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Chapter 1

Linear theory of open channel flow

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Abstract

The linearized solution of the St. Venant equations is discussed for the general case of any shape of channel and any friction law. The upstream and downstream transfer functions are derived analytically for a channel reach of finite length. For the limiting case of a semi-infinite channel the properties of the upstream transfer function are studied using the cumulants, and amplitude and phase spectra. A number of simplified forms of the St. Venant equations are presented.

1. Introduction

In the unsteady motion of water in canals and rivers, the strategy is followed of proposing simplified models and comparing their performance with field data and with the result of simulations by means of the St. Venant equations. The main aim is to obtain convenient predictive models. Although non-linear systems modelling is already extensively developed, it does not seem useful for stochastic hydrology or for optimal control problems when flow routing in channels must be repeated many times for different scenarios. When the starting point in such a strategy is the linearization of a set of non-linear equations, and moreover the aim is to gain insight into the nature of the complete non-linear problem, the problem then belongs to a very special class. That class has been termed by Polish hydrologists Doogeology in recognition of Prof. Dooge's contribution to the linear theory of open channel flow.

The most important problems in the linear theory of open channel flow are:

(1) The two-point boundary problem in which both upstream and downstream boundary conditions are taken into account.

(2) The downstream problem, i.e. the prediction of the flood characteristics at a downstream section on the basis of a knowledge of the flow characteristics at an upstream section and the hydraulic characteristics of the channel between the two sections.

(3) The steady-state rating curve at a downstream control.

(4) The tributary problem which involves predicting the effect of tributary inflow on conditions in the main channel both upstream and downstream of the point of entry.

(5) The lateral inflow problem in which there is a distributed inflow to the channel reach. The above classification and description applies only to a tranquil (subcritical) flow in which the Froude number is less than one. For rapid (supercritical) flow there is no upstream effect. The present paper concentrates on the first two problems of Doogeology only.

2. Linearization of the St. Venant equations

When only one space dimension is taken into account, the equation of continuity for the unsteady flow in an open channel in the absence of lateral inflow is given by:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{1}$$

where $Q(x,t)$ is the discharge, $A(x,t)$ is the cross-sectional area, x is the distance from the upstream boundary, and t is the elapsed time.

If the assumption is made that only acceleration in the direction of motion needs to be taken into account then the equation for the conservation of linear momentum in this direction can be written in terms of the same variables (Dooge et al., 1982):

$$(1 - F^2) g \bar{y} \frac{\partial A}{\partial x} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} = gA(S_o - S_f) \tag{2}$$

where F is the Froude number, $\bar{y}$ is the hydraulic mean depth (the ratio of the area of flow to the width of the channel at the water surface), S_o is the bottom slope and S_f is the friction slope defined by the equilibrium condition for steady uniform flow.

The friction slope depends on the type of friction law assumed, the shape and roughness of the cross-section, the flow of the section and the area of flow. It can be written in very general form as:

$$S_f = f(A, Q, \text{shape, roughness}) \tag{3}$$

The problem of unsteady open channel flow involves the solution of the above set of non-linear equations subject to given initial conditions and two appropriate boundary conditions. No analytical solution is available and equations (1) and (2) must be solved by some method of numerical approximation. To determine analytically the solution, the St. Venant equations are simplified by considering the first order (linear) variation from a steady-state trajectory.

To compute the linearized equations we make use of expansion of non-linear terms in equation (2) in a Taylor series around the uniform steady state (Q_o, A_o) and limitation of this expansion to the first order increments $Q'(x,t), A'(x,t)$. The resulting equations are (Dooge and Napiórkowski, 1984):

$$\frac{\partial Q'}{\partial x} + \frac{\partial A'}{\partial t} = 0 \tag{4}$$

$$(1 - F^2)g\bar{y}_o \frac{\partial A'}{\partial x} + 2v_o \frac{\partial Q'}{\partial x} + \frac{\partial Q'}{\partial t} = gA_o \left[-\frac{\partial S_f}{\partial Q} Q' - \frac{\partial S_f}{\partial A} A' \right] \tag{5}$$

where $v_o = Q_o/A_o$ is the average velocity of flow, and the derivatives of the friction slope $S_f(Q,A)$ with respect to discharge Q and the area A on the right-hand side of the equation are evaluated at the reference conditions.

The variation of the friction slope with discharge at the reference condition for all frictional formulae for rough turbulent flow could be taken as:

$$\frac{\partial S_f}{\partial Q} = 2\frac{S_o}{Q_o} \tag{6}$$

To express the variation of the friction slope with flow area we define for convenience a parameter m as the ratio of the kinematic wave speed to the average velocity of flow:

$$m = \frac{c_k}{v_o} \tag{7}$$

where c_k is the kinematic wave speed (Dooge and Napiórkowski, 1984):

$$c_k = -\left[\frac{\partial S_f}{\partial A}\right] \Big/ \left[\frac{\partial S_f}{\partial Q}\right] = \frac{dQ}{dA} \tag{8}$$

When equations (6) and (7) are substituted in equation (8) we obtain:

$$\frac{\partial S_f}{\partial A} = 2mv_o\frac{S_o}{Q_o} \tag{9}$$

The parameter m is a function of the shape of channel and of area of flow. For a wide rectangular channel with Chezy friction, m is always equal to 3/2 and with Manning friction is always equal to 5/3. For shapes of channel other than rectangles, m will take on different values.

Since equations (4) and (5) are linear first-order equations in two variables, they are equivalent to a single second-order equation in one variable. The most general form of this second-order equation is that obtained by using the unsteady flow potential (Dooge, 1980). This potential can be defined as the function $U'(x,t)$ whose partial derivatives with respect to distance gives minus the perturbation in the area of flow, that is:

$$\frac{\partial U'}{\partial x} = -A'(x,t) \tag{10a}$$

and whose partial derivative with respect to time gives the perturbation from the reference discharge:

$$\frac{\partial U'}{\partial t} = Q'(x,t) \tag{10b}$$

Consequently, the perturbation potential $U'(x,t)$ automatically satisfies the continuity equation (4). When equations (10) are substituted in equation (5) we obtain the dynamic equation for the unsteady flow potential $U'(x,t)$ in the form:

$$(1-F_o^2)g\bar{y}_o\frac{\partial^2 U'}{\partial x^2}+2v_o\frac{\partial^2 U'}{\partial x\,\partial t}+\frac{\partial^2 U}{\partial t^2}=gA_o\left[-\frac{\partial S_f}{\partial A}\frac{\partial U'}{\partial x}+\frac{\partial S}{\partial Q}\frac{\partial U'}{\partial t}\right] \tag{11}$$

Any linear function of the perturbation potential $U'(x,t)$ will also represent a solution of equation (11). Since differentiation is a linear operation, both $A'(x,t)$ and $Q'(x,t)$ will also be governed by an equation of the same form as equation (11). Similarly, any linear combination of solutions is also a solution of the basic linear equation, e.g. the perturbations of the velocity $v'(x,t)$, the surface width of the channel $T'(x,t)$ and the Froude number $F'(x,t)$. The choice of dependent variable in any given problem will be governed largely by the form in which the boundary conditions are given.

3. Solution for finite channel reach

The basic equation (11) to be solved is hyperbolic in form. Accordingly there are two real characteristics defined by:

$$\frac{dx}{dt}=c_{1,2}=v_o\pm\sqrt{(g\bar{y})} \tag{12}$$

along which the discontinuities in the derivatives of the solution will propagate. We will consider the basic case of tranquil flow (Froude number less than one) in which $A'(x,t)$ will be prescribed both at the upstream boundary $x = 0$ and at the downstream boundary $x = L$. The use of other boundary conditions does not introduce any new principle.

The problem is to solve equation (11) subject to the double initial condition:

$$A'(x,0)=0 \qquad \frac{\partial A'}{\partial t}=0 \tag{13}$$

and subject to the boundary conditions:

$$A'(0,t)=A_u(t) \qquad A'(L,t)=A_d(t) \tag{14}$$

The solution can be sought in terms of the Laplace transform. Equation (11) when transformed to the Laplace transform domain becomes:

$$(1-F_o^2)g\bar{y}\frac{d^2\overline{A}}{dx^2}+\left[-2v_o s+gA_o\frac{\partial S_f}{\partial A}\right]\frac{d\overline{A}}{dx}-\left[s^2+gA_o\frac{\partial S}{\partial Q}s\right]\overline{A}=0 \tag{15}$$

where $\overline{A}(x,s)$ is the Laplace transform of $A'(x,t)$. Equation (15) is a second-order homogeneous ordinary equation, so the solution can be written in the general form:

$$\overline{A}(x,s) = C_1(s)\exp[\lambda_1(s)x] + C_2(s)\exp[\lambda_2(s)x] \tag{16}$$

where λ_1 and λ_2 are the roots of the characteristic equation for equation (16) and are given by:

$$\lambda_{1,2} = es + f \pm \sqrt{(as^2 + bs + c)} \tag{17}$$

Here the parameters a, b, c, e and f are given in terms of hydraulic variables by the following relationships:

$$a = \frac{1}{g\bar{y}_o(1 - F_o^2)^2} \tag{18a}$$

$$b = \frac{2S_o}{v_o\bar{y}_o}\frac{1 + (m-1)F_o^2}{(1 - F_o^2)^2} \tag{18b}$$

$$c = \left(\frac{mS_o}{\bar{y}_o}\right)^2 \frac{1}{(1 - F_o^2)^2} \tag{18c}$$

$$e = \frac{v_o}{g\bar{y}_o}\frac{1}{1 - F_o^2} \tag{18e}$$

$$f = \sqrt{(c)} \tag{18f}$$

Having determined the parameters $C_1(s)$ and $C_2(s)$ from the boundary conditions, one can write $\overline{A}(x,s)$ in terms of the $h_u(x,s)$ and $h_d(x,s)$ which may be defined as the Laplace transforms of the responses of the channel reach to delta-function inputs at the upstream and downstream ends respectively. Accordingly we can write:

$$\overline{A}(x,s) = h_u(x,s)\overline{A}_u(s) + h_d(x,s)\overline{A}_d(s) \tag{19}$$

The linear channel responses to an upstream input $h_u(x,s)$ and to a downstream input are given by (Dooge and Napiórkowski, 1987a):

$$h_u(x,s) = \exp[(es + f)x]\frac{\sinh[\sqrt{(as^2 + bs + c)}(L - x)]}{\sinh[\sqrt{(as^2 + bs + c)}L]} \tag{20}$$

$$h_d(x,s) = \exp[-(es + f)(L - x)]\frac{\sinh[\sqrt{(as^2 + bs + c)}x]}{\sinh[\sqrt{(as^2 + bs + c)}L]} \tag{21}$$

The original function in the time domain is determined from the corresponding boundary condition through the relationship:

$$A'(x,t) = h_u(x,t) * A_u(t) + h_d(x,t) * A_d(t) \tag{22}$$

The explicit formulation for the transfer functions $h_u(x,t)$ and $h_d(x,t)$ in the time domain have been obtained by Dooge and Napiórkowski (1987a). The transfer function due to an upstream input is found to have two distinct parts so that we can write:

$$h_u(x,t) = h_u^1(x,t) + h_u^2(x,t) \tag{23}$$

The first part of the solution, which may be termed the head of the wave, is given by:

$$h_u^1(x,t) = \sum_{n=0}^{\infty} \exp(-2nL\alpha_1 - \alpha_2 x)\,\delta(t - nt_o - x/c_1)$$

$$-\sum_{n=1}^{\infty} \exp(-2nL\alpha_1 + \alpha_3 x)\,\delta(t - nt_o - x/c_2) \tag{24}$$

The celerities c_1 and c_2 are defined in equation (12) and the parameters α_1, α_2, α_3 and t_o (functions of the channel parameters) are given by:

$$\alpha_1 = b/2\sqrt{(a)} \tag{25a}$$

$$\alpha_2 = \alpha_1 - f \tag{25b}$$

$$\alpha_3 = \alpha_1 + f \tag{25c}$$

$$t_o = L/c_1 - L/c_2 \tag{25d}$$

It can be seen that the head of the wave moves downstream at the dynamic speed c_1 in the form of a delta function of exponentially declining volume proportional to $\exp(-\alpha_2 x)$. At $x = L$ the delta function is reflected with an inversion of sign and is propagated upstream at the speed c_2 and with a heavier damping factor $\exp[-\alpha_3(L - x)]$. Then it is reflected again at $x = 0$ to move in a downstream direction etc.

The second part of the upstream response, which may be termed the body of the wave, is:

$$h_u^2(x,t) = \sum_{n=0}^{\infty} \exp(-\beta_1 t + \beta_2 x)(h/c_1 - h/c_2)(2nL + x) \cdot$$

$$\frac{I_1\{2h\sqrt{[(t - nt_o - x/c_1)(t + nt_o - x/c_2)]}\}}{\sqrt{[(t - nt_o - x/c_1)(t + nt_o - x/c_2)]}}\, U[t - nt_o - x/c_1]$$

$$-\sum_{n=1}^{\infty} \exp(-\beta_1 t + \beta_2 x)(h/c_1 - h/c_2)(2nL - x) \cdot$$

$$\frac{I_1\{2h\sqrt{[(t - nt_o - x/c_2)(t + nt_o - x/c_1)]}\}}{\sqrt{[(t - nt_o - x/c_2)(t + nt_o - x/c_1)]}}\, U[t - nt_o - x/c_2] \tag{26}$$

where $I_1\{\,\}$ is a modified Bessel function of the first kind, $U[\,]$ is a unit step function, and the remaining parameters are given by:

$$\beta_1 = b/2a \tag{27a}$$

$$\beta_2 = f - be/2a \tag{27b}$$

$$h = \sqrt{(b^2/4 - ac)}/2a \tag{27c}$$

As in the case of the head of the wave, the body of the wave is subject to successive reflection at both the downstream and upstream boundaries but moves and dissipates more slowly than the head of the wave.

For the downstream transfer function, the head of the wave is given by:

$$h_{\mathrm{d}}^1(x,t) = \sum_{n=0}^{\infty} \exp[-2nL\alpha_1 - \alpha_3(L-x)]\,\delta[t - nt_{\mathrm{o}} + (L-x)/c_2]$$

$$- \sum_{n=0}^{\infty} \exp[-(2n+1)L\alpha_1 - fL - \alpha_2 x]\,\delta(t - nt_{\mathrm{o}} + L/c_2 - x/c_1) \tag{28}$$

and is subject to reflection at the two ends of the reach as in the case of $h_{\mathrm{u}}^1(x,t)$. The body of the wave is given by:

$$h_{\mathrm{d}}^2(x,t) = \sum_{n=0}^{\infty} \exp[-\beta_1 t - \beta_2(L-x)](h/c_1 - h/c_2)\,[2nL + (L-x)]\cdot$$

$$\frac{I_1\{2h\sqrt{[(t - nt_{\mathrm{o}} + L/c_2 - x/c_2)(t + nt_{\mathrm{o}} + L/c_1 - x/c_1)]}\}}{\sqrt{[(t - nt_{\mathrm{o}} + L/c_2 - x/c_2)(t + nt_{\mathrm{o}} + L/c_1 - x/c_1)]}}\,U[t - nt_{\mathrm{o}} - x/c_1]$$

$$- \sum_{n=0}^{\infty} \exp[(-\beta_1 t - \beta_2(L-x)](h/c_1 - h/c_2)\,[2(n+1)L + x]\cdot$$

$$\frac{I_1\{2h\sqrt{[(t + nt_{\mathrm{o}} + L/c_1 - x/c_2)(t - nt_{\mathrm{o}} + L/c_2 - x/c_1)]}\}}{\sqrt{[(t + nt_{\mathrm{o}} + L/c_1 - x/c_2)(t - nt_{\mathrm{o}} + L/c_2 - x/c_1)]}}\,U[t - nt_o - x/c_1] \tag{29}$$

Note that:

(1) For the case where there is both an upstream and a downstream boundary condition the unsteady wave motion produced by each of the boundary conditions will be successively reflected at each end of the channel reach and thus will require representation by an infinite series.

(2) Allowance for a downstream boundary condition thus has a double effect, since it produces the reflection of the movement due to the upstream input as well as a direct effect on the channel reach of the downstream boundary condition.

(3) The reflection of the two sets of wave motion at opposite ends of the channel from the point of generation, results in representing each of the linear channel responses by two infinite series, one representing each direction of propagation.

Considering that the modified Bessel function is itself represented by an infinite series, the solution is in the form of a double infinite series which seems too complicated for practical application in river flow forecasting. However, due to heavy damping, only the first few terms of the transfer functions would be required, and the polynomial approximation of the first order modified Bessel function is sufficiently accurate and can be calculated easily (Napiórkowski and Dooge, 1988). It was found moreover, that:

(1) The total volume of the contribution to the input response due to a delta function input at the upstream boundary and a delta function input at the downstream boundary sum to unity for any point in the channel reach.

(2) The rate of convergence in the two infinite series characterizing the response to an upstream input and the two infinite series characterizing the response to a downstream input is the same in all four cases and is equal to $\exp(-2fL)$.

4. The generalized linear channel response

In the present paragraph we will concentrate on the downstream wave which is of primary importance in flood routing. To filter out the downstream wave we can set $L \rightarrow \infty$ in equation (20) and deal with the limiting case of downstream flow for a semi-infinite reach (i.e. the case where the downstream control is so distant from the section of interest that the downstream boundary has no influence). In such a case the upstream transfer function in Laplace transform domain and in the time domain is given respectively by:

$$h_u(x,s) = \exp(exs + fx - x\sqrt{(as^2 + bs + c)} \tag{30}$$

$$h_u(x,t) = \exp(-\alpha_2 x)\,\delta(t - x/c_1) +$$

$$\exp(-\beta_1 t + \beta_2 x)(h/c_1 - h/c_2)\frac{I_1\{2h\sqrt{[(t - x/c_1)(t - x/c_2)]}\}}{\sqrt{[(t - x/c_1)(t - x/c_2)]}}\,U[t - x/c_1] \tag{31}$$

which corresponds to use of only the first term in equations (24) and (26). The above expressions are generalized forms (Dooge, 1980; Dooge et al., 1987a) of those obtained by Dooge and Harley (1967a, b) for the special case of a wide rectangular channel with Chezy friction.

4.1. Cumulants of generalized channel response

The use of cumulants (or moments) has been widely used both in unit hydrograph analysis and in flood routing to study the properties of linear responses and to compare the various models proposed for use in representing the linear channel response or the unit hydrograph (Dooge, 1973). If the corresponding cumulants (or moments) of the impulse responses are equal to each other, up to and including N, then both systems respond identically to input signals which are at most Nth order polynomial functions of time.

In establishing a theoretical relationship, it is more convenient to use the cumulants, rather then the moments which are related to them. While the moments are generated by the Laplace transform, the cumulants are generated by the logarithm of the Laplace transform. Accordingly they are given by:

$$k_R[h(x,t)] = (-1)^R \frac{\mathrm{d}^R}{\mathrm{d}s^R}\{\ln[h(x,s)\}_{s=0} \tag{32}$$

The first cumulant is identical with the first moment about the origin; the second cumulant is identical to the second moment about the centre; the third cumulant is identical to the third moment about the centre; but in the case of the higher moments and cumulants this identity does not exist.

It is clear from the above that in case of the linear channel response it is possible to derive the values of the moments or of the cumulants from the solution in the Laplace transform domain given by equation (30) even if the explicit solution in the time domain given by equation (31) is very complicated. Substituting from equation (30) in equation (32) we obtain for the cumulants:

$$k_R[h(x,t)] = (-1)^R \frac{\mathrm{d}^R}{\mathrm{d}s^R}\left\{[-\surd(as^2 + bs + c) + es + f]x\right\}_{s=0} \tag{33}$$

where the parameters a, b, c, e and f have the values given by equation (18).

The first two cumulants are relatively easy to obtain from implicit equation (33):

$$k_1 = \frac{x}{mv_o} \tag{34a}$$

$$k_2 = \frac{1}{m}[1 - (m-1)^2 F_o^2]\left[\frac{x}{mv_o}\right]\left[\frac{\bar{y}_o}{S_o x}\right] \tag{34b}$$

To calculate the cumulants of higher order ($R > 2$) one can use a general expression recently derived by Romanowicz et al. (1988):

$$k_R[h(x,t)] =$$

$$\left[\frac{x}{mv_o}\right]^R \left[\frac{\bar{y}_o}{S_o x}\right]^{R-1} m^{-R}\frac{[1 + (m-1)F_o^2]^R}{1 - F_o^2}\sum_{i=0}^{[R/2]}\gamma(R,i)\left\{\frac{m^2F_o^2}{[1 + (m-1)F_o^2]^2}\right\}^i \tag{35}$$

where the coefficients $\gamma(R,i)$ can be expressed as:

$$\gamma(R,i) = (-1)^i \frac{r!(2R - 2i - 3)!}{(R - i - 2)!(R - 2i)!\,i!}(1/2)^{R-2} \tag{36}$$

The general expression for Rth cumulant as given by equation (35) was found to depend on the time taken for a kinematic wave to traverse the length of the channel $[x/(mv_o)]$, the dimensionless channel length $(S_o x/\bar{y}_o)$, the ratio of the kinematic wave speed to the average velocity of flow (m), and the Froude number of flow at the reference conditions (F_o).

4.2. Amplitude and phase characteristics

In many other branches of geophysics, the properties of systems are discussed in relation to their response to a harmonic rather than impulsive input. This can be done either by means of wave number analysis (Kundzewicz and Dooge, 1989) or by means of frequency analysis. In this paper the second approach is presented.

The amplitude and phase spectra are used as the basis of comparison of various models and of the comparison of models with data. These spectra are also used to gain insight into the characteristics of the model or of the system being simulated.

The amplitude and phase spectra can be derived from the Fourier transform of the system response. This transform is readily obtained by taking only the imaginary part of the argument of the system function:

$$h(x \cdot i\omega) = h(x,s)|_{s=i\omega} = A(x,\omega)\exp[i\varphi(x,\omega)] \tag{37}$$

The quantities $A(x,\omega)$ and $\varphi(x,\omega)$ are called the amplitude characteristic and the phase characteristic, respectively. The frequency characteristics of the generalized downstream channel response are (Dooge et al., 1987b):

$$A(x,\omega) = \exp\left[xf - x\sqrt{\{\sqrt{[b^2\omega^2 + \sqrt{(-a\omega^2 + c)}]} - a\omega^2 + c\}/\sqrt{2}}\right] \tag{38a}$$

$$\varphi(x,\omega) = xe\omega - x\sqrt{\{\sqrt{[b^2\omega^2 + \sqrt{(-a\omega^2 + c)}]} + a\omega^2 - c\}/\sqrt{2}} \tag{38b}$$

where the parameters a, b, c, e and f are defined by equations (18).

It is instructive to examine the form of the amplitude and phase spectra for the limiting values of the frequency ω. For very low frequencies, i.e. very long waves, the upstream boundary condition in the form of a harmonic oscillation:

$$A_u(t) = A_o\cos(\omega t) \tag{39}$$

results in a harmonic oscillation at the point x:

$$A(x,t) = A_o\cos(\omega t - \omega x/mv_o) \tag{40}$$

Note, that the phase velocity of the above wave $c_k = mv_o$ corresponds to the kinematic wave speed.

At the other extreme of very high frequencies, i.e. very short waves, the upstream boundary condition (41) results in a harmonic oscillation:

$$A(x,t) = A_o\exp(-\alpha_2 x)\cos(\omega t - \omega x/c_1) \tag{41}$$

Clearly, the result corresponds to the head of the wave [see equations (24) and (31)] traveling with the phase velocity $c_1 = v_o + \sqrt{(g\bar{y}_o)}$ and attenuation $\exp(-\alpha_2 x)$.

5. Simplified forms of St.Venant equations

A number of models of simplified forms of the complete St. Venant equation given by equation (11) have been proposed in the hydrological literature. If all three of the second-order terms on the left-hand side of that equation are neglected and the dependent variable is A' we obtain:

$$gA_o\left[-\frac{\partial S_f}{\partial A}\frac{\partial A'}{\partial x}+\frac{\partial S}{\partial Q}\frac{\partial A'}{\partial t}\right]=0 \tag{42}$$

which is equivalent to:

$$c_k\frac{\partial A'}{\partial x}+\frac{\partial A'}{\partial t}=0 \tag{43}$$

where c_k is kinematic wave speed defined by equation (8). The solution of this linear equation is:

$$A'(x,t)=f(t-x/c_k) \tag{44}$$

which represents a pure translation. The system function of this solution is:

$$H(s)=\exp(-sx/c_k) \tag{45}$$

The first cumulant $k_1=x/c_k$ reproduces exactly the first cumulant of the complete solution as is given by equation (34a) and all the higher order cumulants are zero. Equation (43) therefore represents an adequate first order approximation and can be used as the basis of a first order analysis of flood waves.

A new-order approximation can be obtained by using equation (44) to approximate two of the terms on the left hand side of equation (11) in terms of the remaining third term. These approximation are:

$$\frac{\partial^2 A'}{\partial x^2}=\frac{1}{c_k^2}f''(\,)\qquad \frac{\partial^2 A'}{\partial x\partial t}=\frac{1}{c^k}f''(\,)\qquad \frac{\partial^2 A'}{\partial t^2}=f''(\,) \tag{46}$$

If the second and third terms are expressed in terms of the first we have:

$$\frac{\partial^2 A'}{\partial x\partial t}=c_k\frac{\partial^2 A'}{\partial x^2}\qquad \frac{\partial^2 A'}{\partial t^2}=c_k^2\frac{\partial^2 A'}{\partial x^2} \tag{47}$$

Substitution from equation (47) into equation (11) gives diffusion analogy models:

$$D\frac{\partial^2 A'}{\partial x^2}=a\frac{\partial A'}{\partial x}+\frac{\partial A'}{\partial t} \tag{48}$$

where the advcctive parameter a and the diffusion parameter D are given by:

$$a=c_k\qquad D=0.5[1-(m-1)^2F_o^2]\frac{v_o\bar{y}_o}{S_o} \tag{49}$$

The diffusion analogy model gives the correct value for the second cumulant for any value of F_o. Further comparison for higher cumulants reveals that the diffusion analogy model is identical to the complete solution for the special case of $F_o = 0$. A number of forms of the diffusion analogy approximation to the linearized St. Venant equation, the solution of the two-point boundary problem were discussed by Dooge and Harley (1967b), Dooge and Napiórkowski (1984, 1987a, b), Dooge et al. (1983).

If the alternative approach is taken of expressing all the second-order terms as cross-derivatives, then we have the relationship:

$$\frac{D}{a}\frac{\partial^2 A'}{\partial x \partial t} + a\frac{\partial A'}{\partial x} + \frac{\partial A'}{\partial t} = 0 \tag{50}$$

where D and a have the same values as defined by equation (49). The solution of this equation has the system function which is identical in form to the system function for the distributed Muskingum model (Strupczewski et al., 1989).

6. Conclusions

The class of problems that can be termed Doogeology, is defined. The results of research concerning two most important problems in Doogeology, namely, the two-point boundary problem and the downstream problem, are presented.

References

Dooge, J.C.I., 1973. Linear theory of Hydrologic systems. U.S. Dep. Agric., Tech. Bull., No. 1468.

Dooge, J.C.I., 1980. Lectures notes on flood routing in channels. Dep. Civ. Eng., University College Dublin.

Dooge, J.C.I. and Harley, B.M., 1967a. Linear theory of open channel flow. Dept. Civ. Eng., University College Cork (unpublished memorandum).

Dooge, J.C.I. and Harley, B.M., 1967b. Linear routing in uniform open channels. Proc. Int. Hydrol. Symp., Fort Collins, Colo., 1: 57–63.

Dooge, J.C.I. and Napiórkowski, J.J., 1984. Effect of downstream control in diffusion routing. Acta Geophys. Pol., 32: 363–373.

Dooge, J.C.I. and Napiórkowski, J.J., 1987a. The effect of the downstream boundary condition in the linearized St. Venant equations. Q.J. Mech. Appl. Math., 40: 245–256.

Dooge, J.C.I. and Napiórkowski, J.J., 1987b. Applicability of diffusion analogy in flood routing. Acta Geophys. Pol., 35: 65–75.

Dooge, J.C.I., Napiórkowski, J.J. and Strupczewski, J.J., 1982. Hydrodynamic derivation of storage parameters of the Muskingum model. J. Hydrol., 54: 371–387.

Dooge, J.C.I., Kundzewicz, Z.W. and Napiórkowski, J.J., 1983. On backwater effects in linear diffusion routing. Hydrol. Sci. J., 28: 391–402.

Dooge, J.C.I., Napiórkowski, J.J. and Strupczewski, J.J., 1987a. The linear downstream response of a generalized uniform channel. Acta Geophys. Pol., 35: 277–291.

Dooge, J.C.I., Napiórkowski, J.J. and Strupczewski, J.J., 1987b. Properties of the generalized downstream channel response. Acta Geophys. Pol., 35: 405–418.

Kundzewicz, J.C.I. and Dooge, J.C.I., 1989. Attenuation and phase shift in linear flood routing. Hydrol. Sci. J., 34(1): 121–139.

Napiórkowski, J.J. and Dooge, J.C.I., 1988. Analytical solution of channel flow model with downstream control. Hydrol. Sci. J., 33: 269–287.

Romanowicz, R.J., Dooge, J.C.I. and Kundzewicz, Z.W., 1988. Moments and cumulants of linearized St.Venant equation. Adv. Water Resour., 11: 92–100.

Strupczewski, W.G., Napiórkowski, J.J. and Dooge, J.C.I., 1989. The distributed Muskingum model. J. Hydrol., 111: 235–257.