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Application of Volterra series to modelling of rainfall-runoff systems and flow in open channels*

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ABSTRACT The method presented combines a black box analysis with a conceptual model approach. To describe the nonlinear behaviour of the systems, a model in the form of a second-order approximation of a cascade of nonlinear reservoirs was used. Such a model is equivalent to the first two terms of the Volterra series. Applications of the Volterra model based on a nonlinear cascade to the modelling of flow in open channels and of surface runoff systems indicate that the proposed model can be used to represent systems with nonlinear dynamic and linear static behaviour.

Application des séries de Volterra à la mise en modèle des systèmes pluies-débits ou de l'écoulement en canaux découverts

RESUME La méthode présentée combine une analyse type boîte noire avec une approche de modèle conceptuel. Pour décrire le comportement non linéaire des systèmes on a utilisé un modèle se présentant sous la forme d'une approximation du second ordre d'une cascade de réservoirs non linéaires. Un tel modèle est équivalent aux deux premiers termes des séries de Volterra. Les applications du modèle de Volterra basées sur une cascade non linéaire à la mise en modèle de l'écoulement dans les canaux découverts et des systèmes d'écoulement superficiel montrent que le modèle proposé peut être utilisé pour représenter les systèmes avec comportement dynamique non linéaire et comportement statique linéaire.

INTRODUCTION

Three general categories of approach have been employed to formulate nonlinear input-output relationships in hydrology: (1) methods of mathematical physics; (2) black box analysis; and (3) conceptual models.

In the case of surface runoff from a natural catchment or flow in an open channel, an accurate application of the hydraulic approach would require a detailed topographical survey and determination of roughness parameters. In order to avoid these difficulties,

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modellers use an alternative approach e.g. via conceptual models or black box models.

Nonlinear black box analysis is concerned with representing a system by a functional Volterra series in the form of a sum of convolution integrals. The conceptual model approach is to simulate the nature of the catchment response or the channel response by a relatively simple nonlinear model built up from simple nonlinear elements.

The method discussed in this paper combines black box analysis with the conceptual model approach. To describe the nonlinear behaviour of systems, a model in the form of a second-order approximation of a cascade of nonlinear reservoirs is used. Such a model is equivalent to the first two terms of a Volterra series. The two equations for the kernels are linked through the use of parameters and an auxiliary function that are common to both functions. The equations fulfil a number of requirements formulated previously (Diskin & Boneh, 1972) for kernel functions of conservative systems.

MATHEMATICAL DEFINITION OF THE PROBLEM

The paper is concerned with the modelling of hydrological processes relating effective rainfall to runoff, or inflow to outflow in a river reach, by means of a nonlinear integral model:

$$y(t) = \int_0^t h_1(r)x(t-r)dr + \int_0^t\int_0^t h_2(r_1,r_2)x(t-r_1)x(t-r_2)dr_1dr_2 + \int_0^t\int_0^t\int_0^t h_3(r_1,r_2,r_3)x(t-r_1)x(t-r_2)x(t-r_3)dr_1dr_2dr_3 + \ldots \quad (1)$$

In the above equation $x(t)$ is the input to the system (effective rainfall or flow at the upstream end of the channel), $y(t)$ is the output from the model (surface runoff or flow at the downstream end of the channel), $h_1(r)$ is the first-order kernel which reflects the linear properties of the system, $h_2(r_1,r_2)$ is the second-order kernel which reflects the quadratic properties, and so on.

The functional power series of equation (1) was used for the first time in mathematics by Volterra in 1887 (Volterra, 1959). Since then it has been applied in many fields of science and engineering (Barrett, 1977). A Volterra series model was introduced to hydrological modelling by Amorocho (Amorocho, 1963).

The subject of this paper is the identification of the kernels of the Volterra series. The use of a two-term series is discussed, but the approach presented can be applied in a similar way to a series with more terms. However for higher order terms, much more accurate data are required.

The first difficulty associated with the determination of the kernels of a Volterra series using a finite length of input and output data is the determination of the system response due to the initial condition when the system is not initially relaxed. One possible way to overcome this difficulty is through the fulfilment of the assumption that the system considered has a finite memory, T_s.

The term memory is understood as the length of past input history that has an effect on the present operation of the system. It is associated with all the transient states whose influence is carried forward in time.

The identification problem may be stated: find the best estimates of the first and second order kernels, $h_1(r)$, $h_2(r_1,r_2)$, of the quadratic model with memory T_s:

$$y(t) = \int_0^{T_s} h_1(r)x(t - r)dr + \int_0^{T_s}\int_0^{T_s} h_2(r_1,r_2)x(t - r_1)x(t - r_2)dr_1dr_2 \quad (2)$$

which minimize the mean square residual output error using output records, $z(t)$, observed in a finite interval, T:

$$J(h_1,h_2) = \int_0^T [z(t) - y(t)]^2dt \quad (3)$$

Note that the length of the output should be much greater than the memory of the system (the extra data are used for noise smoothing) and that the length of the input is $[-T_s,T]$. The functions $z(t)$, $y(t)$, and $x(t)$ might be considered as sequences of individual records in the case of many independent input-output pairs.

EXISTENCE AND UNIQUENESS OF OF THE SOLUTION

In order to simplify the notation, Y and H will be used to denote the observation space and the solution space respectively. Y is the square integrable function space in an interval $[0,T]$, more precisely the Hilbert space $L^2[0,T]$. The solution space, H, is a vector function space with elements of the form:

$$H \ni \underline{h} = [h_1(r_1),h_2(r_1,r_2)]^T \qquad r_1,r_2 \in [0,T_s] \quad (4)$$

H is the product of the square integrable function space of one variable in an interval $[0,T_s]$ (more precisely $L^2[0,T_s]$) and the square integrable function space of two variables $L^2([0,T_s] \times [0,T_s])$ with inner product:

$$\langle \underline{h},\underline{g} \rangle_H = \int_0^{T_s} h_1(r)g_1(r)dr + \int_0^{T_s}\int_0^{T_s} h_2(r_1,r_2)g_2(r_1,r_2)dr_1dr_2 \quad (5)$$

Using the model equation (2) one can define the linear bounded operator L mapping the space of solutions H into the space of observation Y (Hsieh, 1964):

$$L\underline{h} = \left[\int_0^{T_s} \ldots x(t - r)dr, \int_0^{T_s}\int_0^{T_s} \ldots x(t - r_1)x(t - r_2)dr_1dr_2 \right] \begin{bmatrix} h_1(r) \\ h_1(r_1,r_2) \end{bmatrix} \quad (6)$$

The adjoint operator to L which maps the space of observation Y into the space of solutions H satisfies:

$$<z, L\underline{h}>_Y = <L^*z, \underline{h}>_H \tag{7}$$

and is defined by

$$L^*z = \begin{bmatrix} \int_o^T \ldots x(t - r_1)dt \\ \int_o^T \ldots x(t - r_1)x(t - r_2)dt \end{bmatrix} z(t) \tag{8}$$

The objective function which now takes the form of the norm in Y can be expanded according to the definition of the norm and inner product:

$$\begin{aligned} J(\underline{h}) &= \|L\underline{h} - z\|_Y^2 \\ &= <L\underline{h}, L\underline{h}>_Y - 2<L\underline{h}, z>_Y + <z, z>_Y \end{aligned} \tag{9}$$

Hence the problem of identification can be transformed from the space Y to H using the definition of the adjoint operator:

$$J(\underline{h}) = <L^*L\underline{h}, \underline{h}>_H - 2<L^*z, \underline{h}>_H + \|z\|_Y^2 \tag{10}$$

From the necessary condition for optimum (the gradient with respect to $\underline{h}$ must be equal to zero) we get the Wiener-Hopf type equation:

$$L^*L\underline{h} = L^*z \tag{11}$$

so

$$\underline{h} = (L^*L)^{-1}L^*z \tag{12}$$

We need to ask whether:

(a) the inverse of the operator L*L exists; and whether

(b) the operator $(L^*L)^{-1}$ is continuous.

The properties of the operator L*L (discussed in detail by Napiórkowski & Strupczewski (1984)) lead to the conclusion that even if the inverse of L*L exists, $(L^*L)^{-1}$ is not continuous. It follows from the above that one may observe large errors of the estimates $\hat{h}_1(r)$, $\hat{h}_2(r_1, r_2)$ since the solutions do not change continuously with the data. So the identification of the kernels of the Volterra series is a typical example of an ill-posed problem in the sense of Tichonov (1963).

The example presented below explains why small errors in measurement may result in large errors in the solution. Let us consider a simple linear model:

$$y(t) = \int_o^{T_s} h(r)x(t - r)dr \tag{13}$$

and let h_o be the solution which gives $J = 0$ with perfect measurements z = y. Now perturb h_o as follows:

$$h(r) = h_o(r) + A\sin(wr) \qquad (14)$$

Substituting in equation (13) we get:

$$z(t) = y(t) + A\int_o^{T_s} \sin(wr)x(t - r)dr \qquad (15)$$

The measure of how z(t) differs from y(t) is given by:

$$||z - y||_Y^2 = A^2 \int_o^T [\int_o^{T_s} \sin(wr)x(t - r)dr]^2 dt \qquad (16)$$

One can see that, for any A, if $w \to \infty$ then $||z - y||_Y^2 \to 0$. The function in the square brackets in equation (16) approaches zero as w tends to infinity.

On the other hand

$$||h - h_o||_H^2 = \int_o^{T_s}[h(r) - h_o(r)]^2 dr = A^2 \int_o^{T_s} \sin^2(wr)dr$$

$$= 0.5\,A^2\,(T_s - \sin(wT_s)\cos(wT_s)/w) \approx 0.5\,A^2 T_s \qquad (17)$$

Hence the difference between $h_o(r)$ and h(r) can be made arbitrarily large.

The essential conclusion from the above consideration is that very good fitting of the output from the model to the observed data may be completely misleading.

PRACTICAL METHODS OF SOLUTION

The problem of identification of the kernels of the Volterra series, as defined earlier, is ill-posed because of the class of functions within which the solution is sought is too wide. That class has to be reduced, on the basis of some mathematical and physical characteristics, to such a sub-class for which the identification problem has a unique, stable solution in the case when the measurement values are contaminated with errors. More precisely the solution has to depend continuously on the measurement data and therefore:

$$||z - y|| \to 0 \quad \text{implies} \quad ||h - h_o|| \to 0$$

Physical constraints

Some conditions which have to be fulfilled by the kernels of the Volterra series describing conservative systems (passive and lossless) were specified by Diskin & Boneh (1972). The properties of the first-order kernel were found to be identical with those of the instantaneous unit hydrograph (IUH), which is the kernel for a strictly linear system:

$$\int_0^\infty h_1(r)\, dr = 1 \qquad (18)$$

$$h_1(r) \geq 0$$

For the second-order kernel, a significant finding was that the surface integral of the kernel function over its plane of definition must be zero:

$$\int_0^\infty \int_0^\infty h_2(r_1, r_2)\, dr_1 dr_2 = 0 \qquad (19a)$$

Further it was shown that the integral of the function along any line parallel to the main diagonal must be zero:

$$\int_0^\infty h_2(r, r + C)\, dr = 0 \quad \text{for all } C \geq 0 \qquad (19b)$$

This result means that the output from the second-order term is negative for some intervals of time. It follows that to avoid negative total output it must be assumed that the inputs to the system should have an upper bound, which corresponds to the condition for convergence of the infinite series (Diskin & Boneh, 1972).

The regularization method

The regularization method (Tichonov, 1963; Tichonov & Arsenin, 1974) leads to a stable approximate solution of the identification problem by the solution of the well-posed problem:

$$J^\alpha(\underline{h}) = ||y - z||_Y^2 + \alpha\Omega(\underline{h}) \qquad (20)$$

where α is a parameter and $\Omega(\underline{h})$ is the so called regularizing functional. As a regularizing functional one can choose, for example:

$$\Omega(\underline{h}) = \left(\int_0^{T_s} \{[h_1(r)]^2 + [\frac{dh_1(r)}{dr}]^2\} dr \right)^{0.5} + \left(\int_0^{T_s}\int_0^{T_s} \{[h_2(r_1, r_2)]^2 + [\frac{\partial h_2(r_1,r_2)}{\partial r_1}]^2 + [\frac{\partial h_2(r_1,r_2)}{\partial r_2}]^2\} dr_1 dr_2 \right)^{0.5} \qquad (21)$$

which includes the derivatives in such a way that any high frequency components in $\underline{h}$ contribute strongly to the objective function.

The parameter α in equation (20) is determined empirically. The sequence $\alpha_k = \alpha_o q^k$ $(q > 0)$ is constructed. For any α_k one gets the corresponding solution, $\underline{h}\alpha_k$, and for the solution of the identification problem that element is chosen for which $||y - z||$ is equal to the error of measurements. The regularization technique was applied recently by Bruen & Dooge (1984) to linear surface runoff systems.

Direct optimization of the ordinates

The method based on direct optimization of the ordinates was applied by Diskin & Boneh (1973) to the modelling of nonlinear surface runoff systems. The discretization method adopted in their work is such that each function in equation (2) is represented by a series of pulses at regular grid-points at the same interval, Δt, along the time axis. The second-order kernel is represented by an array of pulses on a square grid at the same interval, Δt, as that used for the first-order function. With that discretization, the integrals are replaced by summations of products. The relationship between the pulses is given by:

$$Y(i) = \Sigma_{k=1}^{N_s} H_1(k)X(i-k) + \Sigma_{k=1}^{N_s} \Sigma_{k=1}^{N_s} H_2(k,\ell)X(i-k)X(i-\ell) \tag{22}$$

$$i = 1,2,\ldots,N_T; \quad N_T \simeq T/\Delta t; \quad N_s \simeq T_s/\Delta t$$

where N_s *is the number representing the memory of the system.* The discrete variables are derived from the corresponding continuous quantities by the following equations:

$$X(i) = \int_{(i-1)\Delta t}^{i\Delta t} x(t)dt \qquad i = -N_s + 1,\ldots,N_T \tag{23a}$$

$$Y(j) = \int_{(j-1)\Delta t}^{j\Delta t} y(t)dt \qquad j = 1,\ldots,N_T \tag{23b}$$

$$H_1(k) = \int_{(k-1)\Delta t}^{k\Delta t} h_1(r)dr \qquad k = 1,\ldots,N_s \tag{23c}$$

$$H_2(k,\ell) = \frac{1}{\Delta t} \int_{(k-1)\Delta t}^{k\Delta t} \int_{(\ell-1)\Delta t}^{\ell\Delta t} h_2(r_1,r_2)dr_1dr_2 \qquad k,\ell = 1,\ldots,N_s \tag{23d}$$

Note that in the case when the memory of the system is longer than the time base of the input (T_x) the second-order kernel $H_2(k,\ell)$ cannot be identified outside the region $|k - \ell| \leq T_x/\Delta t$, since the available records do not make use of $H_2(k,\ell)$ outside this region in the convolution defined by equation (22) (Diskin & Boneh, 1973).

In the method based on direct optimization of the ordinates, the solution space is a product of two Euclidian spaces, $H = R^{N_s} \otimes R^{N_s} \times R^{N_s}$, and the observation space is the Euclidian space $Y = R^{N_T}$. One has to determine $0.5(N_s + 3)N_s$ parameters in the first-order kernel and the symmetric second-order kernel. The operator L*L in equation (11) is now a matrix. The solution is unique if the matrix L*L is positive definite, and the distribution of eigenvalues determines whether the discrete problem is well- or ill-conditioned. From the practical point of view, the length of the output should be much greater than the number of unknown parameters, since it reduces the condition number and the extra data are used for noise smoothing. Note that the additional physical constraints (equations 18 and 19):

$$\Sigma_{k=1}^{N_s} H_1(k) = 1 \tag{18}$$

$$\Sigma_{k=1}^{N_s-|\ell|} H_2(\ell,\ell + k) = 0 \qquad |k| \leqslant N_s \tag{19b}$$

reduce the dimensionality of the problem.

Kernel expansion in orthonormal polynomials

The most popular method of identification expands the kernels in orthonormal polynomials (Amorocho & Brandstetter, 1971; Kuchment, 1972; Papazafiriou, 1976). The solution is searched for in a subset:

$$h_1(r) = \Sigma_{i=1}^{N} a_i \phi_i(r) \tag{24a}$$

$$h_2(r_1,r_2) = \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} a_{ij} \phi_i(r_1) \phi_j(r_2) \tag{24b}$$

where a_i and a_{ij} are unknown parameters and $\{\phi_i(r)\}$ are orthonormal polynomials in $[0,T_s]$. Any set of functions orthonormal over a finite interval may be used for the expansion. However, as advocated by Dooge (1965), orthonormal polynomials with an exponentially decreasing weighting function lend themselves particularly well to hydrological applications. The coefficients of the polynomials are generated by the three-term recurrence relation (Davis & Rabinowitz, 1975):

$$p_{n+1}(t) = (t - a_n) p_n(t) - b_n p_{n-1}(t) \tag{25}$$

where $p_{-1} = 0$, $p_o = 1$,

$$a_n = \frac{<tp_n, p_n>_\beta}{<p_n, p_n>_\beta} \qquad n = 0,1,...$$

$$b_n = \frac{<tp_n, p_{n-1}>_\beta}{<p_{n-1}, p_{n-1}>_\beta} \qquad n = 1,2,...$$

and the inner product is defined as:

$$<a,b>_\beta = \int_o^{T_s} a(t)\ b(t) \exp(-\beta t)dt \tag{26}$$

The orthonormal form of these polynomials is obtained from the relation:

$$Q_n(t) = p_n(t)/<p_n, p_n> \tag{27}$$

Hence the orthonormal functions used to approximate the kernels take the form:

$$\phi_{i+1}(t) = Q_i(t)\exp(-\beta t/2) \qquad (28)$$

and meet the condition:

$$\int_0^{T_s} \phi_i(t)\,\phi_j(t)\,dt = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \qquad (29)$$

For brevity of notation let

$$q_i(t) = \int_0^{T_s} \phi_i(r)\,x(t-r)\,dr \qquad (30)$$

Then for the symmetric second-order kernel the problem of identification can be reduced to the minimization of the following expression:

$$J(\underline{h}) = \int_0^{T} [z(t) - \Sigma_{i=1}^{N}[a_i q_i(t) + a_{ii} q_i^2(t)]$$

$$- 2\Sigma_{i=2}^{N}\ \Sigma_{j=1}^{i-1}\ a_{ij} q_i(t) q_j(t)]^2\ dt \qquad (31)$$

The necessary condition for a minimum of equation (31) is the following set of requirements for the expansion coefficients

$$a_1\langle q_1, q_1\rangle + a_2\langle q_1, q_2\rangle + \dots + a_{NN}\langle q_1, q_N q_N\rangle = \langle q_1, z\rangle$$

$$a_1\langle q_2, q_1\rangle + a_2\langle q_2, q_2\rangle + \dots + a_{NN}\langle q_2, q_N q_N\rangle = \langle q_2, z\rangle$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_i\langle q_N q_N, q_1\rangle + a_2\langle q_N q_N, q_2\rangle + \dots + a_{NN}\langle q_N q_N, q_N q_N\rangle = \langle q_N q_N, z\rangle \qquad (32)$$

where the inner products are computed in Y.

The algebraic set of equations (32) is well-conditioned in the case of rapidly varying input signals because such signals cause an increase in the diagonal elements and a decrease in the nondiagonal elements. For the idealized case of a Dirac delta input one gets

$$q_i(t) = \begin{cases} \phi_i(t) & \text{for } t \leqslant T_s \\ 0 & \text{for } t > T_s \end{cases} \qquad (33)$$

and thus:

$$\langle q_i, q_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{34}$$

On the other hand, slowly varying signals may lead to the ill-conditioned set of equations. In the particular case of a constant input function, all columns of the matrix in equation (32) are linearly dependent and the solution of the problem is not unique.

The physical constraints of equations (18) and (19) result in N + 1 additional equations, which reduce the dimensionality of the problem. The method presented in this section is characterized by a relatively small number of parameters but it arbitrarily assumes a particular structure for the kernels.

The model based on a nonlinear cascade

Napiórkowski & Strupczewski (1979, 1981) analytically obtained the first two kernels of the Volterra series for the simplest quasi-physical nonlinear model, namely a cascade of identical nonlinear reservoirs. The relationship between storage (S) and outflow (Q) for each such reservoir is specified by the following quadratic equation:

$$Q = a\,S + b\,S^2 \tag{35}$$

The parameters of this equation, a and b, together with the number (n) of reservoirs in the cascade specify the model. For that model, which combines linear static and nonlinear dynamic characteristics, the structure of the kernels was shown to be (Napiórkowski & Strupczewski, 1979):

$$h_1(r) = a\,H_n(r) \tag{36}$$

$$h_2(r_1,r_2) = b\{H_n(r_1)\,\Sigma_{k=1}^{n}\,H_k(r_2) + H_n(r_2)\,\Sigma_{k=1}^{n}\,H_k(r_1) - H_n[\max(r_1,r_2)]\} \tag{37}$$

where

$$H_n(r) = (ar)^{n-1}\exp(-ar)/(n-1)! \tag{38}$$

The two equations are linked through the fact that two parameters, a and n, appear in the equations of both kernels. One can see that $h_1(r)$ given by equation (36) is the well known transfer function for a cascade of linear reservoirs. The second-order kernel described by equation (37) meets the conditions of equations (18) and (19) specified by Diskin & Boneh (1972). The plot of the dimensionless second order kernel is given in Fig.1. Note that $h_2(r_1,r_2)$ does not possess derivatives along the main diagonal $r_1 = r_2$. Accordingly, the method which expands the kernels in orthonormal polynomials should be modified. For example, $h_2(r_1,r_2)$ can be repre-

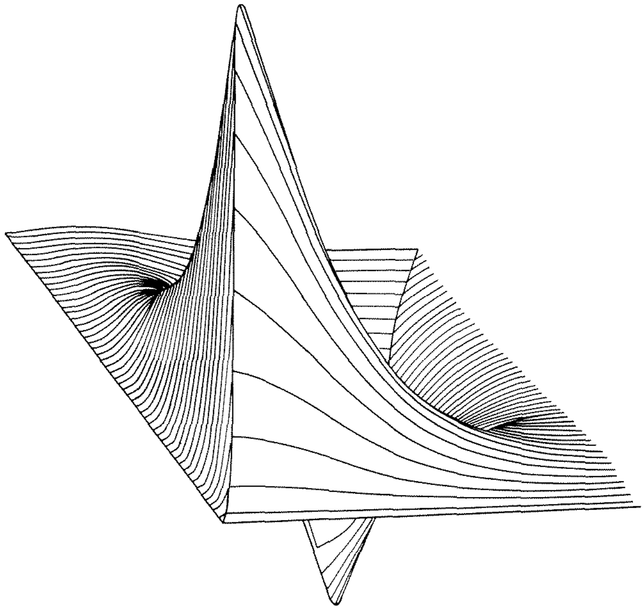

Fig. 1 The dimensionless second-order kernel for n = 3.

sented by:

$$h_2(r_1,r_2) = \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} a_{ij} \phi_i(r_1) \phi_j(r_2) + \Sigma_{i=1}^{N} c_i \phi_i[\max(r_1,r_2)] \tag{39}$$

The previous equation (24b) results in damping which is twice as strong as that corresponding to equation (37) and gives good results in the case of systems with a nonlinear no-memory part. The main feature of the model described by equations (36) to (38) is the small number of parameters to be determined in comparison with the methods presented in the previous sections.

The use of the two-term Volterra series model based on the cascade of nonlinear reservoirs is subject to certain restrictions on the parameters. These are needed to ensure copositivity of the model (a positive output response to a positive input). The sufficient copositivity condition:

$$X = \max x(t) \leqslant 0.5\ a^2/b \tag{40}$$

is weaker than the sufficient convergence condition for the infinite Volterra series (Napiórkowski & Strupczewski, 1979):

$$X = \max x(t) \leqslant 0.25\ a^2/b \tag{41}$$

APPLICATION OF THE MODEL BASED ON A NONLINEAR CASCADE

The examples which illustrate the applicability of the second-order

Volterra model based on the nonlinear cascade to the modelling of flow in open channels and surface runoff systems are presented below. In both cases the problem to be solved is to find the best estimates of the parameters n, a and b of the model given by equations (36) to (38). The optimization problem:

$$J(n,a,b) = \int_0^T [z(t) - y(t)]^2 \, dt \tag{42}$$

where z(t) is the flow at the downstream end of the reach of the direct runoff, can be reduced to optimization with respect to one variable ($a > 0$) only.

Let us denote by $\sigma y(t)$ the linear response of the model, and by $\sigma^2 y(t)$ the quadratic response for $b = 1$. Then due to linearity of the second term with respect to b the objective function takes the form:

$$J(n,a,b) = \int_0^T [z(t) - \sigma y(t) - b\, \sigma^2 y(t)]^2 \, dt \tag{43}$$

Note that the functions $\sigma y(t)$ and $\sigma^2 y(t)$ depend on the parameters n and a but do not depend on the parameter b. Hence b can be determinated from the necessary condition for an extremum, $\partial J/\partial b = 0$:

$$b = \frac{\int_0^T [z(t) - \sigma y(t)]\sigma^2 y(t)\,dt}{\int_0^T [\sigma^2 y(t)]^2 dt} \tag{44}$$

The following steps are therefore required in the overall optimization of the model:

(1) assuming $b = 0$ compute the initial values of the parameters n^* and a^* as in linear analysis e.g. by moment matching (Nash, 1959; Dooge, 1973);

(2) assuming an integral value of the parameter n close to n^* and a suitable value of the scale parameter (a) compute the functions $\sigma y(t)$ and $\sigma^2 y(t)$;

(3) compute directly the optimum value of the parameter b from the necessary condition for the optimum (equation (44));

(4) maintaining the same value of the parameter n and varying the parameter a repeat the procedure of steps (2) and (3) to determine the optimal set of values of (a,b) for the assumed integral value of n; and

(5) assuming a range of values of n repeat the procedure of steps (2) and (3) for each n to determine the optimal set of the three parameters (n,a,b).

Flow in open channels

The prototype flow was taken to be numerical solutions of the St Venant equations for the simple case of a rectangular prismatic channel.

The two-term Volterra model was used to simulate the flow deviation from the steady state of $Q_o = 200\ m^3s^{-1}$. The following parameter values were used: bottom slope I = 0.000 248, Chézy coefficient C = 44.9, channel width B = 100 m, depth of flow in the steady state $y_o = 2$ m. Calculations were carried out for a reach of x = 40 km. In the numerical experiment, the parameters n, a and b were identified for an inflow increment in the form of a rectangular pulse function ($\Delta Q = 200\ m^3s^{-1}$ for $t \in [0,6000\ s]$). The optimal values of the parameters were found to be:

$$n = 6$$
$$a = 0.217 \times 10^{-3}\ [s^{-1}]$$
$$b = 94 \times 10^{-12}\ [s^{-1}m^{-3}]$$

The degree of fit to the prototype by the optimized model is shown in Fig.2.

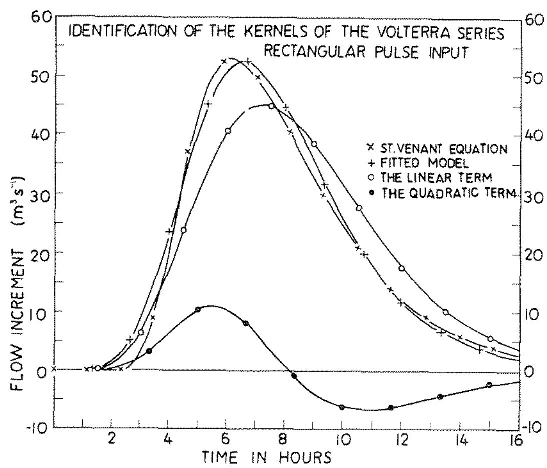

Fig. 2 Identification of the Volterra model at 40 km using a rectangular pulse function.

The model thus obtained was tested by applying an input completely different from the input used for calibration. The input chosen was a smooth bimodal function with maximum amplitude $\Delta Q' = 250\ m^3s^{-1}$ (Napiórkowski *et al.*, 1983). The resulting fit is shown in Fig.3 and can be considered good.

Surface runoff systems

The two-term Volterra model was fitted to the records of eight storms whose quadratic response was previously determined by:

(a) direct optimization of 74 values in the linear and quadratic kernels (Diskin & Boneh, 1973); and

(b) applying the discrete version of the model based on a nonlinear cascade (Diskin *et al.*, 1984).

The catchment is that of the Cache River at Forman in southern Illinois covering 630 km^2, with gentle slopes and a well-developed drainage network. The optimal values of the model parameters were found to be:

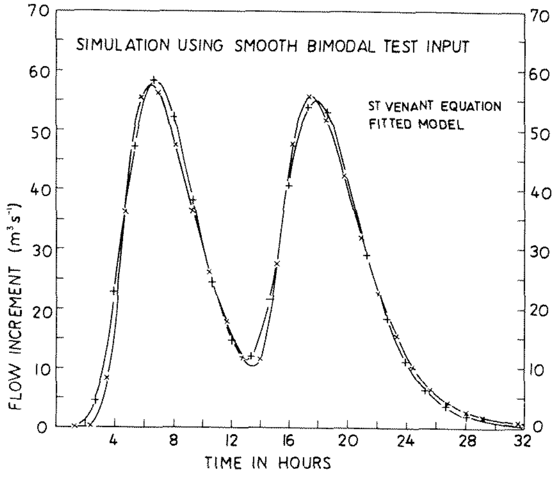

Fig. 3 Simulation of St Venant model at 40 km using a smooth input.

$n = 3$
$a = 0.75$ [day^{-1}]
$b = 6.84 \times 10^{-3}$ [$day^{-1} mm^{-1}$]

Examples of the degree of fit to the observed runoff by the Volterra model are shown in Figs 4(a)-(g).

The agreement between the observed and computed output for the case of the model based on direct optimization of 74 ordinates (objective function $J = 53$) is better than for the case based on the cascade ($J = 238$). However the latter case has only three parameters which ensures that the identification problem is well-conditioned.

The parameters determined by Diskin *et al.* (1984) ($n = 3$, $a = 0.74$ day^{-1}, $b = 6.35 \times 10^{-3}$ $day^{-1} mm^{-1}$) were calculated for the discrete Volterra model. Hence they differ from these obtained for the continuous Volterra model. However, the objective function in both cases is the same.

CONCLUSIONS

The identification of kernels of the Volterra series is a typical example of an ill-posed problem. It follows from the above that one may find large errors of kernel estimates even if measurement errors are very small.

In this paper a number of methods of solving that problem is presented. One of them (the method based on a nonlinear cascade) provides a relatively simple solution. According to this approach the kernels are sought in a subset defined in terms of quasi-physical characteristics.

Applications of the Volterra model based on a nonlinear cascade to the modelling of flow in open channels and of surface runoff systems indicate that the proposed model can indeed be used to represent systems with nonlinear dynamic and linear static behaviour.

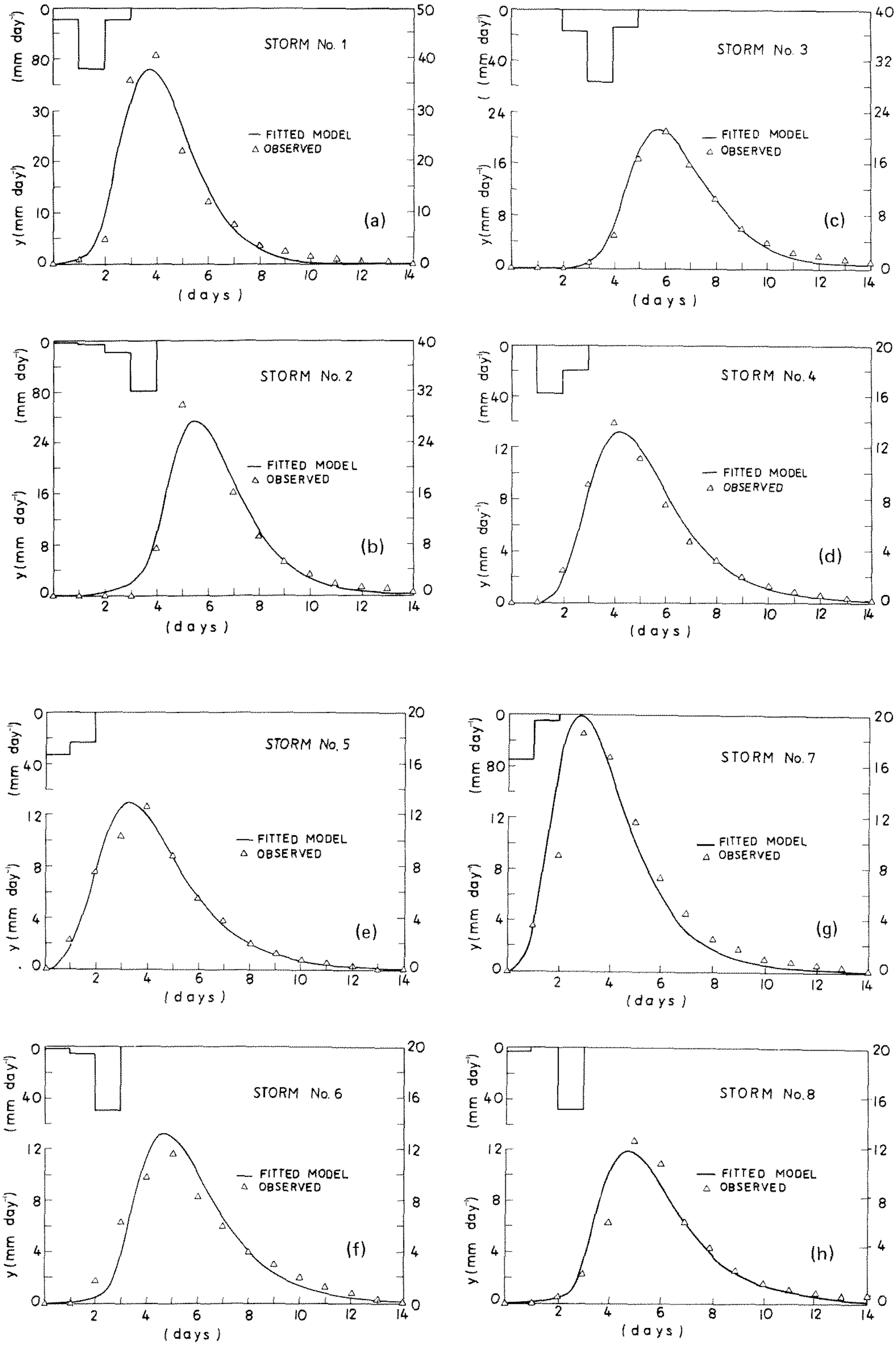

Fig. 4 Comparison of observed runoff and that predicted by the Volterra model.

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