Journal of Hydrology, 52 (1981) 185—198
Elsevier Scientific Publishing Company, Amsterdam — Printed in The Netherlands

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THE PROPERTIES OF THE KERNELS OF THE VOLTERRA SERIES DESCRIBING FLOW DEVIATIONS FROM A STEADY STATE IN AN OPEN CHANNEL

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(Received October 10, 1979; revised and accepted September 2, 1980)

ABSTRACT

Napiórkowski, J.J. and Strupczewski, W.G., 1981. The properties of the kernels of the Volterra series describing flow deviations from a steady state in an open channel. J. Hydrol., 52: 185—198.

The deviation of the flow from a steady state in an open channel is described by a nonlinear state equation. This model is used to derive analytically the kernels of the Volterra series. The properties and the structure of the two first kernels are examined. The condition of convergence of the Volterra series depending on the magnitude of the inflow increase is also discussed.

INTRODUCTION

Solution of the equations of unsteady flow in open channels requires the geometric and hydraulic characteristics of the channel together with the initial and boundary conditions. The difficulties in meeting these requirements and the desire to use a computationally simple, but fairly accurate method, explains why conceptual models and models of the "black box" type have been developed.

In the 1960's the class of linear models was developed, whereas lately the problem of nonlinear models was undertaken. This paper deals just with the nonlinear models and specifically with the modelling of the deviations of the flow from the steady state in an open channel by the Volterra series:

$$y(t) = \int_0^t h_1(\tau)x(t-\tau)\mathrm{d}\tau + \int_0^t\int_0^t h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \ldots$$

$$+ \int_0^t \cdots \int_0^t h_n(\tau_1,\ldots,\tau_n)x(t-\tau_1)\ldots x(t-\tau_n)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n + \ldots \quad (1)$$

Modelling by the Volterra series has become very popular and has devel-

0022-1694/81/0000—0000/$02.50

oped independently of other methods by which the dynamic systems can be described. The problem of series identification was attempted only on the basis of the input and corresponding output records. The method used was to expand the kernels in orthogonal functions (see Jacoby, 1966; Amorocho and Brandstetter, 1971; Kuchment, 1972; Papazafiriou, 1976). It was shown by Napiórkowski (1978) that this procedure can lead to searching for the kernels within a class of functions which is significantly different from the class to which they really belong. Thus the possibility can not be excluded that the derived solution hardly reflects the reality.

The analytical derivation of the kernels of the Volterra series is in our opinion not just another methodological approach. The determination of the structure of the kernels in such a way that the convergence can be examined may be a decisive factor in further application of the Volterra series to modelling and identification of hydrological systems.

THE MODELLING OF THE FLOW DEVIATION FROM A STEADY STATE IN AN OPEN CHANNEL BY MEANS OF STATE EQUATION

Consider the uniform channel of width B and bed slope I, divided in length into n equal reaches. The length L of the reaches is determined according to the method of Kalinin and Miljukov (1957), who introduced the concept of the "characteristic reach". The retention (S) of such a reach and outflow (y) from it, are the functions of the water depth (h) in its center:

$$y = f(h) \qquad \text{and} \qquad S = LBh \tag{2), (3}$$

Kalinin and Miljukov derived the length of the reach from the listed properties, under the assumption of a linear change of the water table height along the channel length. So the length of the characteristic reach is such that the linear increase in the slope of the water level is compensated by the linear increase of the water depth and as a result the flow remains unchanged.

The nonlinear relationship (2) can be derived in an experimental way. We assume here it is of the form:

$$y = \alpha h^m \tag{4}$$

Eq. 4 becomes the Chézy equation for $m = 1.5$, $\alpha = CI^{0.5}B$ and the Manning equation for $m = 5/3$, $\alpha = n_M^{-1} I^{0.5} B$, where C is Chézy's coefficient and n_M is Manning's coefficient.

Let $x_i(t)$ be the inflow, and $y_i(t)$ the outflow for the ith reach. Then the changes in retention can be derived by solving simultaneously the continuity equation:

$$\dot{S}_i(t) = x_i(t) - y_i(t) \tag{5}$$

and the outflow equation resulting from eqs. 3 and 4:

$$y_i(t) = aS_i^m(t) \qquad \text{and} \qquad a = \alpha L^{-m} B^{-m} \tag{6}$$

under the initial condition $S_0 = LBh_0$ corresponding to the retention of the channel reach in a steady state.

Substituting eq. 6 for $y_i(t)$ in eq. 5 and putting $x_i(t) = y_{i-1}(t)$, $i = 2, \ldots, n$; $x_1(t) = x(t)$, $y_n(t) = y(t)$ gives the model of flow deviations from a steady state in an open channel. This model is of the form of state equation:

$$\left.\begin{aligned} \dot{S}_1(t) &= -aS_1^m(t) + x(t) \\ \dot{S}_2(t) &= -aS_2^m(t) + aS_1^m(t) \\ \ldots &= \ldots \\ \dot{S}_n(t) &= -aS_n^m(t) + aS_{n-1}^m(t) \\ y(t) &= aS_n^m(t), \qquad S_i(0) = S_0, \qquad i = 1, \ldots, n \end{aligned}\right| \tag{7}$$

THE ANALYTICAL DETERMINATION OF THE KERNELS OF THE VOLTERRA SERIES DESCRIBING THE FLOW DEVIATIONS FROM A STEADY STATE

We proceed now to determine analytically the kernels of the Volterra series (1) on the basis of a nonlinear state equation (7). As a result we obtain the model of deviation from a steady state flow in an open channel. The model is in the form of an integral series.

Vector state equation (7) can be considered as a definition of some nonlinear operator P which maps the space of inflows into the space of corresponding retentions. In order to determine how P operates for a given inflow $x(t)$, $t \in [0, +\infty)$ it is necessary to solve the set of eqs. 7 under the initial condition $\underline{S}_0$. Eqs. 7 can be written symbolically as follows:

$$\underline{S}(t) = [P_{\underline{S}_0} x](t) \tag{8}$$

Denote by $\underline{S}^0(t) = \text{constant} = \underline{S}_0$, the trajectory of retentions corresponding to the inflow $x(t) = \text{constant} = x_0$ in a steady state and by $\underline{S}(t)$ the trajectory corresponding to the inflow $x(t) = x_0 + \delta x(t)$; [$\delta x(t)$ represents the inflow deviation from a steady state]. The change of state trajectory from $\underline{S}^0(t)$ to $\underline{S}(t)$ can be determined by means of Taylor formula for operators (Findeisen et al., 1977).

$$\begin{aligned}[P_{\underline{S}_0} x](t) - \underline{S}_0 &= [P_{\underline{S}_0} x_0, \delta x](t) + \tfrac{1}{2}[P_{\underline{S}_0} x_0, \delta x^2](t) \\ &\quad + \ldots + \frac{1}{i!}[P_{\underline{S}_0} x_0, \delta x^i](t) + \ldots \end{aligned} \tag{9}$$

where

$$[P_{\underline{S}_0} x_0, \delta x](t) = \delta \underline{S}(t)$$

which is the linear part of a state trajectory increment.

$$\tfrac{1}{2}[P_{\underline{S}_0} x_0, \delta x^2](t) = \delta^2 \underline{S}(t)$$

which is the quadratic part of a state trajectory increment. So the change in the inflow hydrograph from x_0 into $x(t)$ implies the trajectory change from $\underline{S}_0$ into:

$$\underline{S}(t) = \underline{S}_0 + \Delta\underline{S}(t) \tag{10}$$

where

$$\Delta\underline{S}(t) = \delta\underline{S}(t) + \delta^2\underline{S}(t) + \delta^3\underline{S}(t) + \ldots \tag{11}$$

It will be proved below, by computing $\delta S(t)$, $\delta^2\underline{S}(t)$, in this order, that the linear part of the increment of the retention trajectory is represented by the first term of the Volterra series (1), and the quadratic part by its second term.

In order to compute the linear and quadratic increments we will make use of:

(1) The expansion of the retention—outflow relation (2) in Taylor series around the trajectory of a steady state S_{i0} for the increase $\Delta S_i(t), i = 1, \ldots, n$

$$\begin{aligned} y_i(t) &= a[S_{i0} + \Delta S_i(t)]^m = \\ &= aS_{i0}^m + maS_{i0}^{m-1}\Delta S_i(t) + \tfrac{1}{2}m(m-1)aS_{i0}^{m-2}[\Delta S_i(t)]^2 \\ &\quad + \tfrac{1}{6}m(m-1)(m-2)aS_{i0}^{m-3}[\Delta S_i(t)]^3 + \ldots \\ &= y_0 + a_1\Delta S_i + a_2(\Delta S_i)^2 + a_3(\Delta S_i)^3 + \ldots \end{aligned} \tag{12}$$

where

$$a_k = m(m-1)(m-2)\ldots(m-k+1)aS_{i0}^{m-k} \quad \text{and} \quad k = 1, 2, \ldots$$

The above expansion converges if $|\Delta S_i| < S_{i0}$

(2) The expansion in Taylor series of the time derivative of the retention around the trajectory S_{i0}, viz.:

$$\dot{S}_i(t) = \delta\dot{S}_i(t) + \delta^2\dot{S}_i(t) + \delta^3\dot{S}_i(t) + \ldots \tag{13}$$

The function $\delta\underline{S}(t)$ — the linear approximation

Substitution of eqs. 12 and 13 limited to the first-order increments only in eqs. 7 yields the set of equation from which the linear part of an increment of the state trajectory can be determined:

$$\begin{aligned} \delta\dot{S}_1(t) &= -a_1\,\delta S_1(t) + x(t) \\ \delta\dot{S}_2(t) &= -a_1\,\delta S_2(t) + a_1\,\delta S_1(t) \\ \ldots &= \ldots \\ \delta\dot{S}_n(t) &= -a_1\,\delta S_n(t) + a_1\,\delta S_{n-1}(t) \\ \delta\underline{S}(0) &= \underline{0} \end{aligned} \tag{14}$$

The zero initial condition is implied by the lack of changes in the initial condition $\underline{S}_0$ of eqs. 7.

In a matrix notation eqs. 14 can be written as follows:

$$\dot{\underline{S}}(t) = \underline{A}\delta\underline{S}(t) + [1,0,\ldots,0]^{\mathrm{T}}\,\delta x(t) \tag{15}$$

where

$$\underline{A} = \begin{bmatrix} -a_1 & 0 & \dots & 0 \\ a_1 & -a_1 & \dots & 0 \\ \cdot & & & \\ \cdot & & \dots & \\ \cdot & & & \\ 0 & 0 & \dots & -a_1 \end{bmatrix}$$

The above linear stationary set of ordinary differential equations has the solution:

$$\delta \underline{S}(t) = \int_0^t \underline{\Phi}(\tau) \times [1,0,\dots,0]^{\mathrm{T}} \delta x(t-\tau) \mathrm{d}\tau \tag{16}$$

where $\underline{\Phi}(t)$ is the state-transition matrix for eq. 15 (Athans and Falb, 1968).

In order to obtain the matrix $\underline{\Phi}(t)$ it is necessary to obtain the inverse Laplace transform of the matrix $(p\underline{I} - \underline{A})^{-1}$. Matrix $\underline{A}$ is defined in eq. 15, $\underline{I}$ is an identity matrix and p is a complex number.

Following these directions we obtain the following state-transition matrix for the set of equations (15):

$$\left[\begin{array}{c:c:c:c} \mathrm{e}^{-a_1 t} & 0 & \dots & 0 \\ \hdashline (a_1 t)\mathrm{e}^{-a_1 t} & \mathrm{e}^{-a_1 t} & \dots & 0 \\ \hdashline \cdot & & & \\ \cdot & & \dots & \\ \cdot & & & \\ \hdashline \dfrac{(a_1 t)^{n-1}}{(n-1)!}\mathrm{e}^{-a_1 t} & \dfrac{(a_1 t)^{n-2}}{(n-2)!}\mathrm{e}^{-a_1 t} & \dots & \mathrm{e}^{-a_1 t} \end{array}\right] = e^{\underline{A}t} = \underline{\Phi}(t) \tag{17}$$

Having the solution for the state-transition matrix we conclude that the linear increment of a retention in ith reach of a channel can be determined according to the formula:

$$\delta S_i(t) = \int_0^t \frac{(a_1\tau)^{i-1}}{(i-1)!} \mathrm{e}^{-a_1\tau} \delta x(t-\tau)\mathrm{d}\tau = \int_0^t k_1(\tau)\delta x(t-\tau)\mathrm{d}\tau \tag{18}$$

Eq. 18 is of the form of the first term of the Volterra series (1).

The function $\delta^2\underline{S}(t)$ — the quadratic approximation

Substitution of eqs. 12 and 13 limited to the second-order increments in eqs. 7 and subtracting both sides of eqs. 14 from eqs. 7 gives a relation describing the quadratic part of the increment of the storage trajectory:

$$\left.\begin{aligned}
\delta^2\dot{S}_1(t) &= -a_1\delta^2 S_1(t) - a_2[\delta S_1(t)]^2\\
\delta^2\dot{S}_2(t) &= -a_1\delta^2 S_2(t) + a_1\delta^2 S_1(t) - a_2[\delta S_2(t)]^2 + a_2[\delta S_1(t)]^2\\
\dots &= \dots\\
\delta^2\dot{S}_n(t) &= -a_1\delta^2 S_n(t) + a_1\delta^2 S_{n-1}(t) - a_2[\delta S_n(t)]^2 + a_2[\delta S_{n-1}(t)]^2\\
\delta^2\underline{S}(0) &= \underline{0}
\end{aligned}\right| \tag{19}$$

In a matrix notation set (19) can be expressed as

$$\delta^2\dot{\underline{S}}(t) = \underline{A}\times\delta^2\underline{S}(t) + \underline{B}[\delta\underline{S}(t)]^2 \quad \text{and} \quad \delta^2\underline{S}(0) = \underline{0} \tag{20}$$

where

$$\underline{B} = \begin{bmatrix} -a_2 & 0 & \dots & 0\\ a_2 & -a_2 & \dots & 0\\ \cdot & & & \\ \cdot & & \dots & \\ \cdot & & & \\ 0 & 0 & \dots & -a_2 \end{bmatrix}$$

Matrix $\underline{A}$ in eq. 20a is the same as in eq. 15, so the state-transition matrix for eq. 20 is also given by eq. 17. We conclude that the quadratic increment of a state trajectory can be described by

$$\delta^2\underline{S}(t) = \int_0^t e^{-\underline{A}(t-\xi)}\underline{B}[\delta\underline{S}(\xi)]^2 d\xi \tag{21}$$

Substitution of eq. 18 for $\delta S_i(\xi)$ into eq. 21 and double change of the order of integration leads to the conclusion that eq. 21 can be expressed as a second term of the Volterra series. For the ith reach of a channel we get:

$$\delta^2 S_i(t) = \int_0^t\int_0^t \left\{\frac{a_2}{a_1}e^{-a_1(\tau_1+\tau_2)}\left[\frac{(a_1\tau_1)^{i-1}}{(i-1)!}\sum_{k=0}^{i-1}\frac{(a_1\tau_2)^k}{k!} + \frac{(a_1\tau_2)^{i-1}}{(i-1)!}\sum_{k=0}^{i-2}\frac{(a_1\tau_1)^k}{k!}\right]\right.$$

$$\left. - \frac{a_2}{a_1}e^{-a_1\max(\tau_1,\tau_2)}\frac{[a_1\max(\tau_1,\tau_2)]^{i-1}}{(i-1)!}\right\}\delta x(t-\tau_1)\delta x(t-\tau_2)d\tau_1 d\tau_2$$

$$= \int_0^t\int_0^t k_2(\tau_1,\tau_2)\delta x(t-\tau_1)\delta x(t-\tau_2)d\tau_1 d\tau_2 \tag{22}$$

on the basis of solution for the linear and quadratic parts of a state trajectory increment one can derive the increments of higher orders until the required accuracy of the solution is reached. The complete proof that series (1) corresponds to series (9) can be found in Napiórkowski (1978).

The kernels of the Volterra series for the inflow—outflow relations

The outflow from a channel being modelled is a function of a retention of the last reach only. Substituting eqs. 18 and 22 for $\delta S_n(t)$, $\delta^2 S_n(t)$, . . . in eq. 11 for $i = n$ and substituting the results obtained into eq. 12 yields, after reordering the derived expression according to the order of integration, the following equation:

$$y(t) = y_0 + \int_0^t a_1 k_1(\tau)\delta x(t-\tau)\mathrm{d}\tau + \int_0^t\int_0^t [a_1 k_2(\tau_1,\tau_2) + a_2 k_1(\tau_1)k_1(\tau_2)] \times \delta x(t-\tau_1)\delta x(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 + \ldots \tag{23}$$

$$= y_0 + \int_0^t h_1(\tau)\delta x(t-\tau)\mathrm{d}\tau + \int_0^t\int_0^t h_2(\tau_1,\tau_2)\delta x(t-\tau_1)\delta x(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 + \ldots$$

The right-hand side of eq. 23 is the Volterra series describing the inflow—outflow relation. The kernels of this operator are determined on the basis of the known kernels of the Volterra operator describing the inflow—storage relation for the last reach of a channel. It results directly from eq. 23 that the first two kernels are as follows:

$$h_1(t) = a_1 \frac{(a_1 t)^{n-1}}{(n-1)!}\,\mathrm{e}^{-a_1 t} \tag{24}$$

$$h_2(t_1,t_2) = a_2 \mathrm{e}^{-a_1(t_1+t_2)}\left[\frac{(a_1 t_1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\frac{(a_1 t_2)^k}{k!} + \frac{(a_1 t_2)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\frac{(a_1 t_1)^k}{k!}\right] - a_2 \mathrm{e}^{-a_1 \max(t_1,t_2)}\frac{[a_1 \max(t_1,t_2)]^{n-1}}{(n-1)!} \tag{25}$$

Eqs. 24 and 25 describe the first two kernels of the conceptual nonlinear model. Its structure corresponds to the Volterra structure. Relation (24) describes the transfer function (IUH) for the cascade of linear reservoirs. The properties of the latter are not discussed here, as they were analyzed in many publications.

The properties of the second kernel of the Volterra series

It was proved by Napiórkowski (1978) that the second kernel of the Vol-

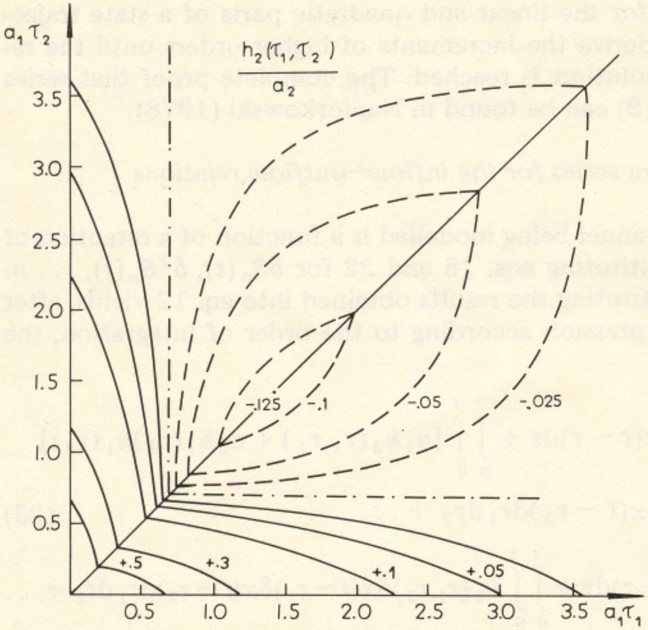

Fig. 1. The second-order kernel of the Volterra series describing the single reach of a channel.

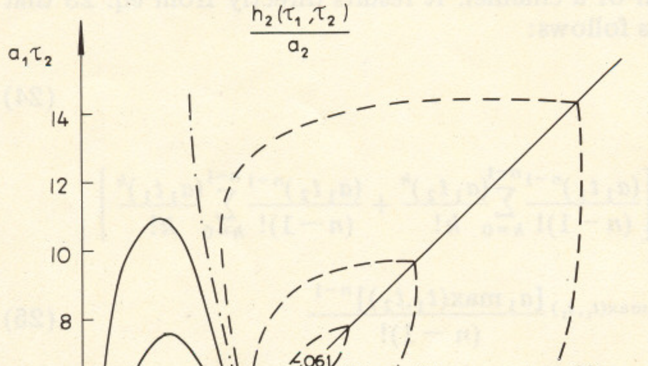

Fig. 2. The second-order kernel of the Volterra series describing five reaches of a channel.

terra series describing the flow in an open channel determined analytically and defined by eq. 25 meets the following conditions:

(a) $h_2(t_1,t_2) = 0$ for either $t_1 < 0$ or $t_2 < 0$

(b) $h_2(t_1,t_2) = 0$ for $t_1 = 0$ or $t_2 = 0$

(c) $|h_2(t_1,t_2)| < M$ for all t_1 and t_2, $M > 0$ constant representing the upper bound

(d) $h_2(t_1,t_2) = h_2(t_2,t_1)$ for all t_1 and t_2

(e) $h_2(t_1,t_2) \to 0$ for either $t_1 \to \infty$ or $t_2 \to \infty$

(f) $\int\limits_0^\infty \int\limits_0^\infty h_2(t_1,t_2)\mathrm{d}t_1\,\mathrm{d}t_2 = 0$

(g) $\int\limits_0^\infty h_2(t,t + C)\mathrm{d}t = 0$ for all $C \geqslant 0$

The above conditions were specified by Diskin and Boneh (1972) in terms of physical laws for a conservative inflow—outflow system described by the Volterra series.

A contour diagram representing the surface $h_2(t_1,t_2)$ are given in Figs. 1, and 2 for one and five reaches of a channel respectively.

Both diagrams are plotted in terms of dimensionless variables $a_1\tau_1$, $a_1\tau_2$, $h_2(\tau_1,\tau_2)a_2^{-1}$

THE EFFECT OF THE INFLOW MAGNITUDE ON THE CONVERGENCE OF THE VOLTERRA SERIES DESCRIBING THE FLOW DEVIATION FROM A STEADY STATE IN AN OPEN CHANNEL

The convergence of the Volterra series implies that the error of approximation of the actual system decreases with increase in number of the terms in the integral series. This aspect of the modelling of hydrologic systems has not so far attracted the attention of many investigators. It was mentioned by Kuchment (1972) who identified the limits of the inflow magnitude $x(t) < +\infty$ with the sufficient condition for the convergence of the series (1). On the other hand, Diskin and Boneh (1972) formulated the condition from physical considerations for the model consisting of the two first terms of the series. Fulfillment of the condition ensures positive outflows provided for positive inflows. However, this condition allows us to model the dynamics of the system by means of the Volterra series in the case of inflows for which the nonlinear model offers a worse approximation than the linear one.

The determination of the set of inflows such that each of them ensures the convergence solution constitutes the basic condition under which the integral series can be used in order to model hydrologic systems. Generally, this problem is extremely complicated. For that reason we will investigate here whether there exists an inflow magnitude limit as a condition for the convergence of the integral series (1). The latter describes the flow deviations from a steady state represented by the set of differential equations (7).

Consider first a single reach of a channel. The following nonlinear first-order differential equation applies:

$$\dot{S}(t) = -aS^m(t) + x(t) \qquad \text{and} \qquad S(0) = S_0 \tag{26}$$

with $a > 0$.

The solution can be expressed as an infinite integral series following the method presented in this paper. The first two terms are derived by substituting $i = n$ in eqs. 18 and 22.

$$S(t) = S_0 + \int_0^t e^{-a_1\tau}\delta x(t-\tau)d\tau + \int_0^t\int_0^t \left[\frac{a_2}{a_1}e^{-a_1(\tau_1+\tau_2)} - e^{-a_1\max(\tau_1,\tau_2)}\right] \times \delta x(t-\tau_1)\delta x(t-\tau_2)d\tau_1 d\tau_2 + \ldots \tag{27}$$

Consider the new steady state after the constant increment with a magnitude x appeared in $t = 0$. We conclude from eqs. 26 that the equation describing the steady state is of the form:

$$0 = -aS^m + (x_0 + \delta x) \tag{28}$$

The obvious solution of this equation is:

$$S = a^{-1/m}(x_0 + \delta x)^{1/m} \tag{29}$$

or the series expansion:

$$S = \frac{x_0^{\ 1/m}}{a^{1/m}} + \frac{x_0^{(1-m)/m}}{a^{1/m}m}\delta x + \frac{(1-m)x_0^{(1-2m)/m}}{2a^{1/m}m^2}(\delta x)^2 + \frac{(1-m)(1-2m)x_0^{(1-3m)/m}}{6a^{1/m}m^3}(\delta x)^3 + \ldots \tag{30}$$

It stems from the determination of the coefficient a_i (eq. 12) and from equation $S_0 = a^{-1/m}x_0^{\ 1/m}$ that relation (30) can be expressed as follows:

$$S = S_0 + \frac{1}{a_1}\delta x - \frac{a_2}{a_1^3}(\delta x)^2 + \left(\frac{2a_2}{a_1^5} - \frac{a_3}{a_1^4}\right)(\delta x)^3 + \ldots \tag{31}$$

The above series is convergent, if:

$$|\delta x| < x_0 \tag{32}$$

Every term of series (27) will converge, under $\delta x(t) = \delta x, t \in [0, +\infty)$ and $t \to \infty$ to the corresponding term of eq. 31. For example:

$$\int_0^t e^{-a_1 \tau} \delta x \, d\tau = \frac{1}{a_1} \delta x (1 - e^{-a_1 t}) \to \frac{1}{a_1} \delta x$$

$$\int_0^t \int_0^t \frac{a_2}{a_1} [e^{-a_1(\tau_1 + \tau_2)} - e^{-a_1 \max(\tau_1, \tau_2)}] (\delta x)^2 \, d\tau_1 \, d\tau_2$$

$$= \frac{(\delta x)^2 a_2}{a_1} \left\{ \left[(1 - e^{-a_1 t}) \frac{1}{a_1} \right]^2 - \frac{2}{a_1^2} [1 - e^{-a_1 t}(1 + a_1 t)] \right\}$$

$$\to - \frac{a_2}{a_1^3} (\delta x)^2$$

So, for $t \to \infty$ series (27) will converge if the inequality (32) is met.

It can easily be shown that for any $t \geqslant 0$ if $| \delta x(t) | < | \delta x |$ every term of series (27) is, in absolute magnitude, less than an absolute value of the corresponding term in series (31). This implies the limits to the storage increment ΔS in eq. 12; it belongs to the interval $(- S_0, (2^{1/m} - 1) S_0)$.

We conclude that inequality (32) is the sufficient condition for convergence of the Volterra series describing the relation between the inflow into ith reach of a channel and its storage.

The modelling of the storage of the last reach in a channel by the convergent Volterra series implies that ΔS, for $m > 1$, meets the condition for convergence of the series (12) that equals to $y(t)$. So we come to the conclusion that the convergence of the series describing the inflow—storage relation implies the convergence of the series describing the inflow—outflow relation.

Condition (32) for the convergence of the Volterra series is a sufficient one and formulated for the most unfavourable case. The inflows can exist such that the condition of maximum amplitude is not met, but the time interval when condition (32) is not fulfilled must be sufficiently short in order to have the finite sum of all terms of the series at any moment t. This conclusion is supported by the results of numerical experiments.

THE RESULTS OF NUMERICAL EXPERIMENTS

The results presented in this paper are illustrated in Figs. 3—5. They show the effect of the type of input signal on the accuracy of approximation of the nonlinear state equation (7), that describes the flow deviations from a steady state by means of the integral Volterra series.

A rectangular channel with width $B = 100$ m, bottom slope $I = 0.000248$,

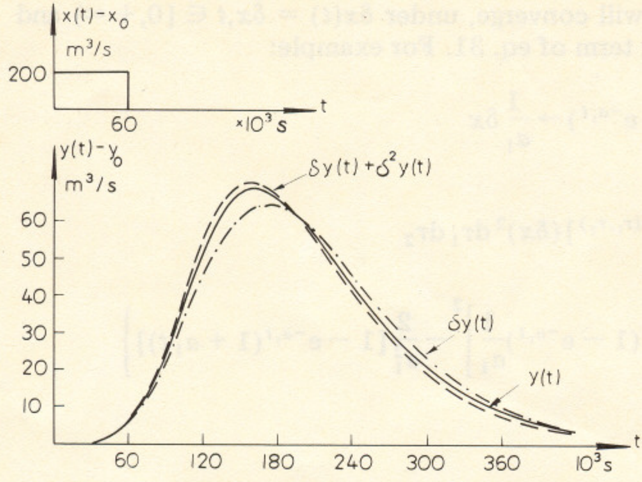

Fig. 3. Comparison of the results of simulation of the outflow by the first $\delta y(t)$ and the first two terms $\delta y(t) + \delta^2 y(t)$ of the Volterra series provided condition (32) is met.

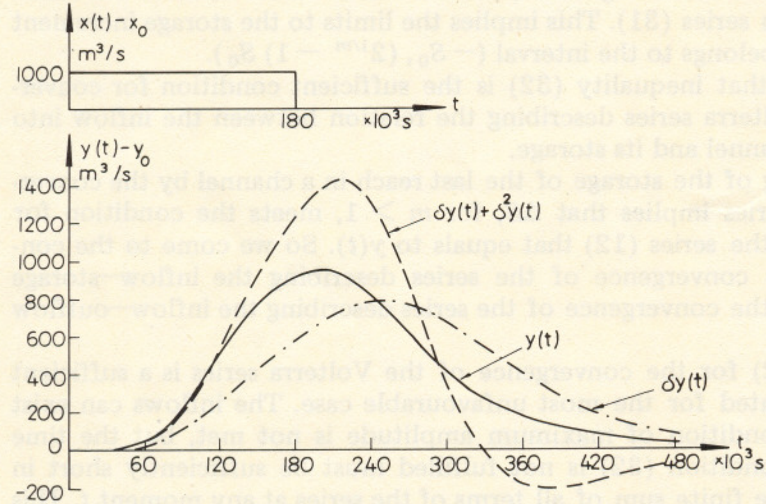

Fig. 4. Comparison of the results of simulation of the outflow by the first $\delta y(t)$ and the first two terms $\delta y(t) + \delta^2 y(t)$ of the Volterra series in the case when condition (32) is not met.

Chézy coefficient $C = 44.9$ was considered and the outflow equation (6) for a "characteristic reach" was approximated according to the Chézy formula.

The following initial conditions were assumed: inflow $x_0 = 200\,\mathrm{m}^3/\mathrm{s}$, depth $h_0 = 2\,\mathrm{m}$. The length of the "characteristic reach" was derived according to the formula $L = \frac{2}{3}(h_0/I)$ proposed by Kalinin and Miljukov (1957).

The simulation of the inflow deviations from a steady state was carried

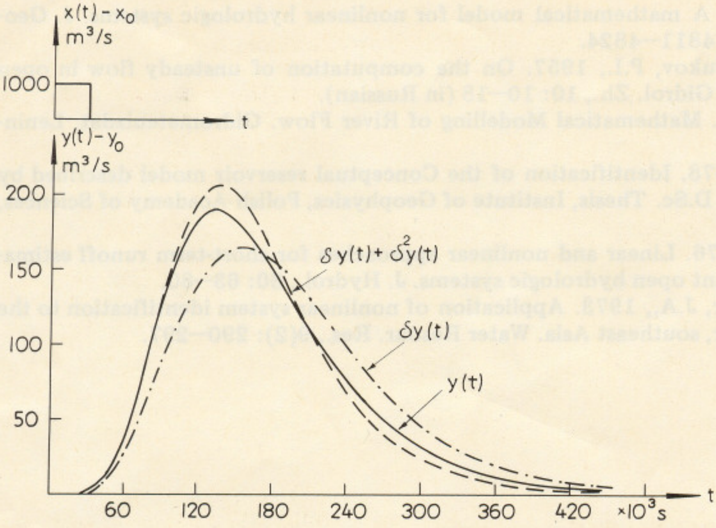

Fig. 5. Comparison of the results of simulation of the outflow by the first $\delta y(t)$ and the first two terms $\delta y(t) + \delta^2 y(t)$ of the Volterra series in the case when condition (32) is not met but the series converges.

out for a channel with the length five times longer than the "characteristic reach". The complete nonlinear set of differential equations was solved numerically by the Runge—Kutta method. The standard IBM® procedure was used, according to which the step of integration is chosen automatically for an imposed computational accuracy. Inflows of the form of rectangular pulses with different duration and amplitude were taken as input signals. The linear and quadratic approximation for the analyzed inflows were determined analytically.

Fig. 3 presents the results of transformation for the case when the inflow amplitude fulfills the convergence condition (32). Fig. 4 shows the results of transformation for the inflow that does not meet the convergence condition. Fig. 5 reflects the case when the sufficient condition (32) is not fulfilled, but the Volterra series converges.

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