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42

8 Towards the physical structure of river flow stochastic process

J. J. NAPIÓRKOWSKI

Institute of Geophysics, Polish Academy of Sciences, Warsaw, Poland

W. G. STRUPCZEWSKI

Department of Civil Engineering, University of Calgary, Alberta, Canada T2N 1N4

ABSTRACT The transformation of white noise and Markov processes through the simplified St. Venant flood routing model is examined. This model has been derived from the linearized St. Venant equation for the case of a wide uniform open channel flow with arbitrary cross-section shape and friction law. The only simplification results in filtering out the downstream boundary condition. The cross-correlation and normalized autocorrelation functions are determined in analytical way.

INTRODUCTION

The development of water resources research has created the need for an extension of mathematical analysis of hydrological data. An awareness of the stochastic structure of hydrologic processes is necessary for modelling water resources systems. The aim of the paper is to investigate the physical structure of the process of outflow from a river reach.

The widely accepted assumption about a structure of an inflow process is that it can be considered as a sum of deterministic and random components. It is assumed that the input signal is weekly stationary (stationarity of the first two moments).

It is assumed that the system behaves linearly. This is the crude simplification granting the compromise between simplicity and accuracy. The structure of the random component transformed by some conceptual linear flood routing models (linear reservoir, Nash, Muskingum) was examined by Strupczewski *et al.* (1975a, b). Some of their results are easily available (e.g. Singh, 1988, p. 240). In the present paper the structure of the random component transformed by the flood routing model based on the St. Venant equations will be analyzed.

The rigorous hydrodynamic description of open channel flow (St. Venant model), requires two boundary conditions and in the case of a tranquil flow one of these is at the downstream end of the channel. In practical flood routing the influence of downstream controls is typically neglected and the routing takes part only in a downstream direction.

The hydrodynamic model used in this paper, called the rapid flow model (RFM), was developed by filtering out the downstream boundary condition to approximate the diffusion term in the St. Venant equations.

DERIVATION OF THE RAPID FLOW MODEL (RFM) FROM THE LINEARIZED ST. VENANT EQUATION

The findings presented in this paragraph borrow heavily from Strupczewski & Napiórkowski (1990). The linearized St. Venant equation for one-dimensional unsteady flow in uniform channel with arbitrary cross-section shape and either of the common friction laws may be written as (Dooge *et al.*, 1987a):

$$(1-F^2)g\bar{y}_0\frac{\partial^2 Q}{\partial x^2}-2v_0\frac{\partial^2 Q}{\partial x\partial t}-\frac{\partial^2 Q}{\partial t^2} = gA_0\left(\frac{\partial S_f}{\partial Q}\frac{\partial Q}{\partial t}-\frac{\partial S_f}{\partial A}\frac{\partial Q}{\partial x}\right) \quad (1)$$

where Q is the perturbation of flow about an initial condition of steady uniform flow Q_0, A_0 is the cross-sectional area corresponding to this flow, F is the Froude number, S_f is the friction slope, $\bar{y}_0$ is the hydraulic mean depth, v_0 is the mean velocity, S_0 is the bottom slope, x is the distance from the upstream boundary, t is the elapsed time and derivatives of the friction slope S_f are evaluated at the reference conditions.

The variation of the friction slope with discharge at the reference condition for either of the common frictional formulas for rough turbulent flow could be expressed as:

$$\frac{\partial S_f}{\partial Q} = 2S_0/Q_0 \tag{2}$$

Define an auxilliary parameter m as the ratio of the kinematic wave speed to the average velocity of flow

$$m = c_k/(Q_0/A_0) \tag{3}$$

where the kinematic wave speed c_k is as given by Lighthill & Whitham (1955)

$$c_k = \frac{dQ}{dA} = -\frac{\partial S_f/\partial A}{\partial S_f/\partial Q} \tag{4}$$

The parameter m is a function of the shape of channel and of a friction law parameter.

Substituting equations (2)–(4) into equation (1) one obtains

$$(1-F^2)\frac{\bar{y}_0}{2mS_0}\frac{\partial^2 Q}{\partial x^2} - F^2\frac{\bar{y}_0}{c_k S_0}\frac{\partial^2 Q}{\partial x \partial t} - F^2\frac{m y_0}{2c_k^2 S_0}\frac{\partial^2 Q}{\partial t^2} = \frac{\partial Q}{\partial x} + \frac{1}{c_k}\frac{\partial Q}{\partial t} \tag{5}$$

The linear equation (5) is a hyperbolic one, i.e. it has two real characteristics. The direction of these characteristics gives the celerity of both the primary and secondary waves. For Froude numbers less than 1, the celerity of secondary wave is in an upstream direction. In order to filter out the downstream boundary condition the small convective term (the first term in equation 5) can be neglected entirely. It provides the exact solution for Froude number equal to one. However, in order to increase the accuracy for the value of the Froude number close to one, one can represent the convective term in equation (5) on the basis of lower order approximation to the solution of the equation. This low order approximation is given by neglecting all terms on the left-hand side of equation (5) to obtain kinematic wave equation

$$\frac{\partial Q}{\partial x} = -\frac{1}{c_k}\frac{\partial Q}{Qt} \tag{6}$$

This lower order solution can be used to approximate the first term on the left-hand side of equation (7) in terms of: the second term:

$$\frac{\partial^2 Q}{\partial x^2} = -\frac{1}{c_k}\frac{\partial^2 Q}{\partial x \partial t} \tag{7}$$

the third term:

$$\frac{\partial^2 Q}{\partial x^2} = \frac{1}{c_k^2}\frac{\partial^2 Q}{\partial t^2} \tag{8}$$

or by the linear combination of the second and third terms:

$$\frac{\partial^2 Q}{\partial x^2} = -C_1\frac{1}{c_k} + C_2\frac{1}{c_k^2}\frac{\partial^2 Q}{\partial t^2} \tag{9}$$

where C_1 and C_2 are coefficient to be determined.

Note, that:

(i) equation (7) is a special case of equation (9) for $C_1 = 1$ and $C_2 = 0$;

(ii) equation (8) is a special case of equation (9) for $C_1 = 0$ and $C_2 = 1$;

(iii) the approximation based on entirely neglecting the diffusion term is for $C_1 = 0$ and $C_2 = 0$.

Substitution of approximations (9) into equation (5) gives the Rapid Flow Model (RFM) in the form

$$-\alpha\frac{\partial^2 Q}{\partial x \partial t} - \beta\frac{\partial^2 Q}{\partial t^2} = \frac{\partial Q}{\partial x} + \frac{1}{c_k}\frac{\partial Q}{\partial t} \tag{10}$$

On general grounds one could expect that the models based on the approximation of the diffusion term through the kinematic wave approximation would be preferable to the one in which this term is neglected. These general considerations are reinforced by comparing some properties of equation (10) with other known results in open channel hydraulics (Strupczewski & Napiórkowski, 1990). All forms of the RFM discussed will exactly predict the first moment or lag of the Linear Channel Response (LCR), i.e. the solution of the equation (1) for semi-infinite channel and for $F < 1$. To get equivalence of second moments of the RFM and the LCR the coefficients C_1 and C_2 should fulfill the relation $C_1 + C_2 = 1$, while for the additional equivalence of third moments $C_1 = 2$. It is suggested that any discussion of the applicability of the RFM should be confined to this form that preserves all three moments of the complete linear equation. Therefore the final values of the parameters α and β in equation (10) are:

$$\alpha = \frac{1}{mc_k}[1 + (m-1)F^2]\frac{y_0}{S_0} \tag{11}$$

$$\beta = \frac{1}{2mc_k^2}[1 + (m^2-1)F^2]\frac{y_0}{S_0} \tag{12}$$

Since the downstream boundary condition was filtered out from the St. Venant equation only upstream boundary condition $Q_u(t) = Q(0, t)$ is required to solve equation (10). Hence, all transfer properties of the hydrodynamic model described by equations (10)–(12) can be described by the impulse response given in the Fourier transform domain as:

$$H^{RFM}(x, j\omega) = \exp\left(-\Delta j\omega - \lambda + \frac{\lambda}{1 + \alpha j\omega}\right) \tag{13}$$

where

$$\lambda = \frac{m}{2}\frac{1-(m-1)^2F^2}{[1+(m-1)F^2]^2}\frac{S_0}{y_0}x \tag{14}$$

$$\Delta = 0.5\,\frac{1+(m^2-1)F^2}{1+(m-1)F^2}\,\frac{x}{c_k} \tag{15}$$

The RFM impulse response in the time domain has a clear conceptual interpretation being the total of the products of the Poisson distribution

$$P_k(\lambda) = \frac{\lambda^k}{k!}\exp(-\lambda) \tag{16}$$

and the impulse response of cascade of k-linear reservoirs (CLR) with a time constant α (equation 11)

$$h_{k,\alpha}^{\mathrm{CLR}}(t) = \frac{1}{\alpha(k-1)!}(t/\alpha)^{k-1}\exp(-t/\alpha) \tag{17}$$

shifted in time by a time delay Δ given in equation (15). This impulse response is thus given by (cf. Strupczewski & Napiórkowski, 1990):

$$h^{\mathrm{RFM}}(x,t) = P_0(\lambda)\,\delta(t-\Delta) + \sum_{k=1}^{\infty} P_k(\lambda)\,h_{k,\alpha}^{\mathrm{CLR}}(t-\Delta)\,1(T-\Delta) \tag{18}$$

The upstream boundary condition is delayed by a linear channel with time lag Δ, divided according to the Poisson distribution with the mean λ, and then transformed by parallel cascades of equal linear reservoirs (with time constant α) of varying lengths.

The Rapid Flow Model can be considered as both, a conceptual and a physical one. On one hand it is a conceptual model with physically derived parameters. On the other it is a rigorous simplification of the linearized St. Venant equations. This simplification results in reducing the number of model parameters and filtering out the downstream boundary condition. The RFM can be applied to any length of channel reach. However, the quality of the Linear Channel Response approximation by the RFM depends on the type of motion, as discussed in Strupczewski & Napiórkowski (1990).

TRANSFORMATION OF STOCHASTIC PROCESSES IN THE RFM

In this section the transformation of stationary random processes in the RFM will be analyzed. This class of processes is important because stationarity provides the possibility of learning the statistical properties under various ergodicity hypotheses. Also the amount of information required to statistically describe stationary processes is greatly reduced. Finally, frequency-domain methods can be used in the analysis of the RFM with stationary input processes. White noise and Markovian noise are the processes assumed as the input stochastic processes in the analysis. They are commonly used in stochastic hydrology due to their simplicity and existing relationship to real processes.

If the stationary random process $X(t)$ is fed to a linear shift-invariant system with the impulse response $h(t)$, then the output random process can be expressed as the convolution integral

$$Y(t) = \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)\,\mathrm{d}\tau \tag{19}$$

Computing the cross-correlation function between the input process and the output process one finds:

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} h(\alpha)R_X(\tau-\alpha)\,\mathrm{d}\alpha = h(\tau) * R_X(\tau) \tag{20}$$

Taking the Fourier transforms of both sides of equation (20) one obtains the frequency domain representation of the cross-correlation function

$$S_{YX}(\omega) = H(\omega)S_X(\omega) \tag{21}$$

Finally, the autocorrelation function of the output is expressed as

$$R_Y(\tau) = \int_{-\infty}^{+\infty} R_{YX}(\tau+\alpha)h(\alpha)\,\mathrm{d}\alpha = R_{YX}(\tau) * h(-\tau) \tag{22}$$

while in the spectral domain via Fourier transformation it becomes

$$S_Y(\omega) = S_{YX}(\omega)H^*(\omega) \tag{23}$$

Combining the preceding results, one obtains a fundamental equation relating the autocorrelation function of the output to the autocorrelation function of the input

$$R_Y(\tau) = h(\tau) * R_X(\tau) * h(-\tau) \tag{24}$$

which in the spectral domain takes the form:

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) \tag{25}$$

TRANSFORMATION OF WHITE NOISE IN THE RFM

Consider the RFM with a white noise input (X_1). The correlation function of the white noise process contains the Dirac delta impulse, i.e.:

$$R_{X_1}(\tau) = \sigma^2\delta(\tau) \tag{26}$$

so that its power spectral density defined as its Fourier transform is a constant

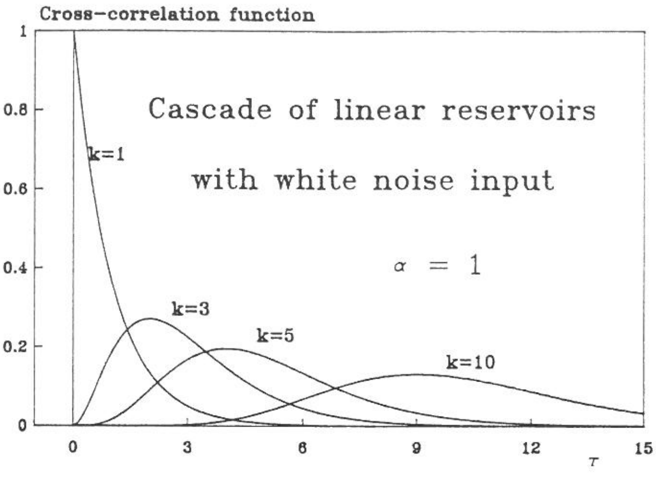

Fig. 1 Cross-correlation function of the cascade of linear reservoirs with white noise input.

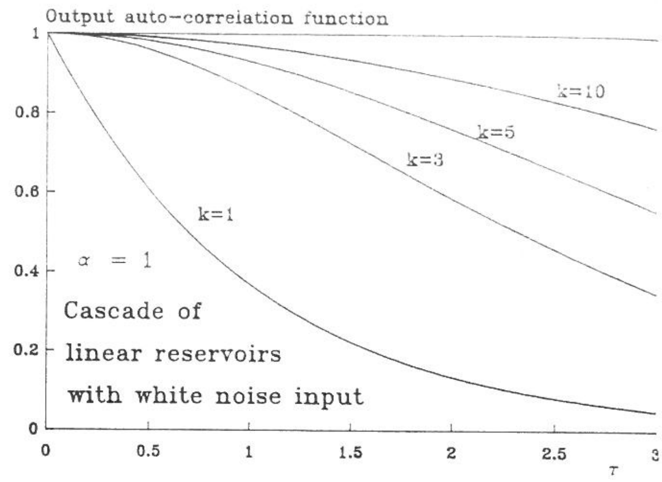

Fig. 2 Normalized auto-correlaton function of the cascade of linear reservoirs with white noise input.

$$S_{X_1}(\omega)=\sigma^2, \qquad -\infty<\omega<+\infty \tag{27}$$

For the RFM transfer function (transform of the impulse response in the Fourier domain) given by equation (13) one gets the following cross- and output-power spectral densities

$$\begin{aligned}S_{YX_1}^{\mathrm{RFM}}(x,\omega)&=H^{\mathrm{RFM}}(x,\omega)S_{X_1}(\omega)\\&=\sigma^2\exp\left(-\Delta j\omega-\lambda+\frac{\lambda}{1+\alpha j\omega}\right)\end{aligned} \tag{28}$$

$$S_{Y_1}^{\mathrm{RFM}}(x,\omega)=|H^{\mathrm{RFM}}(x,\omega)|^2S_{X_1}(\omega)=\sigma^2\mathrm{e}^{-2\lambda}\exp\left(\frac{2\Lambda}{1+\alpha^2\omega^2}\right) \tag{29}$$

It is shown in Appendix A that:

(i) the inverse Fourier transform of equation (28) yields the cross-correlation function

$$R_{YX_1}^{\mathrm{RFM}}(x,\tau)=\begin{cases}P_0(\lambda)\delta(t-\Delta)+\sum\limits_{k=1}^{\infty}P_k(\lambda)h_{k,\alpha}^{\mathrm{CLR}}(t-\Delta) & \tau\geq\Delta\\ 0 & \tau<\Delta\end{cases} \tag{30}$$

where $h_{k,\alpha}^{\mathrm{CLR}}(t)$, given by equation (17) and plotted in Fig. 1, is the impulse response of the CLR, that is cross-correlation function of the CLR with white noise input;

(ii) the inverse Fourier transform of equation (29) yields the output correlation function

$$R_{Y_1}^{\mathrm{RFM}}(x,\tau)=\sigma^2P_0(2\lambda)\delta(\tau)+\sum_{k=1}^{\infty}P_k(2\lambda)\,R_{Y_1}^{\mathrm{CLR}}(k,\alpha,\tau) \tag{31}$$

where $P_k(2\lambda)$ is a Poisson distribution (equation 16) and

$$R_{Y_1}^{\mathrm{CLR}}(k,\alpha,\tau)=\sigma^2\frac{\exp(-|\tau|/\alpha)}{\alpha(k-1)!2^k}\sum_{i=0}^{k-1}\frac{(k+i-1)!}{(k-i-1)!2^i i!}(|\tau|/\alpha)^{k-i-1} \tag{32}$$

is the autocorrelation function of the output from the CLR with the white noise input (see Fig. 2).

It is interesting that the RFM cross-correlation function given by equation (30) is 0 for $\tau<\Delta$. This means that the output Y in time instant t is orthogonal to values of the input X_1 for $t\in(-\Delta,+\infty)$, which is a white noise. This occurs because of three reasons: The part of the model responsible for the modulatory performance is causal, another part of the model can be interpreted as a time shift (pure delay) and the input is a white noise. The system causalty requires that the output does not depend directly on future inputs but only depend directly on present and past inputs. The whiteness of the input X_1 guarantees that the past and present inputs will be uncorrelated with future inputs. Combining three conditions we see that there will be no cross-correlation between the present output and the inputs in time interval $(-\Delta,+\infty)$. If we assume additionally that the input is Gaussian, then the input process is an independent process and the output becomes independent of all future inputs and those in time interval $(-\Delta,0)$. So, the causality of the system prevents the direct dependence of the present output on future inputs, and the independent process input prevents any indirect dependence. These ideas are important to the theory of Markov process in next section.

It is convenient to illustrate the results in terms of dimensionless independent variables defined with the help of bottom slope S_0, the depth y_0, and the velocity v_0 for the steady uniform reference conditions about which perturbation are taken. Thus we can write:

$$x'=x\frac{S_0}{y_0} \tag{33}$$

$$t'=t\frac{v_0S_0}{y_0} \tag{34}$$

Hence, the dimensionless parameters of transfer function are given respectively by:

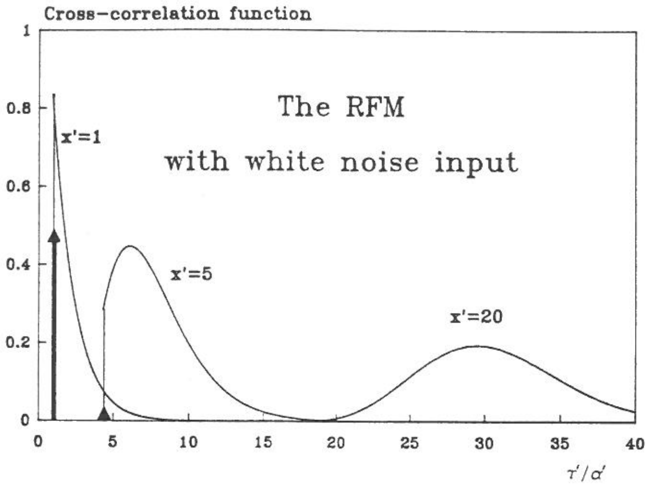

Fig. 3 Cross-correlation function of the Rapid Flow Model with white noise input.

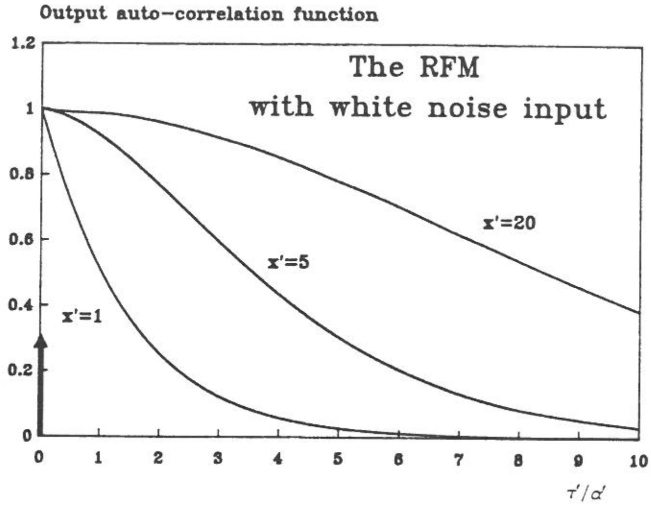

Fig. 4 Normalized auto-correlaton function of the Rapid Flow Model with white noise input.

$$\alpha' = \frac{1}{m^2}[1+(m-1)F^2 \tag{35}$$

$$\lambda' = \frac{m[1-(m-1)^2F^2]}{2[1+(m-1)F^2]^2}x' \tag{36}$$

$$\Delta' = \frac{1+(m^2-1)F^2}{2m[1+(m-1)F^2]}x' \tag{37}$$

For illustration the flow in a broad rectangular channel with Manning friction ($m=5/3$) and the Froude number $F=0.3$ ($\alpha'=0.381$) will be considered. Figs. 3 and 4 show the cross- and output-correlation functions of the Rapid Flow Model described by eqs.30 and 32 for wide rectangular channel of dimensionless lengths: $x'=1$ (short channel), $x'=5$ (intermediate channel), and $x'=20$ (long channel). Both figures are drawn in function of dimensionless time τ'/α'.

TRANSFORMATION OF MARKOVIAN NOISE IN THE RFM

In this section the RFM with Markovian noise input is considered. The correlation function of normal Markovian noise (X_2) is given by:

$$R_{X_2}(\tau) = D^2\exp(-c|\tau|), \qquad -\infty<\tau<+\infty \tag{38}$$

where D^2 is the variance of the input process, so its power spectral density is described by

$$S_{X_2}(\omega) = \frac{2D^2c}{c^2+\omega^2}, \qquad -\infty<\omega<+\infty \tag{39}$$

Accordingly, we have the following cross- and output-power spectral densities

$$\begin{aligned} S_{YX_2}^{\mathrm{RFM}}(x,\omega) &= H^{\mathrm{RFM}}(x,\omega)S_{X_2}(\omega) \\ &= \exp\left(-\Delta j\omega - \lambda + \frac{\lambda}{1+\alpha j\omega}\right)\frac{2D^2c}{c^2+\omega^2} \end{aligned} \tag{40}$$

$$\begin{aligned} S_{Y_2}^{\mathrm{RFM}}(x,\omega) &= |H^{\mathrm{RFM}}(x,\omega)|^2 S_{X_2}(\omega) \\ &= \mathrm{e}^{-2\lambda}\exp\left(\frac{2\lambda}{1+\alpha^2\omega^2}\right)\frac{2D^2c}{c^2+\omega^2} \end{aligned} \tag{41}$$

It is shown in Appendix B that:

(a) the inverse Fourier transform of equation 40 yields the cross-correlation function

$$R_{YX_2}(x,\tau) = P_0(\lambda)D^2\mathrm{e}^{-c|\tau-\Delta|} + \sum_{k=1}^{\infty} P_k(\lambda)R_{YX_2}^{\mathrm{CLR}}(k,\alpha,\tau-\Delta) \tag{42}$$

where

$$R_{YX_2}^{\mathrm{CLR}}(k,\alpha,\tau) = \begin{cases} \dfrac{D^2\mathrm{e}^{-c\tau}}{(1-\alpha c)^k} + D^2\mathrm{e}^{-\tau/\alpha}\displaystyle\sum_{i=0}^{k-1}\frac{(\tau/\alpha)^i}{i!}\left[\frac{1}{(1+\alpha c)^{k-i}} - \frac{1}{(1-\alpha c)^{k-i}}\right] & \tau\geq 0 \\ \dfrac{D^2\mathrm{e}^{c\tau}}{(1+\alpha c)^k}; \quad \tau<0 \end{cases} \tag{43}$$

is the cross-correlation function for the cascade of linear reservoirs with Markovian noise input and is plotted in Fig. 5;

(b) the inverse Fourier transform of equation (41) yields the output correlation function

$$R_{Y_2}^{\mathrm{RFM}}(x,\tau) = P_0(2\lambda)D^2\mathrm{e}^{-c|\tau|} + \sum_{k=1}^{\infty} P_k(2\lambda)\, R_{Y_2}^{\mathrm{CLR}}(k,\alpha,\tau) \tag{44}$$

where $P_k(2\lambda)$ is a Poisson distribution and

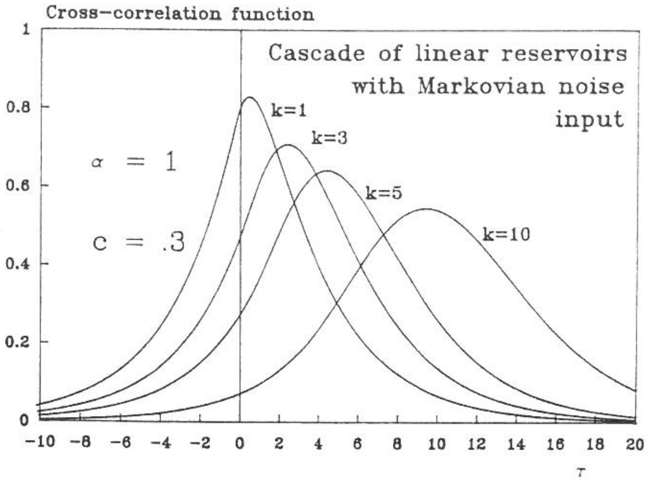

Fig. 5 Cross-correlation function of the cascade of linear reservoirs with Markovian noise input.

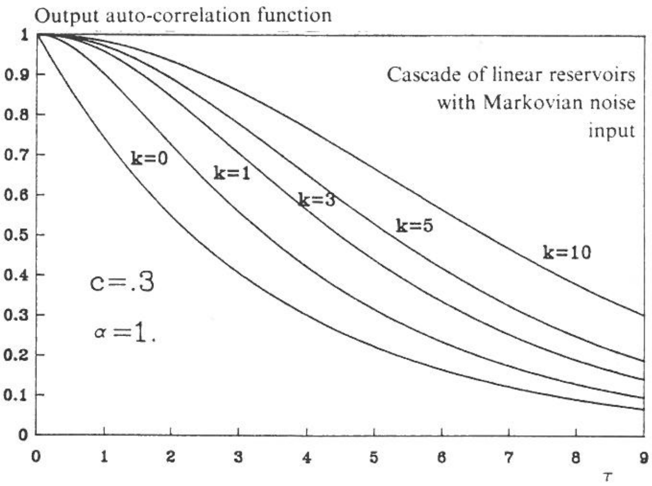

Fig. 6 Normalized auto-correlation function of the cascade of linear reservoirs with Markovian noise input.

$$R_{Y_2}^{\mathrm{CLR}}(k,\alpha,\tau)=\frac{D^2 e^{-c\tau}}{(1-\alpha c)^k(1+\alpha c)^k}+\frac{D^2 e^{-|\tau|/\alpha}}{(k-1)!}\sum_{i=0}^{k-1}\left[\frac{1}{(1+\alpha c)^{k-i}}-\frac{1}{(1-\alpha c)^{k-i}}\right]\sum_{j=0}^{i}\frac{(k+j-1)!(|\tau|/\alpha)^{i-j}}{(i-j)!j!\,2^{k+j}} \tag{45}$$

is the output-correlation function of the cascade of linear reservoirs with Markovian noise input (see Fig. 6).

Figs. 7 and 8 show the cross- and output-correlation functions of the Rapid Flow Model described by equations (42) and (44) for wide rectangular channel of dimensionless lengths: $x'=1$ (short channel), $x'=5$ (intermediate channel), and $x'=20$ (long channel) with the Manning friction $m=5/3$, Froude number $F=0.3$, and dimensionless parameter of Markovian noise $c'=0.3$. Both figures are drawn in function of dimensionless time τ'/α'.

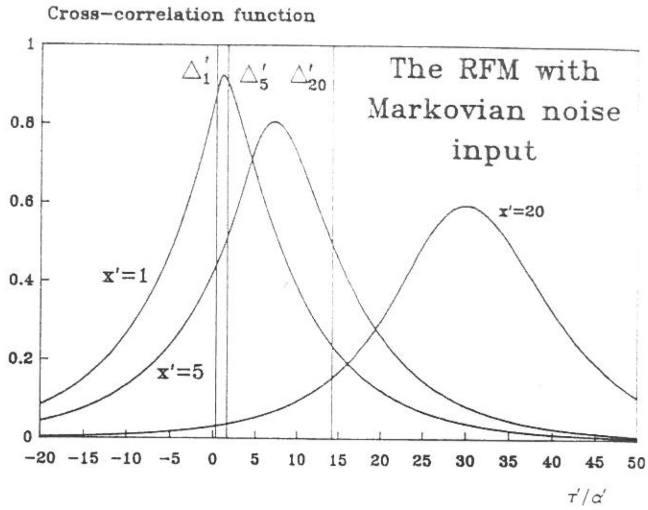

Fig. 7 Cross-correlation function of the Rapid Flow Model with Markovian noise input.

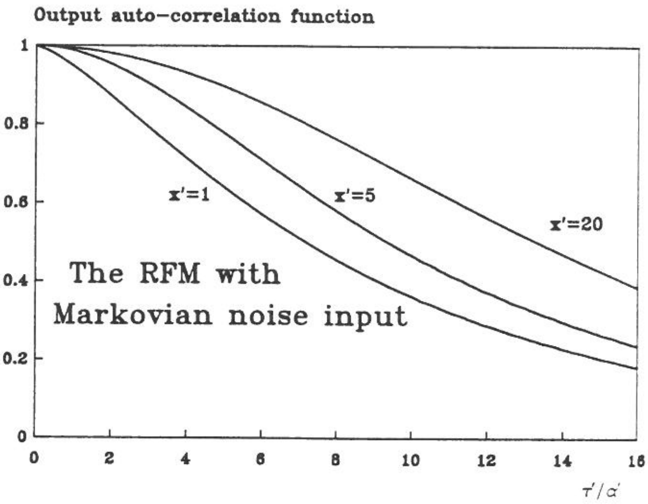

Fig. 8 Normalized auto-correlaton function of the Rapid Flow Model with Markovian noise input.

CONCLUSIONS

The cross- and auto-correlation functions are derived in the analytical way for the simplified linearized St. Venant model with upstream control only, i.e. for the Rapid Flow Model, with white noise and Markovian inputs. Obtained functions are much more complicated than those of the cascade of linear reservoirs. It is also possible to obtain time averaged results in an analytical way.

In the case of the white noise input one can see that the output autocorrelation function is considerably weaker than the one observed in nature (Fig. 4). Consider a wide rectangular channel with depth $y_0 = 1.4$ m, the Manning friction coefficient $n = 0.03$ and the Froude number $F = 0.3$, i.e. the dimensionless α' parameter equal to 0.381. Then the mean velocity and the bottom slope are respectively equal to $v_0 = 1.1$ m/s, and $S_0 = 0.0007$. Accordingly, independent variables x and t are related to the dimensionless variables as follows

$$x = 2000\,x'[\text{m}] \tag{33a}$$

$$t = 12\,t'[\text{min}] \tag{34b}$$

Hence the values of the dimensionless lengths: $x' = 1$, $x' = 5$ and $x' = 20$ in Fig.4 correspond to 2 km, 10 km and 40 km respectively, and the unit of dimensionless time $\tau'/\alpha' = \tau/\alpha$ corresponds to 12 min.

For example, for hourly observations and 40 km length of a reach (above values correspond approximately to $x' = 20$ and $\tau'/\alpha' = 5$) the value of the output auto-correlation function is less than 0.7, while in a real system it is generally greater than 0.99. For the analyzed Markovian noise input and the same flow conditions the value of normalized autocorrelation function is 0.9. It proves, that channel impact on autocorrelation function of river process is strongly limited to a short time. Therefore the long term auto-correlation observed in the river processes must be caused by other contributing processes like watershed alimentation, surface and subsurface runoff.

Streamflow data used in the time series analysis are usually pulse data. In order to make theoretical and empirical results comparable Strupczewski *et al.* (1975b) working on conceptual flood routing models covered also such case. At the cost of additional algebra it is possible to account it also for the RFM with white noise and Markovian inputs and to obtain the cross- and auto-correlation functions for any length of the discretization interval.

More realistically, one can consider a stream network rather than a single river reach with multidimensional input, correlated in space or both in space and time, or alternatively the RFM by complete linear St.Venant equation solved for semiinfinite channel or one can try to tackle with a complex system having rainfall as input. It is obvious that any attempt to an extension of generalization of the presented model would lead to further complication of analytical solution.

Responding to the question, whether the aspiration to establish the physical structure of the stochastic process of river flow is justified, is left to the reader.

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APPENDIX A. DERIVATION OF CROSS- AND OUTPUT-CORRELATION FUNCTION FOR THE RFM WITH WHITE NOISE INPUT

Cross-correlation function

For the case of the RFM with white noise input X_1 the following equation for the cross-power spectral density (equation 28) holds:

$$S_{YX_1}^{\text{RFM}}(x,\omega) = \sigma^2 \exp\left(-\Delta j\omega - \lambda + \frac{\lambda}{1+\alpha j\omega}\right) \tag{A1}$$

Expanding equation (A1) into a convergent series and operating on it term by term one obtains

$$S_{YX_1}^{\text{RFM}}(x,\omega) = = \sigma^2 e^{-\lambda} e^{-\Delta j\omega} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{(1+\alpha j\omega)^k} \tag{A2}$$

Recalling the definition of the Poisson distribution (equation 16) and the system function of the cascade of k-linear reservoirs, namely

$$H_{k,\alpha}^{\text{CLR}}(\omega) = \frac{1}{(1+j\omega\alpha)^k} \tag{A3}$$

equation (A2) may be rewritten in the form

$$S_{YX_1}^{\text{RFM}}(x,\omega) = \sigma^2 P_0(\lambda) e^{-\Delta j\omega} \sum_{k=1}^{\infty} \sigma^2 P_k(\lambda)\, e^{-\Delta j\omega} H_{k,\alpha}^{\text{CLR}}(\omega) \tag{A4}$$

Applying the translation theorem one gets the cross-correlation function

$$R_{YX_1}^{\text{RFM}}(x,\tau) = \begin{cases} \sigma^2 P_0(\lambda)\delta(\tau-\Delta) + \sum_{k=1}^{\infty} \sigma^2 P_k(\lambda) h_{k,\alpha}^{\text{CLR}}(\tau-\Delta) & \tau \geq \Delta \\ 0 & \tau < \Delta \end{cases} \tag{A5}$$

where the impulse response of the cascade of k-linear reservoirs $h_{k,\alpha}^{\text{CLR}}(t)$ is given by equation (17).

Output-correlation function

For the case of the RFM with white noise input the following equation for the power spectral density of the output signal (equation 29) holds:

$$S_{Y_1}^{\mathrm{RFM}}(x,\omega)=\sigma^2 e^{-2\lambda}\exp\left(\frac{2\lambda}{1+\alpha^2\omega^2}\right) \tag{A6}$$

Expanding equation (A6) into a series one gets

$$S_{Y_1}^{\mathrm{RFM}}(x,\omega)=\sigma^2 e^{-2\lambda}\sum_{k=0}^{\infty}\frac{(2\lambda)^k}{k!}\frac{1}{(1+\alpha^2\omega^2)^k} \tag{A7}$$

Since the spectral density of the output signal for the cascade of linear reservoirs with white noise input is given by

$$S_{Y_1}^{\mathrm{CLR}}(k,\alpha,\omega)=\sigma^2|H_{k,\alpha}^{\mathrm{CLR}}(\omega)|^2=\frac{\sigma^2}{(1+\alpha^2\omega^2)^k} \tag{A8}$$

equation (A7) may be rewritten in a way similar to cross-correlation function as

$$S_{Y_1}^{\mathrm{RFM}}(x,\omega)=\sigma^2 P_0(2\lambda)+\sum_{k=1}^{\infty}P_k(2\lambda)\,S_{Y_1}^{\mathrm{CLR}}(k,\alpha,\omega) \tag{A9}$$

and the RFM with white noise input has the autocorrelation function of the output signal given by

$$R_{Y_1}^{\mathrm{RFM}}(x,\tau)=\sigma^2 P_0(2\lambda)\delta(\tau)+\sum_{k=1}^{\infty}P_k(2\lambda)\,R_{Y_1}^{\mathrm{CLR}}(k,\alpha,\tau) \tag{A10}$$

Hence, it remains to invert equation (A8) from the Fourier-transform domain to original domain.

Applying the residue method to evaluate the inverse Fourier transformation (Stark & Woods, 1986), rewrite the right hand side of equation (A8) in terms of complex variable $j\omega$ to obtain

$$S_{Y_1}^{\mathrm{CLR}}(k,\alpha,j\omega)=\frac{\sigma^2}{[1-\alpha^2(j\omega)^2]^k} \tag{A11}$$

Replacing $j\omega$ by s one extends the function $S(j\omega)$ to the entire complex plane (two-sided Laplace transform of the correlation function)

$$S_{Y_1}^{\mathrm{CLR}}(k,\alpha,s)=\frac{\sigma^2}{(1+\alpha s)^k(1-\alpha s)^k} \tag{A12}$$

Evaluating the residues for positive τ (kth order pole, counter-clockwise traversal of the contour), one gets

$$\begin{aligned}R_{Y_1}^{\mathrm{CLR}}(k,\alpha,\tau)&=\mathrm{Res}[S_{Y_1}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-1/\alpha]\\&=\frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left[S_{Y_1}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau}(s+1/\alpha)\right]_{s=-1/\alpha}\\&=\sigma^2\frac{\exp(-\tau/\alpha)}{\alpha(k-1)!2^k}\sum_{i=0}^{k-1}\frac{(k+i-1)!}{(k-i-1)!\,2^i i!}(\tau/\alpha)^{k-i-1}\end{aligned} \tag{A13}$$

while for negative τ (clockwise traversal of the contour) one gets

$$\begin{aligned}R_{Y_1}^{\mathrm{CLR}}(k,\alpha,\tau)&=-\mathrm{Res}[S_{Y_1}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=1/\alpha]\\&=-\frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left[S_{Y_1}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau}(s-1/\alpha)^k\right]_{s=1/\alpha}\\&=\sigma^2\frac{\exp(\tau/\alpha)}{\alpha(k-1)!2^k}\sum_{i=0}^{k-1}\frac{(k+i-1)!}{(k-i-1)!\,2^i i!}(-\tau/\alpha)^{k-i-1}\end{aligned} \tag{A14}$$

combining the results into a single formula, one gets

$$R_{Y_1}^{\mathrm{CLR}}(k,\alpha,\tau)=\sigma^2\frac{\exp(-|\tau|/\alpha)}{\alpha(k-1)!2^k}\sum_{i=0}^{k-1}\frac{(k+i-1)!}{(k-i-1)!\,2^i i!}(|\tau|/\alpha)^{k-i-1} \tag{A15}$$

APPENDIX B. DERIVATION OF CROSS- AND OUTPUT-CORRELATION FUNCTION FOR THE RFM WITH MARKOVIAN NOISE INPUT

Cross-correlation function

For the RFM with Markovian noise input X_2 one gets the following cross-power spectral density (equation 40)

$$S_{YX_2}^{\mathrm{RFM}}(x,\omega)=\exp\left(-\Delta j\omega-\lambda+\frac{\lambda}{1+\alpha j\omega}\right)\frac{2D^2c}{c^2+\omega^2} \tag{B1}$$

Expanding equation (B1) into a series one gets

$$S_{YX_2}^{\mathrm{RFM}}(x,\omega)=e^{-\lambda}e^{-\Delta j\omega}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{1}{(1+\alpha j\omega)^k}\frac{2D^2c}{(c+j\omega)(c-j\omega)} \tag{B2}$$

Since the cross-power spectral density for the cascade of linear reservoirs with Markovian noise input is given by

$$S_{YX_2}^{\mathrm{CLR}}(k,\alpha,\omega)=\frac{1}{(1+\alpha j\omega)^k}\frac{2D^2c}{(c+j\omega)(c-j\omega)} \tag{B3}$$

equation (B3) may be rewritten as

$$S_{YX_2}^{\mathrm{RFM}}(x,\omega)=P_0(\lambda)e^{-\Delta j\omega}S_{X_2}(\omega)+\sum_{k=1}^{\infty}P_k(\lambda)\,e^{-\Delta j\omega}S_{YX_2}^{\mathrm{CLR}}(k,\alpha,\omega) \tag{B4}$$

Applying the translation theorem one gets the cross-correlation function for the RFM with Markovian noise input

$$R_{YX_2}^{\mathrm{RFM}}(x,\tau)=P_0(\lambda)D^2e^{-c|\tau-\Delta|}+\sum_{k=1}^{\infty}P_k(\lambda)\,R_{XY_2}^{\mathrm{CLR}}(k,\alpha,\tau-\Delta) \tag{B5}$$

It remains to invert equation (B3) from the Fourier-transform domain to original domain. Applying the residue method to evaluate the inverse Fourier transformation, we replace $j\omega$ by s to extend the function $S(j\omega)$ to entire complex plane

$$S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)=\frac{1}{(1+\alpha s)^k}\frac{2D^2c}{(c+s)(c-s)} \tag{B6}$$

Evaluating the residues for positive τ (one first order pole, one kth order pole, counterclockwise traversal of the contour), one gets

$$R_{YX}^{\mathrm{CLR}}(k,\alpha,\tau)=\mathrm{Res}[S_{YX}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-1/\alpha] + \mathrm{Res}[S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-c] \tag{B7}$$

Evaluating the residue for $s=-c$, one gets

$$\mathrm{Res}[S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-c]=\frac{D^2}{(1-\alpha c)^k}e^{-c\tau} \tag{B8}$$

while for $s=-1/\alpha$ one has

$$\begin{aligned}\mathrm{Res}[S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-1/\alpha] &=\frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\{S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)\,e^{s\tau}(s+1/\alpha)^k\}_{s=-1/\alpha}\\ &=\frac{2D^2c}{(k-1)!\alpha^k}\frac{d^{k-1}}{ds^{k-1}}\{(c+s)^{-1}(c-s)^{-1}e^{s\tau}\}_{s=-1/\alpha}\\ &=D^2e^{-\tau/\alpha}\sum_{i=0}^{k-1}\frac{(\tau/\alpha)^i}{i!}\left[\frac{1}{(1+\alpha c)^{k-i}}-\frac{1}{(1-\alpha c)^{k-i}}\right]\end{aligned} \tag{B9}$$

Evaluating the residue for negative τ (one first order pole, clockwise traversal of the contour), one gets

$$R_{YX_2}^{\mathrm{CLR}}(k,\alpha,\tau)=-\mathrm{Res}[S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=c]=\frac{D^2e^{c\tau}}{(1+\alpha c)^k} \tag{B10}$$

Combining equations (B7)–(B9) one gets the cross-correlation function of the cascade of linear reservoirs with Markovian noise input as

$$R_{YX_2}^{\mathrm{CLR}}(k,\alpha,\tau)=\begin{cases}\dfrac{D^2e^{-c\tau}}{(1-\alpha c)^k}+D^2e^{-\tau/\alpha}\displaystyle\sum_{i=0}^{k-1}\frac{(\tau/\alpha)^i}{i!}\left[\frac{1}{(1+\alpha c)^{k-i}}-\frac{1}{(1-\alpha c)^{k-i}}\right] & \tau\geq 0\\ \dfrac{D^2e^{c\tau}}{(1+\alpha c)^k}; \quad \tau<0 & \end{cases} \tag{B11}$$

Output-correlation function

For the case of the RFM with Markovian noise input the following output spectral density is obtained (equation 41):

$$S_{Y_2}^{\mathrm{RFM}}(x,\omega)=e^{-2\lambda}\exp\left(\frac{2\lambda}{1+\alpha^2\omega^2}\right)\frac{2D^2c}{c^2+\omega^2} \tag{B12}$$

Expanding equation (B12) into a series one gets

$$S_{Y_2}^{\mathrm{RFM}}(x,\omega)=\frac{2D^2c}{c^2+\omega^2}e^{-2\lambda}\sum_{k=0}^{\infty}\frac{(2\lambda)^k}{k!}\frac{1}{(1+\alpha^2\omega^2)^k} \tag{B13}$$

Since the output-power spectral density for the cascade of linear reservoirs with Markovian noise input is given by

$$S_{Y_2}^{\mathrm{CLR}}(k,\alpha,\omega)=|H_{k,\alpha}^{\mathrm{CLR}}(\omega)|^2S_{X_2}(\omega)=\frac{1}{(1+\alpha^2\omega^2)^k}\frac{2D^2c}{c^2+\omega^2} \tag{B14}$$

equation (B13) may be rewritten as

$$S_{Y_2}^{\mathrm{RFM}}(x,\omega)=P_0(2\lambda)S_{X_2}(\omega)+\sum_{k=1}^{\infty}P_k(2\lambda)\,S_{Y_2}^{\mathrm{CLR}}(k,\alpha,\omega) \tag{B15}$$

and the RFM with Markovian noise input has the correlation function of output given by

$$R_{Y_2}^{\mathrm{RFM}}(x,\tau)=P_0(2\lambda)D^2e^{-c|\tau|}+\sum_{k=1}^{\infty}P_k(2\lambda)\,R_{Y_2}^{\mathrm{CLR}}(k,\alpha,\tau) \tag{B16}$$

It remains to invert equation (B14) from the Fourier-transform domain to original domain. Applying the residue method to evaluate the inverse Fourier transformation, one can replace $j\omega$ by s to extend the function $S(j\omega)$ to entire complex plane

$$S_{Y_2}^{\mathrm{CLR}}(k,\alpha,s)=\frac{2D^2c}{(c+s)(c-s)}\frac{1}{(1+\alpha s)^k(1-\alpha s)^k} \tag{B17}$$

Evaluating the residues for positive τ (one first order pole, one kth order pole, counterclockwise traversal of the contour), one gets

$$R_{Y_2}^{\mathrm{CLR}}(k,\alpha,\tau)=\mathrm{Res}[S_{Y_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-1/\alpha] + \mathrm{Res}[S_{Y_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-c] \tag{B18}$$

Evaluating the residue for $s=-c$, one gets

$$\mathrm{Res}[S_{YX_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-c]=\frac{D^2e^{-c\tau}}{(1-\alpha c)^k(1+\alpha c)^k} \tag{B19}$$

while for $s=-1/\alpha$

$$\begin{aligned}\mathrm{Res}[S_{Y_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau};\ s=-1/\alpha] &=\frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left\{S_{Y_2}^{\mathrm{CLR}}(k,\alpha,s)e^{s\tau}\left(s+\frac{1}{\alpha}\right)^k\right\}_{s=-\frac{1}{\alpha}}\\ &=\frac{2D^2c}{(k-1)!\alpha^k}\frac{d^{k-1}}{ds^{k-1}}\{(c+s)^{-1}(c-s)^{-1}(1-\alpha s)^{-k}e^{s\tau}\}_{s=-1/\alpha}\\ &=\frac{D^2e^{-\tau/\alpha}}{(k-1)!}\sum_{i=0}^{k-1}\left[\frac{1}{(1+\alpha c)^{k-i}}-\frac{1}{(1-\alpha c)^{k-i}}\right]\sum_{j=0}^{i}\frac{(k+j-1)!(\tau/\alpha)^{i-j}}{(i-j)!j!\,2^{k+j}}\end{aligned} \tag{B20}$$

Evaluating the residues for negative τ (one first order pole, one kth order pole, clockwise traversal of the contour), and combining equations (B19) and (B20) one can find the output-correlation function of the cascade of linear reservoirs with Markovian noise input as

$$R_{Y_2}^{\mathrm{CLR}}(k,\alpha,\tau)=\frac{D^2e^{-c|\tau|}}{(1-\alpha c)^k(1+\alpha c)^k}+\frac{D^2e^{-|\tau|/\alpha}}{(k-1)!}\sum_{i=0}^{k-1}\left[\frac{1}{(1+\alpha c)^{k-i}}-\frac{1}{(1-\alpha c)^{k-i}}\right]\sum_{j=0}^{i}\frac{(k+j-1)!(|\tau|/\alpha)^{i-j}}{(i-j)!j!\,2^{k+j}} \tag{B21}$$