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PROBLEMS INVOLVED IN IDENTIFICATION OF THE KERNELS OF VOLTERRA SERIES

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Abstract

The subject of the paper is modelling of hydrologic processes by means of the second-order Volterra series model. The existence and uniqueness of the solution of identification problem is discussed. Examples illustrating the applicability of the Volterra model based on cascade of non-linear reservoirs to the modelling of flow in open channels and of surface runoff systems are presented.

1. INTRODUCTION

In the case of surface runoff from a natural catchment or flow in an open channel an accurate application of the hydraulic approach would require a detailed topographical survey and determination of roughness parameters. Accordingly, it is reasonable to search for a mathematical non-linear model which reflects adequately physical nature of the above processes but which is based only on accurately determined mathematical relationship between the input and output.

In this paper we are concerned with modelling of hydrologic processes relating effective rainfall to runoff and inflow to outflow in a river reach by means of a non-linear integral model:

$$y(t)=\int_0^t h_1(r)x(t-r)\,dr+\int_0^t\int_0^t h_2(r_1,r_2)x(t-r_1)x(t-r_2)\,dr_1\,dr_2+$$
$$+\int_0^t\int_0^t\int_0^t h_3(r_1,r_2,r_3)x(t-r_1)x(t-r_2)x(t-r_3)\,dr_1\,dr_2\,dr_3+\dots, \tag{1}$$

where $x(t)$ is the input to the system (effective rainfall or flow at the upstream end of the channel), $y(t)$ is the output from the model (surface runoff or flow at the downstream end of the channel), $h_1(r)$ is the first-order kernel which reflects the linear properties of the system, $h_2(r_1, r_2)$ is the second-order kernel which reflects the quadratic properties etc.

The functional power series (1) was used for the first time in mathematics by Volterra in 1887 (Volterra, 1959). Since then it has been applied in many fields of science and engineering (Barrett, 1977). The Volterra series model was first used to hydrologic modelling by Amorocho in 1961 (Amorocho, 1963).

2. MATHEMATICAL DEFINITION OF THE PROBLEM

The subject of the paper is identification of the kernels of Volterra series. The use of a two-term Volterra series is discussed in detail, but the presented approach can be applied in a similar way to series with more terms. However, the computations become much more complex for higher order terms and much more accurate data are required.

Let us assume that the system considered has a finite settling time and thus the non-zero initial conditions cause no difficulties. The identification problem consists in searching for the best estimates of the first and second order kernels $h_1(r)$ and $h_2(r_1, r_2)$ of the quadratic model

$$y(t)=\int_0^{T_s} h_1(r)x(t-r_1)dr+\int_0^{T_s}\int_0^{T_s} h_2(r_1, r_2)x(t-r_1)x(t-r_2)dr_1\,dr_2 \tag{2}$$

with the settling time T_s, which minimizes the mean square residual output error using the output records $z(t)$ observed in a finite interval T, longer than the "memory" of the system:

$$J(h_1, h_2)=\int_0^{T} [z(t)-y(t)]^2 dt. \tag{3}$$

Note that the length of the input $x(t)$ should be greater than $2T_s$. In order to simplify mathematical manipulations, the notations of functional analysis are adopted to reformulate the problem (Hsieh, 1964). We use Y to denote the observation space. Y is the square integrable function space in an interval $[0,T]$, more precisely the Hilbert space $L^2\,[0, T]$ with the inner product

$$\langle z, y\rangle_Y=\int_0^{T} z(t)y(t)dt \tag{4a}$$

and the norm

$$\|z\|_Y=\sqrt{\langle z, z\rangle_Y}. \tag{4b}$$

The solution space H is a vector function space with the elements in the form

$$\mathbf{h}=\begin{bmatrix} h_1(r_1) \\ h_2(r_1, r_2)\end{bmatrix}, \quad r_1, r_2 \in [0, T_s]. \tag{5}$$

The space H is the product of the square integrable function space of one variable in an interval $[0, T_s]$ ($L^2\,[0, T_s]$) and the square integrable function space of two variables in

$$[0, T_s]\times[0, T_s](L^2[0, T_s]\times[0, T_s])$$

with the inner product

$$\langle \mathbf{h}, \mathbf{g}\rangle_H=\int_0^{T_s} h_1(r)g_1(r)dr+\int_0^{T_s}\int_0^{T_s} h_2(r_1, r_2)g_2(r_1, r_2)dr_1\,dr_2 \tag{6a}$$

and the norm

$$||\mathbf{h}||_H=\sqrt{\langle \mathbf{h}, \mathbf{h}\rangle_H}. \tag{6b}$$

H is a vector Hilbert space since it is the composition of two Hilbert spaces.

Using model equation (2) we define the linear bounded operator L mapping the space of solutions H into the space of observations Y (see Fig. 1):

$$L\mathbf{h}=\left[\int_0^{T_s} \ldots x(t-r)\,dr,\ \int_0^{T_s}\int_0^{T_s} \ldots x(t-r_1)\,x(t-r_2)\,dr_1\,dr_2\right]\left[\begin{matrix} h_1(r) \\ h_2(r_1, r_2)\end{matrix}\right]. \tag{7}$$

The operator L is bounded because x is bounded and linearity can be easily checked

$$L(\alpha\mathbf{h}+\beta\mathbf{g})=\alpha L\mathbf{h}+\beta L\mathbf{g}. \tag{8}$$

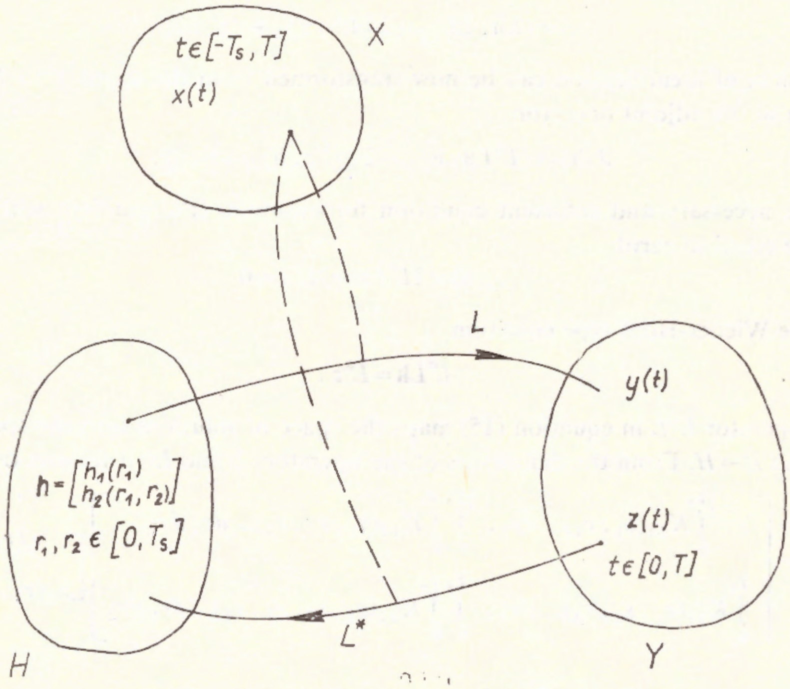


Fig. 1. Linear operator L and its adjoint L^*

The adjoint operator to L, which maps the space of observations Y into the space of solutions H (see Fig. 1), is defined by

$$L^*z=\left[\begin{matrix} \int_0^T \ldots x(t-r_1)\,dt \\ \int_0^T\int_0^T \ldots x(t-r_1)\,x(t-r_2)\,dt \end{matrix}\right] z(t), \tag{9}$$

$$t\in[0, T], \quad r_1, r_2\in[0, T_s]$$

and satisfies

$$\langle z, L\mathbf{h}\rangle_Y = \langle L^* z, \mathbf{h}\rangle_H, \tag{10}$$

as may be verified by substitution.

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

The objective function which now takes form of the norm in Y

$$J(h_1, h_2) = \int_0^T [z(t) - y(t)]^2 dt = ||z - y||_Y^2 = ||L\mathbf{h} - z||_Y^2 \tag{11}$$

can be expanded according to the definition of the norm (4b) and the inner product (4a):

$$J(\mathbf{h}) = ||L\mathbf{h} - z||_Y^2 = \langle L\mathbf{h} - z, L\mathbf{h} - z\rangle =$$

$$= \langle L\mathbf{h}, L\mathbf{h}\rangle_Y - 2\langle L\mathbf{h}, z\rangle_Y + \langle z, z\rangle_Y. \tag{12}$$

The problem of identification can be now transformed from the space Y to H using the definition of the adjoint operator

$$J(\mathbf{h}) = \langle L^* L\mathbf{h}, \mathbf{h}\rangle_H - 2\langle L^* z, \mathbf{h}\rangle_H + ||z||_Y^2. \tag{13}$$

From the necessary and sufficient condition for optimum (the gradient with respect to $\mathbf{h}$ must be equal to zero)

$$J_h(\mathbf{h}) = 2L^* L\mathbf{h} - 2L^* z = 0 \tag{14}$$

we get the Wiener-Hopf type equation

$$L^* L\mathbf{h} = L^* z. \tag{15}$$

The operator L^*L in equation (15) maps the space of solutions into the space of solutions $L^*L : H \to H$. From the definitions of the operators L and L^* it follows that

$$L^* L\mathbf{h} = \begin{bmatrix} \int_0^{T_s} K_{11}(s_1, r_1) \ldots dr_1, & \int_0^{T_s}\int_0^{T_s} K_{12}(s_1, r_1, r_2) \ldots dr_1\, dr_2 \\ \int_0^{T_s} K_{21}(s_1, s_2, r_1) \ldots dr_1, & \int_0^{T_s}\int_0^{T_s} K_{22}(s_1, s_2, r_1, r_2) \ldots dr_1\, dr_2 \end{bmatrix} \begin{bmatrix} h_1(r_1) \\ h_2(r_1, r_2) \end{bmatrix}, \tag{16}$$

where

$$K_{11}(s_1, r_1) = \int_0^T x(t - s_1) x(t - r_1)\, dt, \tag{17a}$$

$$K_{12}(s_1, r_1, r_2) = \int_0^T x(t - s_1) x(t - r_1) x(t - r_2)\, dt, \tag{17b}$$

$$K_{21}(s_1, s_2, r_1) = \int_0^T x(t - s_1) x(t - s_2) x(t - r_1)\, dt, \tag{17c}$$

$$K_{22}(s_1, s_2, r_1, r_2) = \int_0^T x(t - s_1) x(t - s_2) x(t - r_1) x(t - r_2)\, dt, \tag{17d}$$

and the parameters s arise from the repetition of the arguments r.

Having determined the operator L^*L we can discuss its properties. The operator L^*L is:

– compact as a composition of integral operators with the square integrable kernels;

– self-adjoint

$$(L^*L)^* = L^*(L^*)^* = L^*L;$$

– non-negative definite

$$\langle L^*L\mathbf{h}, \mathbf{h}\rangle_H = \langle L\mathbf{h}, L\mathbf{h}\rangle_Y = \|L\mathbf{h}\|_Y^2 \geqslant 0.$$

Under these assumptions it can be proved (Kołodziej, 1970) that L^*L possesses discrete set of real non-negative eigenvalues. Let $\{\lambda_i\}$ be the sequence of non-zero eigenvalues such as $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_n \geqslant \ldots$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be their corresponding orthonormal eigenfunction. Then L^*z can be expanded in a Fourier series (Kołodziej, 1970)

$$L^*z = \sum_{n=1}^{\infty} b_n \varphi_n, \tag{18}$$

where the Fourier coefficients are

$$b_n = \langle L^*z, \varphi_n\rangle_H, \quad n = 1, 2, \ldots \tag{19}$$

Let the solution of the identification problem (13) be sought in the form of a linear combination of the eigenfunctions corresponding to non-zero eigenvalues, e.i. in the subspace spanned by $\{\varphi_i\}$:

$$h_o = \sum_{n=1}^{\infty} c_n \varphi_n. \tag{20}$$

Substituting equation (20) into equation (15) and taking into account equation (18) we have $c_n = b_n/\lambda_n$, so

$$h_o = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} \varphi_n. \tag{21}$$

The solution exists if and only if $\|h_o\| < +\infty$, that is when

$$\sum_{n=1}^{\infty} \frac{b_n^2}{\lambda_n^2} < \infty. \tag{22}$$

If the condition expressed by equation (22) is not met, then there is no element in H which will satisfy equation (15). In the case of slow varying signals, when $\lambda = 0$ is the eigenvalue of the operator L^*L, the solution is not unique, and we cannot uncover the "true" kernel functions of the system with available input-output data.

Note that even if L^*L is positive, the inverse of the compact operator L^*L is not continuous (not bounded). It follows from the above that if the solution

$$\mathbf{h} = (L^*L)^{-1} L^*z \tag{23}$$

exists, it does not change continuously with the data. Hence one may observe large errors of the solution $\mathbf{h}$ even if the measurements errors are very small. So, the identification of the kernels of Volterra series is a typical example of an ill-posed problem in the sense of Tikhonov (1963).

Example. The example presented below explains why small error in measurements may result in large error in the solution of the identification problem. Let us consider a simple linear model

$$y(t)=\int_0^{T_s} h(r)x(t-r)dr, \qquad t\in[0, T]. \tag{24}$$

Let $h_o(r)$ be the solution which gives $J_o=0$ with perfect measurements $z=y$. Now let us perturb h_o as follows

$$h'(r)=h_o(r)+N\sin\omega r. \tag{25}$$

Substituting in equation (24) we find

$$z(t)=y(t)+N\int_0^{T_s}\sin\omega r\cdot x(t-r)dr. \tag{26}$$

The measure of how $z(t)$ differs from $y(t)$ is given by

$$||z-y||_Y^2=N^2\int_0^T\Big[\int_0^{T_s}\sin\omega r\cdot x(t-r)dr\Big]^2dt. \tag{27}$$

One can see that for any N

$$||z-y||_Y^2 \xrightarrow[\omega\to\infty]{} 0. \tag{28}$$

The function in the square bracket approaches zero as ω goes to infinity. This may be seen as follows. If x is the constant function then this is obvious. Since any bounded x can be approximated by a step wise functions the result would be similar.

On the other hand

$$||h'-h_o||_H^2=\int_0^{T_s}[h_o(r)-h'(r)]^2dr=N^2\int_0^{T_s}\sin^2\omega r\,dr=$$

$$=N^2\left(\frac{T_s}{2}-\frac{1}{2\omega}\sin\omega T_s\cos\omega T_s\right)\underset{\omega\to\infty}{\approx} N^2\frac{T_s}{2}. \tag{29}$$

Hence the H norm can be made arbitrarily large

$$||h'-h_o||_H^2 \xrightarrow[N\to\infty]{} \infty, \tag{30}$$

while the Y norm can be made arbitrarily small. So, small error in measurements may results in arbitrarily large error in the identification problem.

The essential conclusion from the above consideration is that very good fitting of the output from the model to the observed data may be completely misleading.

4. APPROXIMATE METHODS OF SOLUTION

The main reason why the problem of identification of the kernels of Volterra series (as defined in chapter 3) is ill-posed is that the class of functions within which the solution is sought is too wide. We have to reduce that class, on the basis of some mathematical and physical characteristics, to such a sub-class M for which the identification problem has a unique, stable solution in the case when the measurement values are contaminated with errors. More precisely, M should be such a subset of H for which the solution depends continuously on the measurements data, and therefore

$$\|z-y\|_Y \to 0 \text{ implies } \|\mathbf{h}-\mathbf{h}_o\|_H \to 0 .$$

Physical constraints. Some conditions which have to be fulfilled by conservative systems described by Volterra series were specified by Diskin and Boneh (1972). For loss-less systems the following conditions are valid:

$$\int_0^\infty h_1(r)\,dr=1, \quad h_1(r)\geqslant 0, \tag{31a}$$

$$\int_0^\infty h_2(r, r+C)\,dr=0 \quad \text{for all } C\geqslant 0. \tag{31b}$$

These constraints reduce the H space, but the solution can be unstable even in this sub-class.

4.1. The quasi-solution method. For the positive operator L^*L the set of solutions is reduced to a sphere ($\|\mathbf{h}\|\leqslant R$) in the space H. The quasi-solution of optimization problem (13) is given by equation (22) if the condition

$$\sum_{n=1}^{\infty} \frac{b^2}{\lambda_n^2} < R^2 \tag{32}$$

is met, that is if $\|\mathbf{h}\|<R$. If condition (32) is not met, $\|\mathbf{h}\|\geqslant R$, we solve the optimization problem (13) under constraint $\|\mathbf{h}\|^2=R^2$. Using the Lagrange multiplier theorem one can reduce the problem to an unconstrained optimization

$$J_\alpha(\mathbf{h})=\langle L^*L\mathbf{h}, \mathbf{h}\rangle_H - 2\langle L^*z, \mathbf{h}\rangle_H + \|z\|_Y^2 + \alpha\langle \mathbf{h}, \mathbf{h}\rangle \tag{33}$$

and then to the solution of the corresponding Wiener-Hopf equation

$$L^*L\mathbf{h}+\alpha\mathbf{h}=L^*z . \tag{34}$$

Substituting equation (21) into equation (34) and taking into account equation (20) one gets

$$h=\sum_{n=1}^{\infty} \frac{b_n}{\alpha+\lambda_n}\varphi_n , \tag{35}$$

where the coefficient α is determined from the condition $\|\mathbf{h}\|^2=R^2$

$$\sum_{n=1}^{\infty} \frac{b_n^2}{(\alpha+\lambda_n)^2}=R^2 . \tag{36}$$

The above quasi-solution method leads to stable solutions (Tikhonov and Arsenin, 1974). However, it should be noted that the determination of eigenfunctions for the operator L^*L is itself a big problem (Krasnov, 1975).

4.2. The regularisation method. The class of functions within which the solution is sought is reduced to a sub-class of functions which possess generalized derivatives. Accordingly, the first-order kernel is sought in the Sobolev space $W_2^1\,[0, T_s]$ and the second-order kernel in the Sobolev space $W_2^1\,\{[0, T_s]\times[0, T_s]\}$. We define a norm in the product space W similar to the norm in H as

$$||\mathbf{h}||_W=\left(\int_0^{T_s}\left\{[h_1(r)]^2+\left[\frac{dh_1(r)}{dr}\right]^2\right\}dr\right)^{0.5}+$$
$$+\left(\int_0^{T_s}\int_0^{T_s}\left\{[h_2(r_1, r_2)]^2+\left[\frac{\partial h_2(r_1, r_2)}{\partial r_1}\right]^2+\left[\frac{\partial h_2(r_1, r_2)}{\partial r_2}\right]^2\right\}dr_1\,dr_2\right)^{0.5} \tag{37}$$

which includes the derivatives in such a way that any high frequency components in $\mathbf{h}$ contribute strongly to $||\mathbf{h}||_W$. Because a sphere in W is compact in H the assumption of Tikhonov regularization method is met (Tikhonov, 1963). We propose as a regularizing functional

$$\Omega(\mathbf{h})=||\mathbf{h}||_W \tag{38}$$

and look for the element in W which minimizes the functional

$$J^\alpha(\mathbf{h})=||y-z||_Y^2+\alpha\cdot\Omega(\mathbf{h}) \tag{39}$$

with α as a parameter.

The parameter α in equation (39) is determined experimentally. The sequence $\alpha_k=\alpha_0\,q^k$ $(q>0)$ is constructed, for any α_k one gets corresponding solution $\mathbf{h}_{\alpha k}$ and as the solution of the identification problem we choose that element for which

$$||y-z||_Y\approx\delta, \tag{40}$$

where δ is the error of output measurements.

The regularization method leads to stable approximate solution of the identification problem by the solution of well-posed optimization problem (39).

4.3. The "selection" method. The most popular method of solving ill-posed problems that leads to stable solutions is the selection method. It is widely used in practice. According to this method the solution is searched for in a subset M of the feasible solutions H, arbitrarily adopted by the user. As the approximate solution the element $\mathbf{h}_o$ is taken for which

$$||L\mathbf{h}_o-z||_Y=\inf_{\mathbf{h}\in M}||L\mathbf{h}-z||_Y. \tag{41}$$

Clearly, M is usually a set of elements depending on a finite number of parameters. The most popular solution subsets are discussed below.

4.3.1. *Direct optimization of the ordinates* (M_1). The method based on direct optimization of the ordinates was applied by Diskin and Boneh (1973) to the modelling of surface runoff systems. The discretization method adopted in that work is such that each function in equation (2) is represented by a series of pulses at regular intervals Δt along the time axes. The second-order kernel is represented by an array of pulses on a square grid at the same interval Δt as that used for other functions.

With that method of discretization the integrals in equation (2) are replaced by summations of products. The relationship between the pulses is given by

$$Y(i)=\sum_{k=1}^{N_s} H_1(k)X(i-k)+\sum_{k=1}^{N_s}\sum_{l=1}^{N_s} H_2(k,l)X(i-k)X(i-l), \quad i=1,\ldots,N_T.$$

The discrete variables are derived from the corresponding continuous quantities in equations (2).

Thus the solution space is now a product of two Euclidian spaces $H=R^{N_s}\otimes R^{N_s}\times R^{N_s}$ and the observation space is the Euclidian space R^{N_T}.

We have to determine $0.5\ N_s\,(N_s+3)$ parameters in the first-order kernel and the symmetric second-order kernel. All discussion concerned with the existence and uniqueness of the solution applies directly by substituting the integrals by summation terms. The operator L^*L is now a matrix. The solution is unique if the matrix L^*L is positive and definite and the distribution of eigenvalues determines whether the problem is well- or ill-conditioned. A good measure of that is a condition number, i.e. the ratio of the largest to the smallest eigenvalue. The relative solution error increases with the increase of the condition number. The condition number increases directly with the number of unknown parameters.

Note that the length of the output should be greater than the number of unknown parameters. This requirement can be weakened by additional physical constrains (31).

4.3.2. *Kernel expansion into orthonormal polynomials* (M_2). The most popular approximate method of the identification is the expansion of the kernels into orthonormal polynomials (Amorocho and Brandstetter, 1971; Jacoby, 1966; Kuchment, 1972; Papazafiriou, 1976). The solution is searched for in a subset $-M_2$:

$$h_1(r)=\sum_{i=1}^{N}\alpha_i\,\varphi_i(r), \tag{42a}$$

$$h_2(r_1,r_2)=\sum_{i=1}^{N}\sum_{j=1}^{N}\beta_{ij}\,\varphi_i(r_1)\,\varphi_j(r_2), \tag{42b}$$

where α_i, β_{ij} are the parameters and $\varphi_i\,(r)$ are orthonormal polynomials, usually Laguerre polynomials in the form

$$\varphi_i(t)=\sqrt{a}\,\exp\left(-\frac{at}{2}\right)\sum_{k=1}^{i}\frac{(-1)^k i!}{k!\,k!\,(i-k)!}(at)^k. \tag{43}$$

This method is characterized by a relatively small number of parameters but it arbitrarily assumes a particular structure for the kernels.

4.3.3. *Kernels of Volterra series describing a cascade of non-linear reservoirs* (M_3). All methods of identification of the kernels of Volterra series should be tested and verified. The best way to test an identification procedure is to apply it to a system for which an analytical solution is available. The simplest non-linear model frequently used in hydrology is a cascade of equal non-linear reservoirs with state-space representation as follows (x – inflow, y – outflow, S_i – retention, n – number of reservoirs)

$$\begin{aligned}\dot{S}_1(t)&=-f[S_1(t)]+x(t),\\ \dot{S}_2(t)&=-f[S_2(t)]+f[S_1(t)],\\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{S}_n(t)&=-f[S_n(t)]+f[S_{n-1}(t)],\end{aligned} \tag{44a}$$

$$y(t)=f[S_n(t)]. \tag{44b}$$

We assume that the system is in a steady state, so without loosing generality the zero initial condition is taken $\mathbf{S}(0)=0$. The function $f[\cdot]$ is unknown. We assume only that it can be expanded into Taylor series around 0 for $S_i(t)\leqslant SM$:

$$f[S_i(t)]=aS_i(t)+b[S_i(t)]^2=e(f) \tag{45}$$

where

$$a=\frac{\partial f}{\partial S_i} \quad \text{and} \quad b=0.5\frac{\partial^2 f}{\partial S_i^2}. \tag{46}$$

The vector differential equation can be considered as the definition of a non-linear operator P mapping a space of inflows into a space of corresponding outflows. Accordingly, the behaviour of the model can be approximated by first two terms of the Taylor series for operators, and $S_i(t)$ and $y(t)$ can be divided into linear part, quadratic part and a residual error:

$$S_i(t)=\delta S_i(t)+\delta^2 S_i(t)+e(S_i), \tag{47a}$$

$$y(t)=\delta y(t)+\delta^2 y(t)+e(y). \tag{47b}$$

Substituting the above equations into the vector differential equation (44) and neglecting in the first step the second and higher order terms one obtains the set of equations for the linear approximation as

$$\delta\dot{\mathbf{S}}(t)=a\Phi\delta\mathbf{S}(t)+[1,0,\ldots,0]^T x(t), \tag{48a}$$

$$\delta y(t)=a\delta S_n(t), \tag{48b}$$

where

$$\Phi=\begin{bmatrix}-1 & 0 & \ldots & 0\\ 1 & -1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & -1\end{bmatrix}. \tag{49}$$

When the second-order increments are taken into account we get the additional equations

$$\delta^2\mathbf{S}(t)=a\Phi\delta^2\mathbf{S}(t)+b\Phi[\delta\mathbf{S}(t)]^2, \tag{50a}$$

$$\delta^2 y(t)=a\delta^2 S_n(t)+b[\delta S_n(t)]^2. \tag{50b}$$

The forcing function used for the second-order increment equations is the solution of the equations for the first-order increments. The transition matrix used for both sets of equations (48) and (50) is the same and is given by

$$\exp(a\Phi t)=\mathcal{L}^{-1}\{[pI-a]^{-1}\}, \tag{51}$$

where p is a complex number and I an identity matrix.

Knowing the transition matrix one can calculate $\delta S(t)$ and $\delta y(t)$ and then $\delta^2 S(t)$ and $\delta^2 y(t)$. It can be proved (Napiórkowski and Strupczewski, 1979, 1981) that the first- and second-order approximations of the output trajectory can be regarded as the first and second terms of Volterra series with the kernels

$$h_1(r)=aH_n(r), \tag{52}$$

$$h_2(r_1, r_2)=b\{H_n(r_1)\sum_{k=1}^{n}H_k(r_2)+H_n(r_2)\sum_{k=1}^{n}H_k(r_1)-H_n[\max(r_1, r_2)]\}, \tag{53}$$

where

$$H_n(r)=\frac{(ar)^{n-1}}{(n-1)!}\exp(-ar). \qquad H_m(p)=\frac{a^{m-1}}{(p+a)^m} \tag{54}$$

Equations (52), (53), and (54) determine the subset M_3 based on non-linear cascade. One can see that $h_1(r)$, given by equation (52), is the well known transfer function for a cascade of linear reservoirs. The second-order kernel for the non-linear cascade (44), described by equation (53), meets the condition specified by Diskin and Boneh (1972). The plot of the second-order kernel is shown in Fig. 2. Note that $h_2(r_1, r_2)$ does not possess derivatives

$$\mathcal{L}[t^{m-1}e^{-at}]=\frac{(m-1)!}{(p+a)^m}$$

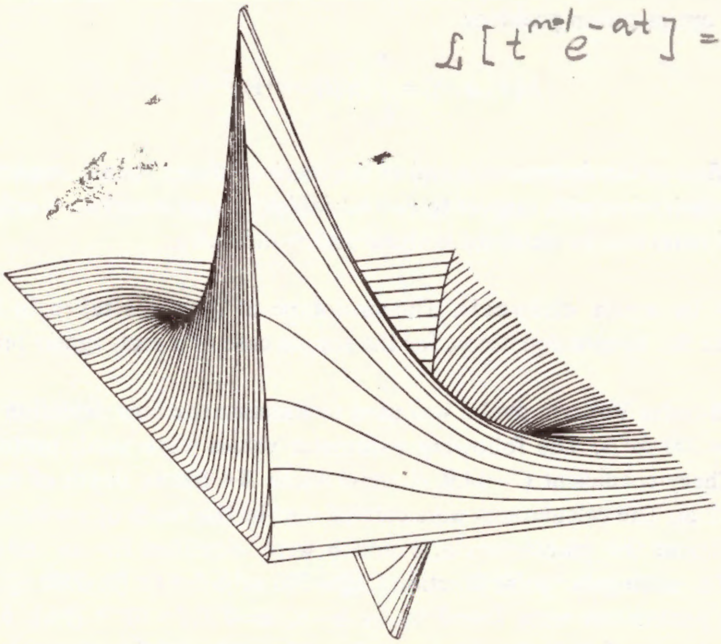


Fig. 2. Dimensionless second-order kernel for $n=3$

along the main diagonal $r_1 = r_2$. Accordingly, the method which expands the kernels into orthonormal polynomials cannot be used, unless additional information about the second-order kernel is taken into account. For example $h_2(r_1, r_2)$ can be approximated by:

$$h_2(r_1, r_2) = \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ij}\, \varphi_i(r_1)\, \varphi_j(r_2) + \sum_{i=1}^{N} \gamma_i\, \varphi_i \max(r_1, r_2). \tag{55}$$

The previous equation (42b) gives good results in the case of systems with non-linear static characteristics.

The details of the approach presented in this chapter and the condition for convergence of the Volterra series based on the nonlinear cascade can be found by Napiórkowski and Strupczewski (1979, 1981). As a practical means of overcoming the problem of determining the range of convergence, Boneh and Golan (1978) derived a criterion for the maximum volume of input which when applied ensures a positive output response to a positive input signals (see also Diskin et al., 1984).

5. A PPLICATIONS OF THE MODEL BASED ON NON-LINEAR CASCADE

In this chapter we present examples which illustrate the applicability of the second-order Volterra model based on non-linear cascade (M_3) to the modelling of flow in open channels and surface runoff systems. In both cases the problem to be solved is to find the best estimates of the parameters n, a and b of the model M_3 given by equations (52), (53), and (54). The optimization problem

$$J(n, a, b) = \int_0^T [z(t) - y(t)]^2\, dt, \tag{56}$$

where $z(t)$ is flow at the downstream end of the reach or direct runoff, respectively, can be reduced to optimization with respect to two variables integer n and $a > 0$ only. Hence, the parameter estimates can be obtained without undue difficulty.

5.1. Flow in open channels. The prototype of flow was taken to be numerical solutions of the St. Venant equations, for the simple case of a rectangular prismatic channel.

The second-order Volterra model was used to simulate the flow deviation from the steady state $Q_o = 200$ m^3/s. The following parameter values were used: bottom slope $I_o = 0.000248$, Chezy coefficient $C = 44.9$, channel width $B = 100$ m, depth of flow in the steady state $y_o = 2$ m. The calculations were carried out for the reach of $x = 40$ km. In the numerical experiment the parameters n, a and b were identified for an inflow increment in the form of a rectangular pulse function ($\Delta Q = 200$ m^3/s for $t \in [0, 6000$ s$]$). The optimal values of the parameters were found to be $n = 6$, $a = 0.217 \times 10^{-3}$ [1/s], $b = 9.4 \times 10^{-12}$ [1/s$\cdot$m^3]. The degree of fit to the prototype by optimized model is shown in Fig. 3.

The model thus obtained was tested by applying an input completely different from

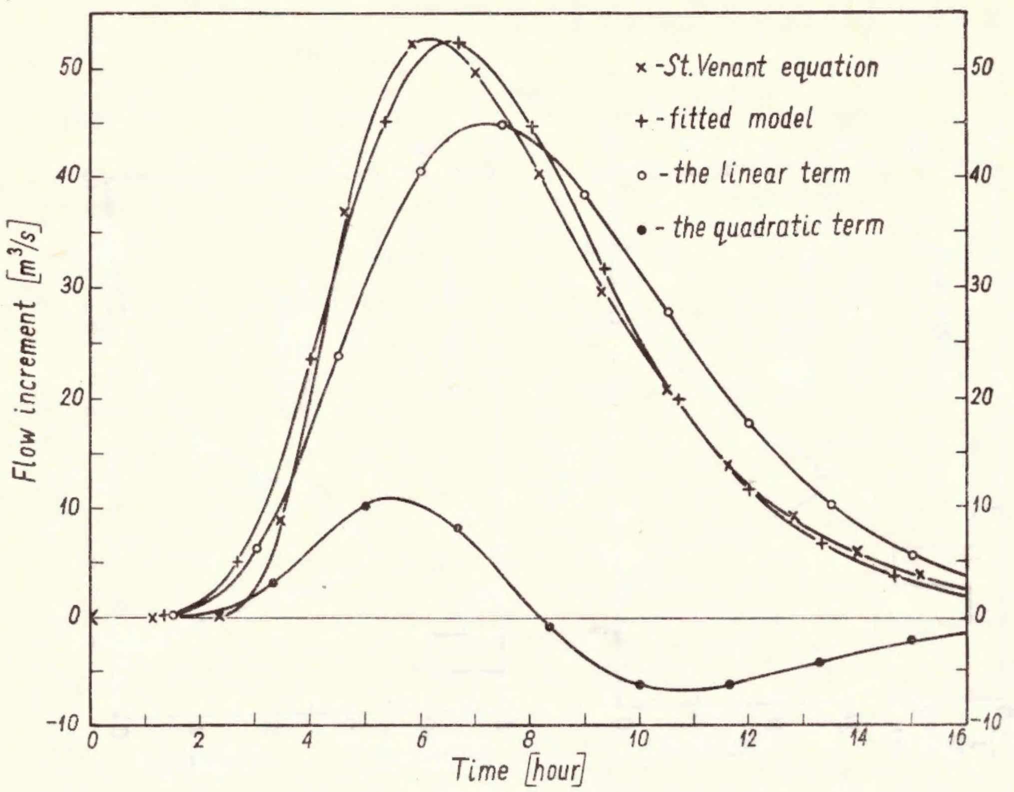


Fig. 3. Identification of the Volterra model at 40 km using a rectangular pulse function

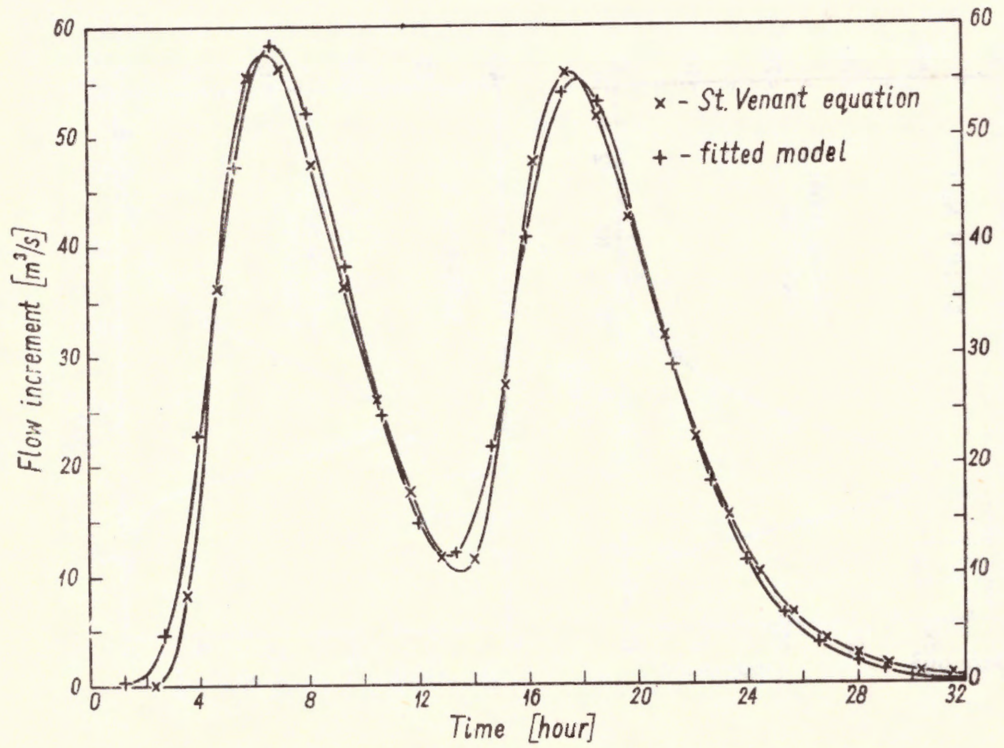
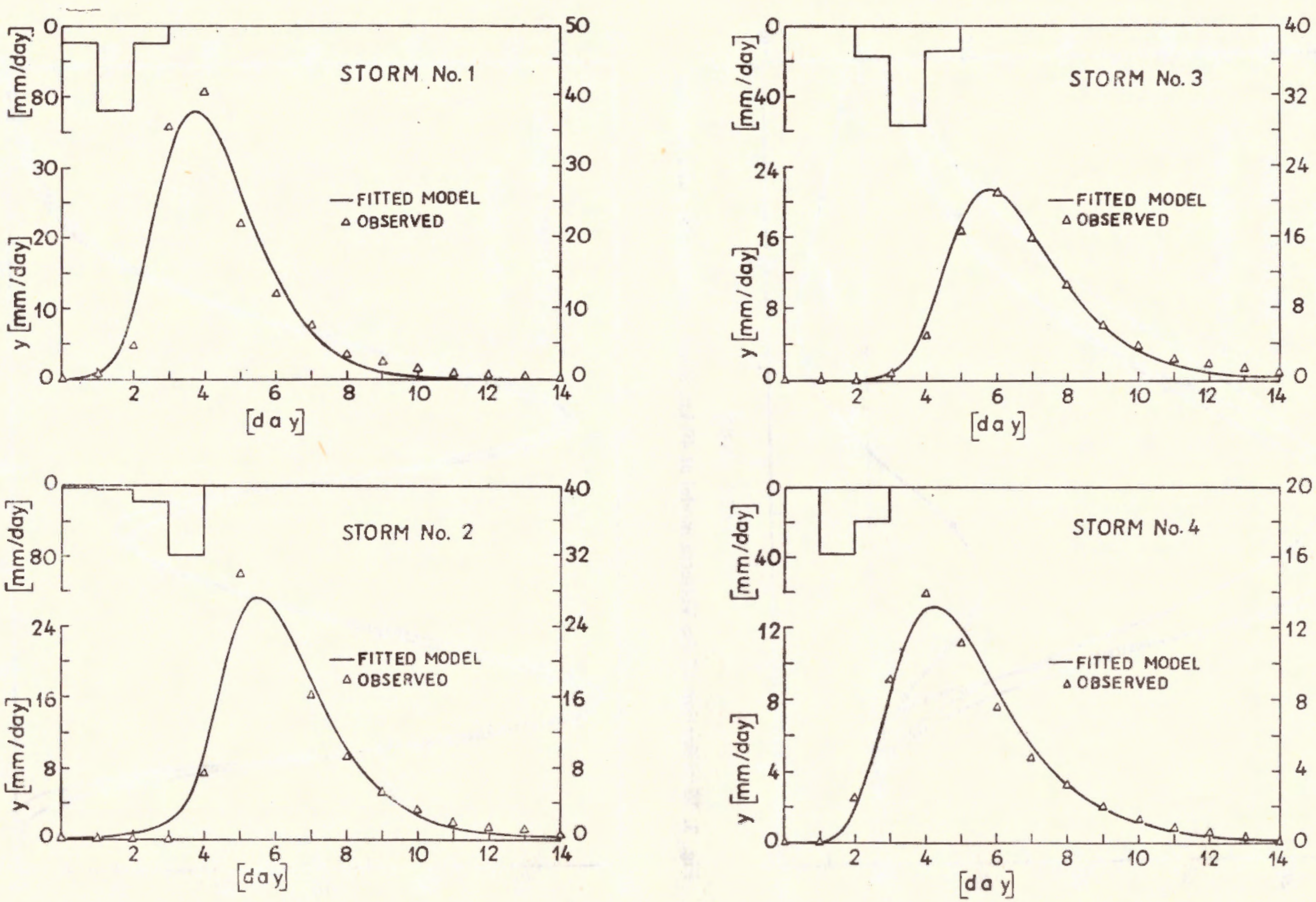


Fig. 4. Simul ation of St. Venant model at 40 km using a smooth input


STORM No. 1
STORM No. 2
STORM No. 3
STORM No. 4
— FITTED MODEL
△ OBSERVED
[mm/day]
y [mm/day]
[day]
0
80
40
10
20
30
50
8
16
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4
12
2
6
14

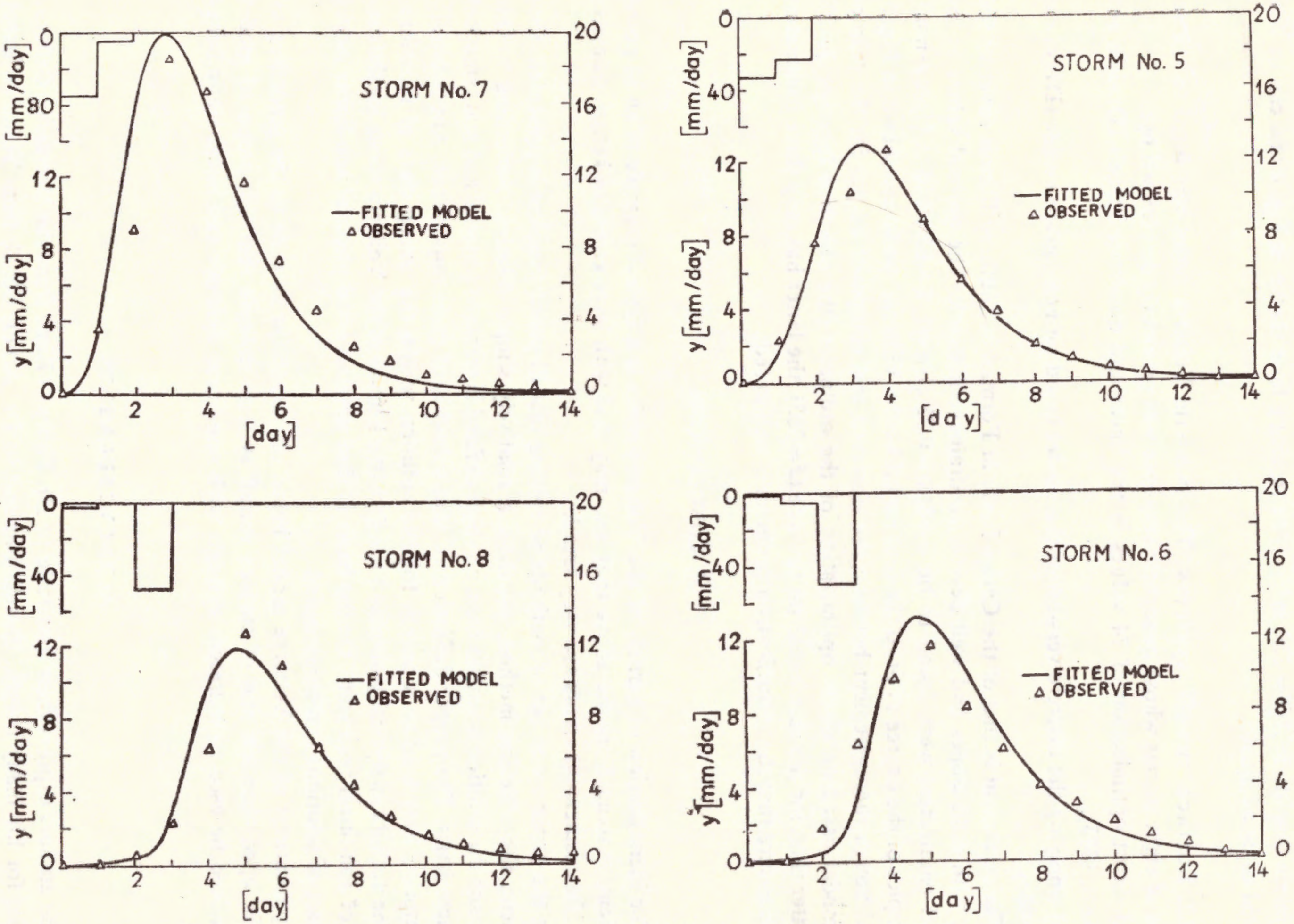

Fig. 5. Comparison of observed runoff and that predicted by the Volterra model

the input used for calibration. The input chosen was the smooth bimodal function with maximum amplitude $\Delta Q'=250$ m^3/s. The resulting fit is shown in Fig. 4 and can be considered to be adequate.

5.2. Surface runoff systems. The two-term Volterra series was fitted to the records of eight storms whose quadratic response was previously determined by:

a) direct optimization of 74 values in the linear and quadratic kernels (Diskin and Boneh, 1973),

b) applying the discrete version of the model based on non-linear cascade (Diskin et al., 1984).

The catchment is that of the Cache River at Forman in Southern Illinois covering 630 km^2, with mild slopes and well developed drainage network. The optimal values of the model parameters were found to be $n=3$, $a=0.75$ [1/day], $b=6.84\times10^{-2}$ [1/day·mm]. Examples of the degree of fit to the real runoff by the Volterra model are shown in Fig. 5.

Although the agreement between the observed and computed output for the case of the model based on direct optimization of the ordinates (the objective function $J=53$) is better than for the case based on cascade ($J=253$), the latter has only three parameters which ensure that the identification problem is well-posed.

6. CONCLUSIONS

The identification of kernels of the Volterra series is a typical example of an ill-posed problem. It stems from the above that one may deal with large errors of kernel estimates even if measurement errors are very small.

In the paper a number of methods of solving that ill-posed problem is presented. One of them, the selection method, provides a relatively simple solution. According to this approach the solution is sought from a subset defined in terms of physical and mathematical characteristics. The proper definition of that subset is of great importance. Any arbitrary assumption may lead to a search for the solution within the wrong class of functions.

The authors analytically obtained the first two kernels of the Volterra series for the cascade of non-linear reservoirs. On the basis of that simplest quasi-physical non-linear model a subset of solutions is recommended.

Applications of the Volterra model based on non-linear cascade to the modelling of flow in open channels and of surface runoff systems indicate that the proposed model can indeed be used to represent systems with non-linear dynamic and linear static behaviour.

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PROBLEMY ZWIĄZANE Z IDENTYFIKACJĄ JĄDER SZEREGU VOLTERRY

Streszczenie

W artykule omawia się modelowanie matematyczne procesów hydrologicznych za pomocą dwóch pierwszych składników szeregu Volterry. Dyskutowane jest istnienie i jednoznaczność rozwiązania zadania identyfikacji. Podano przykłady zastosowania szeregu Volterry do modelowania przepływu w korytach otwartych i procesu opad — odpływ.