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THE DETERMINISTIC IDENTIFICATION OF KERNELS OF THE VOLTERRA SERIES DESCRIBING THE FLOW IN AN OPEN CHANNEL

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A b s t r a c t

While identifying the kernels of the Volterra series we encounter the problem of determination of a subset within which the solution is searched. In the paper the first two kernels of the Volterra series are identified.

This series describes the flow in an open channel. The structure of the kernels stems from the physical description of the process being modelled. When orthogonal polynomials are used the identification problem reduces to the problem of solving the set of linear equations. The stability of solution depending upon the type of an inflow signal is investigated. In the numerical example the Laguerre orthogonal polynomials were applied.

1. Introduction

An optimal identification of the model of flow in an open channel carried out in terms of input-output data is usually based on the assumption that a structure of the model is linear. The straightforward and more comprehensive approach is the use of the Volterra series in which the first term reflects the linear properties of the object, the second term - the quadratic properties, etc.

The problem of great importance is to determine the structure of the kernels of individual terms of the Volterra series that would correspond to the physical object being modelled. In order to achieve this goal the dynamic wave equations were reduced to the simple form of a model with lumped parameters, namely to the cascade of nonlinear reservois (K a l i n i n and M i l y u k o v, 1957). The above model was used to determine the structure of the kernels. The subject of this paper is the identification of the first two kernels of the Volterra series by the least squares method.

2. The mathematical definition of the problem

The problem in question is the deterministic identification of the system. In our case this means the determination of the first two kernels of the integral Volterra series that models the flow deviations from a steady state in an open channel

$$y_m(t) = \int_0^{T_s} h_1(\tau)\, \delta x(t-\tau)\, d\tau + \int_0^{T_s}\int_0^{T_s} h_2(\tau_1,\tau_2)\, \delta x(t-\tau_1)\, \delta x(t-\tau_2)\, d\tau_1\, d\tau_2 \qquad (1)$$

where: $\delta x(t)$ is the flow deviation from a steady state; $h_1(t)$, $h_2(t_1,t_2)$ are the kernels of the series; $y_m(t)$ is the outflow from the model; T_s is the model memory.

The model of a flow in an open channel has the form of a cascade of nonlinear reservoirs described by the nonlinear state equation

$$\begin{aligned}
\dot{S}_1(t) &= - f[S_1(t)] + x(t) \\
\dot{S}_2(t) &= - f[S_2(t)] + f[S_1(t)] \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{S}_n(t) &= - f[S_n(t)] + f[S_{n-1}(t)]
\end{aligned} \qquad (2)$$

$$y(t) = + f[S_n(t)]; \quad \underline{S}(0) = \underline{S}_0$$

where: $x(t)$ is the inflow into the cascade, $y(t)$ is the outflow from the cascade, $S_i(t)$ is the retention of the i-th reservoir.

The basic problem of identification we are going to discuss in this paper is shown in Fig. 1.

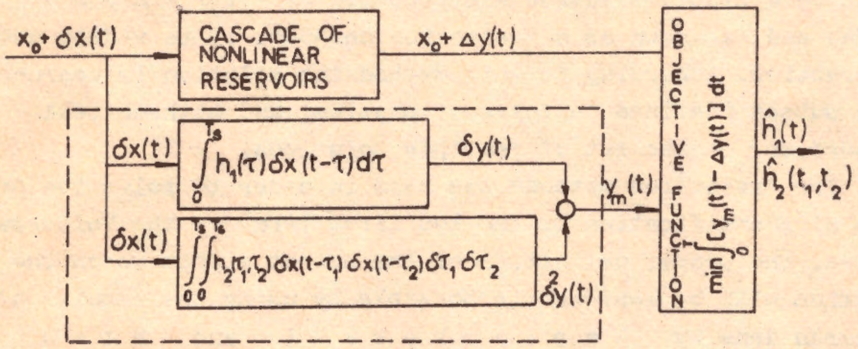

Fig. 1. The identification of the first two kernels of the Volterra series describing the flow deviations from a steady state.

The problems to be solved is to find out the best (in terms of mean square errors) estimates $h_1(t)$ and $h_2(t_1,t_2)$ on the basis of records of the inflow increment $\delta x(t)$ and the outflow increment $\Delta y(t)$ observed in a finite time interval.

So we are looking for the estimates minimizing

$$Q(h_1, h_2) = \int_0^T [\Delta y(t) - y_m(t)]^2\, dt \ , \tag{3}$$

where T is the time the input is observed; the observation starts at point zero, $T > T_s$; $[-T_s, T]$ is the time the output is observed.

3. <u>The solution of the identification problem by a "selection method"</u>

It was shown by N a p i ó r k o w s k i (1978) that the identification of kernels of the Volterra series is a typical example of an ill conditioned problem. In order to see that, the discussed identification problem was reduced to the Wiener-Hopf equation. It stems from the above that one may deal with the large errors of estimates $h_1(t)$, $h_2(t_1, t_2)$ even if the input and output measurement errors are very small.

The method of solving the ill conditioned problems that leads to a stable solution was proposed by T i k h o n o v (1974) and is known as a "selection method". It is widely used in practice. According to this method the solution is searched in a subset (defined in terms of physical and mathematical properties) of the set of feasible solutions.

The "selection method" was used in order to solve the problem of identification of the two first terms of the Volterra series. The proper definition of the subset within which the solution will be searched is possible by using the results of a research done by N a p i ó r k o w s k i and S t r u p - c z e w s k i (1979). On that basis, the following structure of the kernels was assumed

$$h_1(t) = \sum_{i=1}^{n} a_i \varphi_i(t), \tag{4}$$

$$h_2(t_1,t_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \varphi_i(t_1) \varphi_j(t_2) + \sum_{i=1}^{n} a_i \varphi_i [\max(t_1,t_2)] \tag{5}$$

where $\varphi_i(t)$ is the sequence of orthogonal functions in $L^2[0,T_s]$, a_i, a_{ij}, b_i are the parameters.

It must be emphasized here that the proper definition of the subset within which the approximate solution is searched is of great importance when it comes to the correct formulation of the identification problem. Any arbitrary assumption (without a physical and mathematical analysis) may lead to a search of the solution within the class of functions that is significantly different from the class it actually belongs to. This implies that the derived solution of the identification problem may have nothing to do with an optimal one.

Let

$$q_i(t) = \int_0^{T_s} \varphi_i(\tau)\delta x(t-\tau)\,d\tau$$

and

$$p_i(t) = \int_0^{T_s}\int_0^{T_s} \varphi_i[\max(\tau_1, \tau_2)]\, \delta x(t-\tau_1)\ \delta x(t-\tau_2)\ d\tau_1\ d\tau_2 .$$

Then the problem of identification by the "selection method" of kernels of the Volterra series describing a cascade of non-linear reservoirs can be reduced to the minimization of the following expression:

$$Q(h_1,h_2) = \int_0^T [\Delta y(t) - \sum_{i=1}^{n} a_i q_i(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} q_i(t) q_j(t) + - \sum_{i=1}^{n} b_i p_i(t)]^2\, dt . \qquad (6)$$

The necessary condition for a minimum of (6) is that the expansion coefficients a_i, a_{ij}, b_i; $i,j = 1, \ldots , n$ satisfy the following set of linear equations.

$$\begin{array}{l}
a_1 \langle q_1,q_1\rangle + a_2 \langle q_1,q_2\rangle + \ldots + b_n \langle q_1,p_n\rangle = \langle q_1, \Delta y\rangle \\
a_1 \langle q_2,q_1\rangle + a_2 \langle q_2,q_2\rangle + \ldots + b_n \langle q_2,p_n\rangle = \langle q_2, \Delta y\rangle \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1 \langle q_i q_j,q_1\rangle + a_2 \langle q_i q_j,q_2\rangle + \ldots + b_n \langle q_i q_j,p_n\rangle = \langle q_i q_j, \Delta y\rangle \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1 \langle p_n,q_1\rangle + a_2 \langle p_n,q_2\rangle + \ldots + b_n \langle p_n,p_n\rangle = \langle p_n, \Delta y\rangle
\end{array} \qquad (7)$$

where the scalar products are computed in $L^2\,[0, T]$

The problems of identification of kernels of the first two terms of the Volterra series by the "selection method" can be reduced to the problem of solving the set of linear equations

$$A\, \underline{z} = \underline{b} . \qquad (8)$$

4. The impact of the type of the input signal on the stability of the solution of the identification problem

The basic requirement that must be satisfied by any identification method is the stability of the solution including the case when we deal with measurement errors. In order to meet this condition while identifying the kernels of the Volterra series the Tikhonov "selection method" was used. Recall that the Volterra series in question describes the cascade of nonlinear reservoirs on the basis of finite input and corresponding output records. Using the "selection method" it is possible to reduce the identification problem expressed in terms of an infinite number of dimensions into the algebraic set linear equations (7). The matrix of this set depends directly on the input signal $\delta x(t)$ and the stability of the solution of these equations depends not only on the measurement accuracy but also on the type of input variability. The enclosed in this paragraph theoretical analysis of what causes the errors in a solution made it possible to formulate the criterion for a selection of such a pair of input-output data from long-term recordings that the set of linear equations is well conditioned.

It should be emphasized that the stability of solution of the identification problem of kernels of the Volterra series in the case of measurement errors is ensured only after two conditions are satisfied, namely:

1) the subset within which the solution is searched is well defined,

2) the algebraic set of linear equations is well conditioned.

We will discuss below what causes the errors in a solution of the identification problem when it is reduced to the set of linear equations of form (8).

Let the errors of matrix A and vector b be δA and δb, respectively. Then eq. (8) becomes

$$(A + \delta A)(\underline{z} + \delta \underline{z}) = \underline{b} + \delta \underline{b} \qquad (9)$$

We assume that A and $(A + \delta A)$ are nonsingular.

Denote

$$\alpha = \frac{\|\delta A\|}{\|A\|}, \qquad \xi = \frac{\|\delta \underline{z}\|}{\|\underline{z}\|}, \qquad \beta = \frac{\|\delta \underline{b}\|}{\|\underline{b}\|}.$$

Since Az = b, eq. (9) can be simplified to

$$A\,\delta\underline{z} + \delta A\,\delta\underline{z} = \delta\underline{b} - \delta A\underline{z} \tag{10}$$

The norm of the left-hand side of the above equation is estimated under the assumption that the errors in coefficients are smaller than the coefficients themselves.

$$\|A\,\delta\underline{z} + \delta A\,\delta\underline{z}\| \geq \|A\,\delta\underline{z}\| - \|\delta A\,\delta\underline{z}\| \geq \|A^{-1}\|^{-1}\,\|\delta\underline{z}\| - \|\delta A\|\,\|\delta\underline{z}\|.$$

Then we estimate the norm of the right-hand side of eq.(10)

$$\|\delta\underline{b} - \delta A\underline{x}\| \leq \|\delta\underline{b}\| + \|\delta A\|\cdot\|z\|.$$

We conclude that

$$\|A^{-1}\|^{-1}\,\|\delta\underline{z}\| - \|\delta A\|\,\|\delta\underline{z}\| \leq \|\delta\underline{b}\| + \|\delta A\|\cdot\|\underline{z}\|. \tag{11}$$

We multiply both sides of inequality (11) by $\dfrac{\|A^{-1}\|}{\|\underline{z}\|}$

$$\frac{\|\delta\underline{z}\|}{\|\underline{z}\|} - \|A^{-1}\|\cdot\|\delta A\|\cdot\frac{\|\delta\underline{z}\|}{\|\underline{z}\|} \leq \|A^{-1}\|\,\frac{\|\delta\underline{b}\|}{\|\underline{z}\|} + \|A^{-1}\|\cdot\|\delta A\| \tag{12}$$

Using the symbols introduced earlier we have

$$\|A^{-1}\|\cdot\|\delta A\| = \|A^{-1}\|\cdot\|A\|\cdot\alpha = \gamma(A)\cdot\alpha, \tag{13}$$

where $\gamma(A)$ is a condition number of the matrix, e.g. the ratio of the greatest to the smallest absolute eigen values.

$$\frac{\|\delta\underline{b}\|}{\|\underline{z}\|} = \frac{\|\delta\underline{b}\|}{\|\underline{b}\|}\cdot\frac{\|A\underline{z}\|}{\|\underline{z}\|} = \beta\cdot\frac{\|A\underline{z}\|}{\|\underline{z}\|} \leq \beta\|A\| \tag{14}$$

Substitution of (13) and (14) into (12) gives

$$\xi - \|A^{-1}\|\,\|\delta A\|\,\xi \leq \gamma(A)\cdot(\alpha+\beta)$$

so

$$(1 - \gamma(A)\cdot\alpha) \leq \gamma(A)\cdot(\alpha+\beta).$$

By assuming $\gamma(A) \cdot \alpha \leqslant 1$ we have

$$\xi \leqslant \frac{\gamma(A)}{1 - \gamma(A)} (\alpha + \beta) .$$

We obtained the following estimate of the relative solution error

$$\frac{\|\delta \underline{z}\|}{\|\underline{z}\|} \quad \frac{\gamma(A)}{1 - \gamma(A) \frac{\|\delta A\|}{\|A\|}} \left(\frac{\|\delta \underline{b}\|}{\|\underline{b}\|} + \frac{\|\delta A\|}{\|A\|} \right).$$

The above estimate is usually a pesimistic one, if the right-hand side of (13) is not exactly known.

There is no precise definition that would make it possible to judge when the system is well conditioned and when it is not However, it stems from (13) that,

1) the relative solution error increases with the increase of condition number $\gamma(A)$,

2) the solution error is a linear increasing function of δA and $\delta \underline{b}$.

So, if we have several input-output flood hydrographs at our disposal and we want to identify the kernels of the Volterra series we should choose the pair of records for which the set of linear equations is well conditioned. The criterion that should be applied here is the minimum of the ratio of the greatest to smallest absolute eigen values of the matrix of the set of linear equations (7).

Obviously, "variable input signals" in a form similar to the Dirac delta function (e.g. rectangular pulse function) will lead to the well conditioned set of linear equations. It is so because these signals cause an increase in the diagonal elements and minimize the nondiagonal elements (for the Dirac delta function $\langle q_i, q_j \rangle = 0$, $i \neq j$). On the other hand, the "invariable input signals" may lead to the ill conditioned set of equations. In a particular case of a constant function all columns of the matrix of the set of eqs. (7) are lineary dependent and the identification problem does not have a unique solution.

5. <u>The numerical example</u>

We will present below an example illustrating the applicability of the suggested method. The point is to solve the problem of identification of the first two kernels of the Volterra series describing the flow deviations from a steady state in a rectangular uniform channel. This channel is modelled by the nonlinear state eq. (2).

The steady flow in a channel and the channel itself were charactericed by the following parameters: flow $x_o = 200\ m^3/s$, depth $h_o = 2$ m, velocity $v_o = 1$ m/s, bottom slope I = 0.000248, width B = 100 m, Chezy coefficient c = 44.9, length of a reach interpreted as one reservoir $L = 2/3\ h_o/I = 5376$ m (according to K a l i n i n and M i l y u k o v, 1957). The nonlinear function appearing on the right-hand sides of (2) was as follows

$$f[S(t)] = c \frac{I^{1/2}}{B^{1/2}\ L^{3/2}} S^{3/2}(t) \ .$$

It was our intention to assume a rectangular uniform channel for only in such a case it was possible to compare the identification results with the results derived analytically (N a p i ó r - k o w s k i and S t r u p c z e w s k i, 1979).

In order to ensure that the set of linear eqs. (7) is well condition the inflow increment was assumed as rectangular pulse function.

The outflow from a cascade of nonlinear reservoirs for the input like that was calculated by means of the Runge-Kutta method. The standard IBM procedure was used.

The orthonormal Laguerre polynomials were taken as orthonormal functions

$$\varphi_n(t) = \sqrt{\alpha}\ e^{-\frac{\alpha t}{2}} \sum_{i=0}^{n} \frac{(-1)^i\ n!}{i!\ i!\ (n-1)!} (\alpha t)^i; \quad n = 0,1,\ldots$$

The model memory T_s was chosen in such a way, that

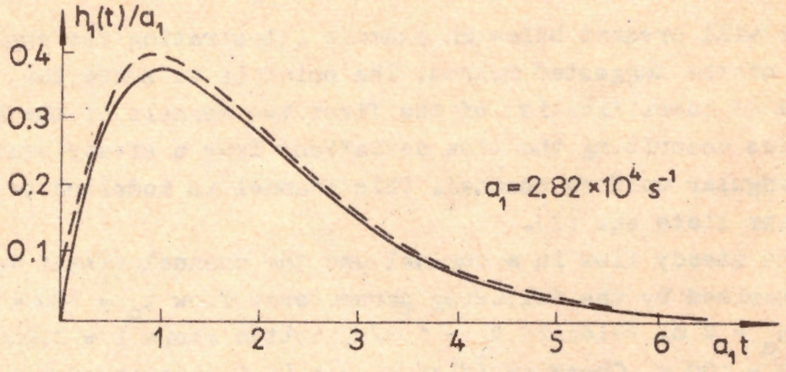

Fig. 2. Actual —— and computed – – first order kernels. Diagram is plotted in terms of the dimensionless variables $a_1 t$ and h_1/a_1.

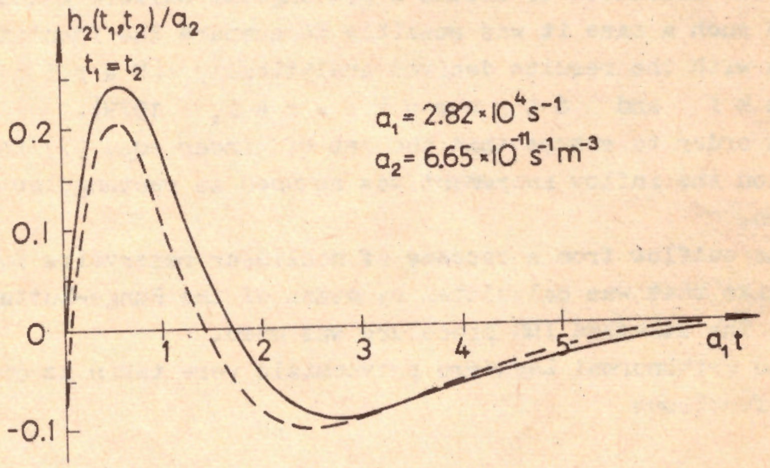

Fig. 3. Actual —— and computed – – second order kernels. Diagram is plotted in terms of the dimensionless variables $a_1 t$ and h_2/a_2.

$$\int_0^{T_s} \varphi_m(t) \cdot \varphi_n(t)\, dt = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

The calculations were carried out for the cascade of the nonlinear reservoirs. During the identification process there were determined the parameters of the set of linear equations and the parameter α of the Laguerre polynomials. The obtained results are presented in Figs. (2) and (3).

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DETERMINISTYCZNA IDENTYFIKACJA JĄDER SZEREGU VOLTERRY
OPISUJĄCEGO PRZEPŁYW W KORYCIE OTWARTYM

S t r e s z c z e n i e

Przy identyfikacji jąder szeregu Volterry napotykamy problem określenia podzbioru, w którym poszukiwane jest rozwiązanie. W pracy identyfikowane są dwa pierwsze jądra szeregu Volterry, opisującego przepływ w korycie otwartym. Struktura ich wynika z opisu fizycznego modelowanego procesu. Przy zastosowaniu wielomianów ortonormalnych zadania identyfikacji sprowadzono do rozwiązania układu równań liniowych. Zbadano wpływ charakteru sygnału wejściowego na stabilność rozwiązania. W przykładzie numerycznym zastosowano wielomiany ortogonalne Laguerree'a.