from Eq. 7a and then from equations 13 (a) to 13 (d) the parameters D_1 and D_2 for use in Eq. (18). If there is a steady state in a river reach ($Q_{10} = Q_{20}$) the initial value b_0 will be equal to

Step "n"

A new value of b is determined from formula

$$\{b\}^{n+1} = \{b\}^{n} + \delta\Delta b$$

in which $\{b\}^n$ is known from the previous time interval and b is calculated from Eq. 12. The new values of the parameters D_1, D_2 for use in Eq. 18 are evaluated from Eqs. 13(a) and 13 (b).

At the end of this chapter authors would like to emphasise that methods presented in this paper reduces to Discrete Muskingum Method with variable parameters when linearisation around the steady state instead of lineariation around the transient state is used in the derivation of Eq. 18. The former model Eqs. 16a 14 and 15 and in that case Eq. 18 takes form

$$\{Q_2\}^{n+1} = \frac{-K\delta + \Delta t/2}{K(1-\delta) + \Delta t/2}\{Q_1\}^{n+1} + \frac{K\delta + \Delta t/2}{K(1-\delta) + \Delta t/2}\{Q_1\}^{n} + \frac{K(1-\delta) - \Delta t/2}{K(1-\delta) + \Delta t/2}\{Q_2\}^{n} \tag{18a}$$

where K, α are the Muskingum Model parameters. Eq. (18a) is exactly the same as were used in the methods of Koussis (1978) and Ponce and Yevjevich (1978).

7. RESULTS OF NUMERICAL EXPERIMENTS

The effect of updating the values of the routing parameters is illustrated for a particular case by a comparison of figure 2 (no updating) and figure 3 (updating at each step). In the numerical computations, the channel is taken as rectangular with a width of 100 m, with a bottom slope of $S_0 = 0.000248$ and a Manning roughness of n = 0.025. The initial condition was taken as a steady flow in the reach of 200 m^3/s and the upstream inflow as

$$Q_1\ (t) = 200 + t.\ \exp\ (-t/C_2)\ /C_1$$

with $C_1 = 16.377\ s^2/m^3$ and C_2 = 13,355 seconds. The downstream outflow was calculated for a reach length of 10 km both for (a) routing parameters based on the initial steady flow of 200 m/s as described in section 5 above and (b) routing parameters updated at each step as described in section 6, thus taking account of the increase in flow from 200 m^3/s.

Case (a) is illustrated in figure 2 in which the full line represents an accurate numerical solution of the complete non - linear St. Venant equations and the dotted line the lumped solution with the linearisation and consequently the routing parameters based on the initial flow. Case (b) is illustrated in figure 3 in which the dotted line represents the lumped solution based on a linearisation which is updated at time step ie. every 3 minutes. In both cases, the lumped solution predicts negative outflows during the early part of the event. This arises because the length of the channel reach is 10 km which is greater than the initial characteristic reach length of 4.8 km (corresponding to a flow of 200 m^3/s). For longer lengths of channel, this negative contribution would become even more marked. For case (a) shown in figure 2, the peak value is well predicted but the predicted time to peak is later than for the complete distributed model. For case (b) shown in figure 3 the predicted output is indistinguishable from that given by the complete distributed model except at the beginning of outflow.

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