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*Jarosław J. Napiórkowski, Witold G. Strupczewski (Warsaw)*

# THE ANALYTICAL DETERMINATION OF THE KERNELS OF THE VOLTERRA SERIES DESCRIBING THE CASCADE OF NONLINEAR RESERVOIRS

## 1. INTRODUCTION

The mathematical description of the relationships between the input and output of a hydrologic system is one of the most important tasks of contemporary hydrology. The current knowledge of hydrologic processes as well as the quality of measurement techniques do not allow the complete solution of the above problem in terms of physical interpretation of the processes in a watershed or a river, e.g. by momentum, mass and energy equations. This is why the conceptual or "black box" models are becoming more and more popular. They are simpler and can be more easily applied. They are based not upon a physical interpretation of dynamic processes but upon the precisely determined relationships between input and output quantities or processes. The linear models were developed in the sixties, whereas the nonlinear ones have not often been discussed in the literature until recently.

This paper deals only with a nonlinear model. In particular our point of interest is the modelling of hydrologic processes (like "effective rainfall-runoff", flood routing) by means of the conceptual model of cascade of nonlinear reservoirs described by the Volterra series. The kernel's structure and convergence of the series is analysed as well as the kernel's dependence on physical parameters.

The modelling of hydrologic processes by means of the Volterra series has so far been developed independently of other methods of description of dynamic systems in particular by a state equation formulation. The problem of series identification (determination of its kernels) has been solved by numerical methods applied to an input record and a corresponding output by means of kernels expanding (under an arbitrary assumption as to their structure) in orthogonal polynomials, see Amorocho and Brandstetter (1971), Jacoby (1966), Kuchment (1972), Papazafiriou (1976). As it was pointed out by Napiórkowski (1978), the fact that the structure of kernels is not known may imply their searching within the class of functions that is significantly different from the class they actually belong to. This in turn may lead to a solution of the identification problem that has nothing to do with a true one. For that reason, the analysis of the connections between the description by state equation and integral series in relation to a cascade of nonlinear reservoirs is, in our opinion, of great importance. This approach makes possible the analytic determination of the kernels structure and for that reason we consider it to be a significant contribution to the theory of nonlinear hydrologic models.

## 2. DESCRIPTION OF A CASCADE OF RESERVOIRS BY NONLINEAR OPERATORS

It was assumed in models discussed until recently in the hydrologic literature that an output is a superposition of signals corresponding to individual input signals. However, in case of many practically important systems the hypothesis of linearity holds only within some range of input signals variation or it must be rejected altogether. In such cases we have to apply the theory of nonlinear systems in order to describe processes in question with an accuracy required. The nonlinear operations that are most frequently used for a hydrologic system's description will be given below. These systems are considered to be dynamic ones with lumped parameters. The cascade of nonlinear reservoirs will be used as an example.

**Description of one reservoir — Nemycki's operator.** The changes in retention $S(t)$ of a hydrologic system in time depend on inflow $x(t)$ and outflow $y(t)$. The relation between these three functions can be expressed by the following continuity equation

$$\dot{S}(t) = x(t) - y(t), \tag{1}$$

where the dot over $S$ represents $d/dt$.

In order to solve effectively the above equation it is necessary to introduce an additional relation between outflow and retention. This relation

$$y(t) = f[S(t), t] \tag{2}$$

is known as Nemycki's operator. The parametric dependence on time reflects a nonstationarity of outflow in a retention function. Equations (1) and (2) give the complete description.

**Nonlinear state equation.** The description of dynamic systems by state equation is one of the basic approaches (Athans and Falb, 1969). Then the lumped dynamic model can be represented by a set of ordinary differential equations. The state vector for the given dynamic system can be determined in different ways. The set of components of the state vector, e.g. state variables, is the mininum set of numbers that must be known in $t=0$ in order to give a complete description of the states of the system for any $t \geqslant 0$ and for any input signals from the given input signals' set. A state vector's dimension does not depend on how the state variables are determined and it equals to the number of system's energy accumulators. The above conclusions will now be used in order to describe the classical conceptual model, namely the cascade of reservoirs (Fig. 1) given by equations (1), (2). These conclusions are:

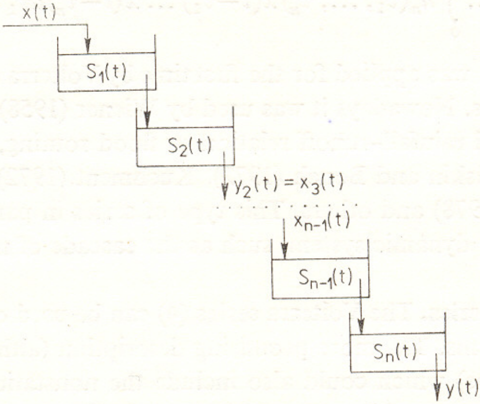


Fig. 1. The cascade of nonlinear reservoirs

(i) The dimension of a state vector describing the cascade equals to the number of reservoirs.

(ii) The simplest approach is to assume that storages of individual reservoirs are the components of the state vector.

(iii) The cascade has one input $x(t)=x_1(t)$ and one output $y(t)=y_n(t)$. Substituting (2) for $y(t)$ into (1) and putting $x_i(t)=f_{i-1}[S_{i-1}(t), t]$, $i=2, \ldots, n$ we get a set of equations that describe changes in storages of the cascade's reservoirs

$$\begin{aligned}
\dot{S}_1(t) &= -f_1[S_1(t), t]+x(t), \\
\dot{S}_2(t) &= -f_2[S_2(t), t]+f_1[S_1(t), t], \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\dot{S}_n(t) &= -f_n[S_n(t), t]+f_{n-1}[S_{n-1}(t), t], \\
y(t) &= f_n[S_n(t), t], \qquad \mathbf{S}(0)=\mathbf{S}_0 .
\end{aligned} \tag{3}$$

**Volterra series.** The description of dynamic systems by the Volterra series is a generalization of the concept of the transfer function, which is of a great importance in the analysis and design of linear systems. The Volterra series represents an explicit input–output relation for nonlinear dynamic systems and consists of an infinite series composed of terms of the form of convolution integrals. The first term is the convolution integral of the first order kernel and the input function, the $n$th order term is an $n$-fold convolution integral containing the $n$th order kernel multiplied by an $n$th order product of the input function

$$y(t)=\int_0^t h_1(\tau)x(t-\tau)d\tau+\int_0^t\int_0^t h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2+\ldots$$
$$\ldots+\int_0^t\ldots\int_0^t h_n(\tau_1,\ldots,\tau_n)x(t-\tau_1)\ldots x(t-\tau_n)d\tau_1\ldots d\tau_n+\ldots \tag{4}$$

This type of series was applied for the first time by Volterra and Frechet in 1910 on functional equations. Nowadays it was used by Wiener (1958), Flake (1963), and, in hydrology, to model rainfall–runoff relation or flood routing, by Amorocho and Brandstetter (1971), Diskin and Boneh (1972), Kuchment (1972), Zand and Harder (1973), Papazafiriou (1978) and others. This type of series in particular can be used to describe a nonlinear dynamic system, such as the cascade of nonlinear reservoirs.

**Lichtenstein–Lapunov series.** The Volterra series (4) can be used only for the description of stationary systems. The more promising description (although more difficult in terms of applications) which could also include the nonstationarity of modelled processes seems to be one making use of Lichtenstein–Lapunov series (Kudrewicz, 1976). Its kernels depend directly on time

$$y(t)=\int_0^t h_1(t,\tau)x(\tau)d\tau+\int_0^t\int_0^t h_2(t,\tau_1,\tau_2)x(\tau_1)x(\tau_2)d\tau_1 d\tau_2+\ldots$$
$$\ldots+\int_0^t\ldots\int_0^t h_n(t,\tau_1,\ldots,\tau_n)x(\tau_1)\ldots x(\tau_n)d\tau_1\ldots d\tau_n+\ldots \tag{5}$$

If the kernels of this series depend on the differences $t-\tau_i$ between arguments it describes the stationary model. Then the relation (5) is equivalent to the relation (4). (To see that it is enough to perform some elementary operations).

## 3. METHOD OF DETERMINATION OF THE KERNELS OF INTEGRAL SERIES DESCRIBING THE CASCADE OF NONLINEAR RESERVOIRS

The determination of the kernels of nonlinear integral series is of great importance in an application of the theory of modelling of hydrologic processes. The kernels are usually determined for a given real system by optimization methods. The latter are based on a finite input and corresponding output records. However, what is lacking in the literature is putting together the description by state equations and the description by integral series. This would make possible the determination of the structure of the kernels. As it was already mentioned, the fact that the structure of the kernels of the integral series is known enables the correct formulation of their identification. Then the identification problem resolves itself in to the estimation of a small number of parameters in the state equations.

**The kernels of a series describing the relation: inflow-storage of reservoirs.** The idea presented here as to how to determine the kernels of Lichtenstein–Lapunov series (5) which describes the cascade of nonlinear reservoirs stems from the following. The vector differential equation (3) can be considered as a definition of some nonlinear operator $P$ mapping a space of inflows in a space of corresponding storages. In order to determine how $P$ operates for a given inflow's hydrograph $x(t)$, $t \in [0, +\infty)$ it is necessary to solve the set of equations (3) under the initial condition $\boldsymbol{S}_0$. It can be symbolically expressed as follows

$$\boldsymbol{S}(t) = [P_{S_0} x](t). \tag{6}$$

We postulate here that the assumptions of Picard's theorem are met, so we make sure that the solution of the set of ordinary differential equations exists and is unique (Matveev, 1972). In further discussions it is assumed that the functions on right hand sides of an equality sign in (3) are differentiable so many times as required as to their arguments.

Let us denote by

$$\boldsymbol{S}^0(t) = [P_{S_0} \theta](t)$$

the solution of a set (3) corresponding to a zero inflow in the entire time interval and the initial condition $\boldsymbol{S}_0$. Let

$$\boldsymbol{S}(t) = [P_{S_0} x](t)$$

be a solution corresponding to a given inflow's hydrograph $x(t)$ and the same initial condition. The change of state trajectory from $\boldsymbol{S}^0(t)$ into $\boldsymbol{S}(t)$ can be determined by means of Taylor series for operators (Findeisen, 1977):

$$\begin{aligned}&[P_{S_0} x](t) - [P_{S_0} \theta](t)\\ &\quad = [P_{S_0} \theta, x](t) + \frac{1}{2}[P_{S_0} \theta, x^2](t) + \ldots + \frac{1}{i!}[P_{S_0} \theta, x^i](t) + \ldots,\end{aligned} \tag{7}$$

where

$$[P_{S_0} \theta, x](t) = \delta \boldsymbol{S}(t)$$

is the linear part of the state trajectory's increment, the first order Frechet differential of operator $P$,

$$\frac{1}{2}[P_{S_0} \theta, x^2](t) = \delta^2 \boldsymbol{S}(t)$$

is the quadratic part of the state trajectory's increment, the second order Frechet differential of operator $P$, etc.

So the change in the inflow hydrograph from zero level into $x(t)$ implies the trajectory's change from $\boldsymbol{S}^0(t)$ into

$$\boldsymbol{S}(t) = \boldsymbol{S}^0(t) + \Delta \boldsymbol{S}(t), \tag{8}$$

where

$$\Delta \boldsymbol{S}(t) = \delta \boldsymbol{S}(t) + \delta^2 \boldsymbol{S}(t) + \delta^3 \boldsymbol{S}(t) + \dots \tag{9}$$

We will prove below, by computing functions $\boldsymbol{S}^0(t)$, $\delta \boldsymbol{S}(t)$, $\delta^2 \boldsymbol{S}(t)$ (in this order) that the linear part of the increment of the storage function is represented by the first term of the Lichtenstein–Lapunov series (5), whereas the quadratic part — by the second term.

**Function $\boldsymbol{S}^0(t)$.** Function $\boldsymbol{S}^0(t)$ is a solution of a homogeneous set of equations (3) which corresponds to the zero inflow in the entire time interval and the initial condition $\boldsymbol{S}_0$

$$\begin{aligned}
\dot{S}_1^0(t) &= -f_1[S_1^0(t), t],\\
\dot{S}_2^0(t) &= -f_2[S_2^0(t), t] + f_1[S_1^0(t), t],\\
&\dots\dots\dots\dots\dots\dots\\
\dot{S}_n^0(t) &= -f_n[S_n^0(t), t] + f_{n-1}[S_{n-1}^0(t), t],\\
\boldsymbol{S}^0(0) &= \boldsymbol{S}_0.
\end{aligned} \tag{10}$$

The above homogeneous set of equations has unique solution under the imposed assumptions. This solution depends on the initial condition. The latter describes the process of emptying of the cascade of reservoir provided there is no inflow into the first reservoir. In order to compute the linear and quadratic increments we make use of:

(a) the expansion of the retention — outflow relation (2) in Taylor series around the inflow trajectory $S_i^0(t)$ for the increase $\Delta S_i(t)$, $i = 1, \dots, n$,

$$y_i(t) = f_i[S_i^0(t), t] + \frac{\partial f_i[S_i^0(t), t]}{\partial S_i} \Delta S_i(t) + \\ + \frac{1}{2} \frac{\partial^2 f_i[S_i^0(t), t]}{\partial S_i^2} [\Delta S_i(t)]^2 + \frac{1}{6} \frac{\partial^3 f_i[S_i^0(t), t]}{\partial S_i^3} [\Delta S_i(t)]^3 + \dots \tag{11}$$

(b) the expansion in Taylor series the time derivative of the retention around the trajectory $S_i^0(t)$

$$\dot{S}_i(t) = \dot{S}_i^0(t) + \delta \dot{S}_i(t) + \delta^2 \dot{S}_i(t) + \delta^3 \dot{S}_i(t) + \dots \tag{12}$$

**Function $\delta \boldsymbol{S}(t)$.** Substitution of (11), (12) and (9) (limited to the first-order increments only) into (3) yields following set of equations

$$\dot{S}_1^0(t)+\delta\dot{S}_1(t) = -f_1[S_1^0(t), t]-\frac{\partial f_1}{\partial S_1}\delta S_1(t)+x(t),$$

$$\dot{S}_2^0(t)+\delta\dot{S}_2(t)+ = -f_2[S_2^0(t), t]-\frac{\partial f_2}{\partial S_2}\delta S_2(t)+f_1[S_1^0(t), t]+\frac{\partial f_1}{\partial S_1}\delta S_1(t),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{13}$$

$$\dot{S}_n^0(t)+\delta\dot{S}_n(t) = -f_n[S_n^0(t), t]-\frac{\partial f_n}{\partial S_n}\delta S_n(t)+f_{n-1}[S_{n-1}^0(t), t]+\frac{\partial f_{n-1}}{\partial S_{n-1}}\delta S_{n-1}(t)$$

with the initial condition

$$\boldsymbol{S}^0(0) = \boldsymbol{S}_0, \quad \delta\boldsymbol{S}(0) = \boldsymbol{0},$$

as there is a zero increment in the initial condition. The arguments of the partial derivatives are omitted in order to shorten the notation.

By subtracting the set of equations (10) (corresponding to $\boldsymbol{S}^0(t)$ solution) from (13) we get the relation representing the linear part of the storage trajectory's increment

$$\delta\dot{S}_1(t) = -\frac{\partial f_1}{\partial S_1}\delta S_1(t)+x(t),$$

$$\delta\dot{S}_2(t) = -\frac{\delta f_2}{\delta S_2}\delta S_2(t)+\frac{\partial f_1}{\partial S_1}\partial S_1(t),$$

$$\ldots\ldots\ldots\ldots\ldots \tag{14}$$

$$\delta\dot{S}_n(t) = -\frac{\partial f_n}{\partial S_n}\delta S_n(t)+\frac{\partial f_{n-1}}{\partial S_{n-1}}\partial S_{n-1}(t),$$

$$\delta\boldsymbol{S}(0) = \boldsymbol{0}.$$

Equation (14) can be expressed in a matrix notation as follows

$$\delta\dot{\boldsymbol{S}}(t) = \boldsymbol{A}(t, \boldsymbol{S}_0)\times\delta\boldsymbol{S}(t)+\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}x(t), \tag{15}$$

$$\delta\boldsymbol{S}(0) = \boldsymbol{0}.$$

The elements of matrix $\boldsymbol{A}(t, \boldsymbol{S}_0)$ are the functions of time and the initial condition. So we got the linear nonstationary set of differential equations with the zero initial condition. It has the following solution

$$\delta\boldsymbol{S}(t) = \int_0^t \boldsymbol{K}_1(t, \boldsymbol{S}_0, \tau)\,x(\tau)\,d\tau, \tag{16}$$

where

$$\boldsymbol{K}_1(t, \boldsymbol{S}_0, \tau) = \Phi_{n\times n}(t, \boldsymbol{S}_0, \tau)\times \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $\Phi$ is a state-transition matrix for (14) (Athans and Falb, 1968).

It can be concluded that the linear part of the increment of the storage trajectory in Taylor expansion (7) can be put down as the first term of the Lichtenstein–Lapunov series.

**Function** $\delta^2\boldsymbol{S}(t)$. Substitution of (11), (12) and (9) into (3) (limited to the second-order increments) and in turn substraction (13) from the both sides, make use to arrive to a relation describing the quadratic part of the increment of the storage trajectory

$$\begin{aligned}
\delta^2\dot{S}_1(t) &= -\frac{\partial f_1}{\partial S_1}\,\delta^2 S_1(t) - \frac{1}{2}\,\frac{\partial^2 f_1}{\partial S_1^2}\,[\delta S_1(t)]^2,\\
\delta^2\dot{S}_2(t) &= -\frac{\partial f_2}{\partial S_2}\,\delta^2 S_2(t) + \frac{\partial f_1}{\partial S_1}\,\delta^2 S_1(t) - \frac{1}{2}\,\frac{\partial^2 f_2}{\partial S_1^2}\,[\delta S_2(t)]^2+\\
&\quad + \frac{1}{2}\,\frac{\partial^2 f_1}{\partial S_1^2}\,[\delta S_1(t)]^2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta^2\dot{S}_n(t) &= -\frac{\partial f_n}{\partial S_n}\,\delta^2 S_n(t) + \frac{\partial f_{n-1}}{\partial S_{n-1}}\,\delta^2 S_{n-1}(t) - \frac{1}{2}\,\frac{\partial^2 f_n}{\partial S_n^2}\,[\delta S_n(t)]^2+\\
&\quad + \frac{1}{2}\,\frac{\partial^2 f_{n-1}}{\partial S_{n-1}^2}\,[\delta S_{n-1}(t)]^2
\end{aligned} \tag{17}$$

with the zero initial condition, exactly as it was in case of function $\delta\boldsymbol{S}(t)$.

The set of equations (17) can be expressed in a matrix notation as follows.

$$\delta^2\dot{\boldsymbol{S}}(t) = \boldsymbol{A}(t, \boldsymbol{S}_0)\times\delta^2\boldsymbol{S}(t) + \boldsymbol{B}(t, \boldsymbol{S}_0)\times[\delta\boldsymbol{S}(t)]^2. \tag{18}$$

The matrix $\boldsymbol{A}(t, \boldsymbol{S}_0)$ in (18) is the same as in (15), the elements of matrix $\boldsymbol{B}(t, \boldsymbol{S}_0)$ are functions of time and the initial condition. Having the solution for $\delta\boldsymbol{S}(t)$ from equation (16) we can insert $[\delta\boldsymbol{S}(t)]^2$ in equation (18)

$$[\delta\boldsymbol{S}(t)]^2 = \int_0^t\int_0^t \boldsymbol{K}_1(t, \boldsymbol{S}_0, \lambda_1)\,\boldsymbol{K}_1(t, \boldsymbol{S}_0, \lambda_2)\,x(\lambda_1)\,x(\lambda_2)\,d\lambda_1\,d\lambda_2. \tag{19}$$

Denote by $\boldsymbol{\varphi}(t, \boldsymbol{S}_0, \lambda_1, \lambda_2)$ the product

$$\boldsymbol{B}(t, \boldsymbol{S}_0)\times[\boldsymbol{K}_1(t, \boldsymbol{S}_0, \lambda_1)\ \boldsymbol{K}_1(t, \boldsymbol{S}_0, \lambda_2)].$$

Substitution of (19) into (18) yields the nonstationary linear set of ordinary differential equations corresponding to the quadratic increment of the state trajectory:

$$\delta^2 \dot{\boldsymbol{S}}(t) = \boldsymbol{A}(t, \boldsymbol{S}_0) \times \delta^2 \boldsymbol{S}(t) + \int_0^t \int_0^t \boldsymbol{\varphi}(t, \boldsymbol{S}_0, \lambda_1, \lambda_2)\, x(\lambda_1)\, x(\lambda_2)\, d\lambda_1\, d\lambda_2 \tag{20}$$

with the zero initial condition. The solution of this set is defined by the following equation

$$\delta^2 \boldsymbol{S}(t) = \int_0^t \boldsymbol{\Phi}(t, \boldsymbol{S}_0, \xi) \times \int_0^\xi \int_0^\xi \boldsymbol{\varphi}(\xi, \boldsymbol{S}_0, \lambda_1, \lambda_2)\, x(\lambda_1)\, x(\lambda_2)\, d\lambda_1\, d\lambda_2\, d\xi, \tag{21}$$

where $\boldsymbol{\Phi}(t, \boldsymbol{S}_0, \xi)$ is the state-transition matrix for (20). This matrix is the same as for equation (15).

The double change of the order of integration results in

$$\delta^2 \boldsymbol{S}(t) = \int_0^t \int_0^t \left[ \int_{\lambda_1}^t \boldsymbol{\Phi}(t, \boldsymbol{S}_0, \xi) \times \boldsymbol{\varphi}(\xi, \boldsymbol{S}_0, \lambda_1, \lambda_2)\, 1(\xi - \lambda_2)\, d\xi \right] x(\lambda_1)\, x(\lambda_2)\, d\lambda_1\, d\lambda_2, \tag{22}$$

where $1(t)$ is the unit step function.

Substitution

$$\boldsymbol{K}_2(t, \boldsymbol{S}_0, \lambda_1, \lambda_2) = \int_{\lambda_1}^t \boldsymbol{\Phi}(t, \boldsymbol{S}_0, \xi) \times \boldsymbol{\varphi}(\xi, \boldsymbol{S}_0, \lambda_1, \lambda_2)\, 1(\xi - \lambda_2)\, d\xi$$

gives us the solution for the quadratic part of the increment of the state trajectory in formula (7). This solution is of a form of the second term in the Lichtenstein–Lapunov series (5)

$$\delta^2 \boldsymbol{S}(t) = \int_0^t \int_0^t \boldsymbol{K}_2(t, \boldsymbol{S}_0, \lambda_1, \lambda_2)\, x(\lambda_1)\, x(\lambda_2)\, d\lambda_1\, d\lambda_2. \tag{23}$$

Having determined functions $\boldsymbol{S}^0(t)$, $\delta \boldsymbol{S}(t)$, $\delta^2 \boldsymbol{S}(t)$ one can obtain the 3rd-order increment of the storage trajectory by expansions of the set (3) up to 3rd-order increments. This increment of the storage is represented by the third term in the Lichtenstein–Lapunov series, etc. The complete proof of the equivalency between series (7) and (5) can be found in Napiórkowski (1978).

**The kernels of the inflow-outflow operator.** The outflow from the cascade of reservoirs is a function of the retention of the last reservoir only. Substitution of the last rows in formulas (16), (23), ... for $S_n(t)$, $\delta^2 S_n(t)$, ... in (11) for $i = n$ gives us

$$y(t) = f_n[S_n^0(t), t] + \frac{\partial f_n}{\partial S_n} \int_0^t K_{1n}(t, \boldsymbol{S}_0, \lambda)\, x(\lambda)\, d\lambda +$$

$$+\frac{\partial f_n}{\partial S_n}\int_0^t\int_0^t K_{2n}(t, S_0, \lambda_1, \lambda_2)\, x(\lambda_1)\, x(\lambda_2) d\lambda_1 d\lambda_2 +$$

$$+\frac{1}{2}\frac{\partial^2 f_n}{\partial S_n^2}\int_0^t\int_0^t K_{1n}(t, S_0, \lambda_1) K_{1n}(t, S_0, \lambda_2)\, x(\lambda_1)\, x(\lambda_2) d\lambda_1 d\lambda_2 + \dots \tag{24}$$

The right-hand side of (24) is the Lichtenstein–Lapunov operator describing the inflow–outflow relation. The kernels of this operator are determined by means of the known kernels of the Lichtenstein–Lapunov operator describing the inflow--retention relation for the last reservoir. We conclude from (24) that the first two kernels are as follows

$$h_1(t, S_0, \lambda) = \frac{\partial f_n}{\partial S_n} K_{1n}(t, S_0, \lambda), \tag{25a}$$

$$h_2(t, S_0, \lambda_1, \lambda_2) = \frac{\partial f_n}{\partial S_n} K_{2n}(t, S_0, \lambda_1, \lambda_2) +$$

$$+\frac{1}{2}\frac{\partial^2 f_n}{\partial S_n^2} K_{1n}(t, S_0, \lambda_1) K_{1n}(t, S_0, \lambda_2). \tag{25b}$$

The application of the analytical method of the determination of the integral series' kernels by means of the state equation will be shown in the next paragraph. As an example the cascade of the identical stationary nonlinear reservoirs modelling the flow in an open channel will be used.

## 4. THE RELATIONSHIP BETWEEN THE VOLTERRA SERIES AND NONLINEAR STATE EQUATIONS IN THE CASE OF THE CASCADE OF RESERVOIRS MODELLING THE FLOW IN AN OPEN CHANNEL

Consider the prismatic channel with a rectangular cross-section characterized by the following parameters: $B$ — width of the channel, $I$ — bottom slope, $n^*$ — Manning's roughness coefficient, $h$ — average depth (equals aproximatly to the hydraulic radius) divided into $n$ reaches of the length $L$. The retention of the reach with length $L$ can be described by the following equation.

$$V = BhL. \tag{26}$$

The outflow from the reach can be aproximately expressed (using Manning's formula) as follows

$$Q = vBh = aV^{5/3}, \tag{27}$$

where $v$ — average flow's velocity at the downstream cross-section, $Q$ — flow at the downstream cross-section,

$$a = \frac{1}{n^*} I^{1/2} B^{-2/3} L^{-5/3}.$$

The deviation of the outflow from the steady-state outflow

$$y(t) = Q(t) - Q_0(t)$$

as a function of the retention changes

$$S(t) = V(t) - V_0(t)$$

can be aproximately determined by the expansion of (27) in Taylor series

$$y(t) = \left.\frac{\partial Q}{\partial V}\right|_{V=V_0} S(t) + \frac{1}{2} \left.\frac{\partial^2 Q}{\partial V^2}\right|_{V=V_0} S^2(t) + \ldots = a_1 S(t) + a_2 S^2(t) + \ldots, \tag{28}$$

where $V_0$ is the retention of the reach in the steady state.

The following are the first two coefficients in the formula (28) aproximating the increment of outflows from the reach in a retention increment function

$$a_1 = \frac{5}{3} a V_0^{2/3}, \tag{29}$$

$$a_2 = \frac{5}{9} a V_0^{-1/3}. \tag{30}$$

According to (28), the deviation from the steady-state flow in an open channel will be modelled by means of the cascade of nonlinear reservoirs and each of them is described by the following set of equations

$$\dot{S}_i(t) = x_i(t) - y_i(t), \tag{31}$$

$$y_i(t) = a_1 S_i(t) + a_2 S^2(t) + \ldots \tag{32}$$

with the zero initial condition (see formulas (1), (2)).

At this point, we can derive the relation between the deviation from the steady state in the open channel. This relation has a form of the integral series (Fig. 2). The results obtained in the third paragraph will be used here.

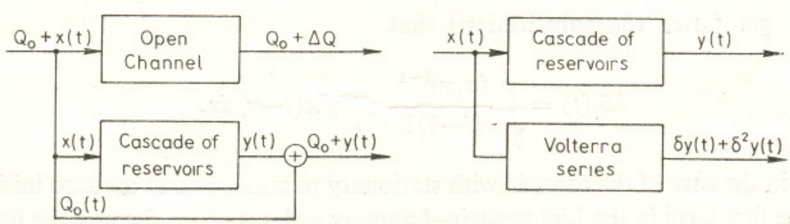


Fig. 2. The conceptual model simulating the flow's deviation from the steady state in open channels

**Vector $S^2(t)$.** Recall that we analyze the deviation from the steady state. Under this assumption the initial condition with respect to the cascade of reservoirs will also be zero and the solution for set (3) under $x(t) = 0$ is a vector equals to zero within the interval $[0, +\infty)$. So

$$S^0(t) = \mathbf{0}.$$

**Vector $\delta S(t)$ — the linear aproximation.** Substitution of equation (32) for $f_i[S_i(t), t]$ yields from (14) the stationary set of differential equations. This in turn enables us to determine the linear part of the increment of the storage trajectory

$$\begin{aligned}
\delta\dot{S}_1(t) &= -a_1\,\delta S_1(t)+x(t),\\
\delta\dot{S}_2(t) &= -a_1\,\delta S_2(t)+a_1\,\delta S_1(t),\\
&\ldots\ldots\ldots\ldots\\
\delta\dot{S}_n(t) &= -a_1\,\delta S_n(t)+a_1\,\delta S_{n-1}(t),\\
\delta \boldsymbol{S}(0) &= \mathbf{0}.
\end{aligned} \tag{33}$$

In order to solve (33) one has to derive its state-transition matrix. The easiest way to achieve that in case of stationary systems is by Laplace transform. As far as equations (33) are concerned it is necessary to obtain the inverse transform of matrix $(p\boldsymbol{I}-\boldsymbol{A})^{-1}$ (matrix $\boldsymbol{A}$ is defined by (15), $\boldsymbol{I}$ is identity matrix, $p$ is a complex variable). The state-transition matrix derived this way for equations (33) is of the form

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t} = \left[\begin{array}{c|c|c|c}
e^{-a_1 t} & 0 & - & 0\\ \hline
(a_1 t)e^{-a_1 t} & e^{-a_1 t} & - & 0\\ \hline
- & - & - & -\\ \hline
\dfrac{(a_1 t)^{n-1}}{(n-1)!}\,e^{-a_1 t} & \dfrac{(a_1 t)^{n-2}}{(n-2)!}\,e^{-a_1 t} & - & e^{-a_1 t}
\end{array}\right]. \tag{34}$$

Taking the above into account as well as the formula (16) we conclude that the linear part of the storage increment in the $i$th reservoir of the cascade can be determined according to the formula

$$\delta S_i(t) = \int_0^t \frac{[a_1(t-\lambda)]^{i-1}}{(i-1)!}\, e^{-a_1(t-\lambda)}x(\lambda)\,d\lambda. \tag{35}$$

Inasmuch as the kernel of the operator (35) depends only on the difference $t-\lambda$ we get (after the substitution) that

$$\delta S_i(t) = \int_0^t \frac{(a_1\tau)^{i-1}}{(i-1)!}\, e^{-a_1\tau}x(t-\tau)\,d\tau. \tag{36}$$

So, in the case of the cascade with stationary parameters and the zero initial condition the first term in the Lichtenstein–Lapunov series is of the form of the first term in the Volterra series.

**Vector $\delta^2 S(t)$ — the quadratic aproximation.** We can get from the relation (17) the set of differential equations from which the quadratic part of the increment of the storage trajectory can be derived

$$\begin{aligned}
\delta^2\dot{S}_1(t) &= -a_1\delta^2 S_1(t) - a_2[\delta S_1(t)]^2,\\
\delta^2\dot{S}_2(t) &= -a_1\delta^2 S_2(t) + a_1\delta^2 S_1(t) - a_2[\delta S_2(t)]^2 + a_2[\delta S_1(t)]^2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta^2\dot{S}_n(t) &= -a_1\delta^2 S_n(t) + a_1\delta^2 S_{n-1}(t) - a_2[\delta S_n(t)]^2 + a_2[\delta S_{n-1}(t)]^2,\\
\delta^2\boldsymbol{S}(0) &= \boldsymbol{0}.
\end{aligned} \tag{37}$$

The state-transition matrix for equation (37) is the same as for equation (33) so the quadratic part of the increment of the state trajectory is determined by the following relation

$$\delta^2\boldsymbol{S}(t) = \int_0^t e^{-A(t-\xi)} \times \boldsymbol{B}[\delta\boldsymbol{S}(\xi)]^2 d\xi, \tag{38}$$

where matrix $\boldsymbol{B}$ is defined by (18).

The analytical determination of the second kernel of the operator entails tedious computations. For that reason we will not present details of how the quadratic part of the trajectory's increment was derived. So we only note here that substitution into (38) of formula (35) for $[\delta S_i(\xi)]^2$ and double change of the order of integration according to formula (22) and the variables substitution lead to the conclusion that equation (38) can be expressed as the second term in the Volterra series

$$\begin{aligned}
\delta^2 S_i(t) = \int_0^t\int_0^t \Bigg\{ &\frac{a_2}{a_1} e^{-a_1(\tau_1+\tau_2)} \left[ \frac{(a_1\tau_1)^{i-1}}{(i-1)!} \sum_{k=0}^{i-1} \frac{(a_1\tau_2)^k}{k!} + \frac{(a_1\tau_2)^{i-1}}{(i-1)!} \sum_{k=0}^{i-2} \frac{(a_1\tau_1)^k}{k!} \right] - \\
&- \frac{a_2}{a_1} e^{-a_1\max(\tau_1,\tau_2)} \frac{[a_1\max(\tau_1,\tau_2)]^{i-1}}{(i-1)!} \Bigg\} x(t-\tau_1)\,x(t-\tau_2)\,d\tau_1\,d\tau_2.
\end{aligned} \tag{39}$$

All transformations can be found in Napiórkowski (1978). One can now determine, using the solutions for the linear and quadratic parts of the increment of the state trajectory, the higher order increments up to the point the required accuracy is achieved.

**The kernels of the Volterra series for the inflow-outflow relation.** The kernels of the Volterra series describing the inflow–outflow relation can be derived from (25), by which they are expressed in terms of the kernels of the series describing the inflow–

storage relation. The simple algebraic transformation give us the following solution

$$h_1(t) = a_1 \frac{(a_1 t)^{n-1}}{(n-1)!} e^{-a_1 t}, \tag{40}$$

$$h_2(t_1, t_2) = a_2 e^{-a_1(t_1+t_2)} \left[ \frac{(a_1 t_1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \frac{(a_1 t_2)^k}{k!} + \right.$$

$$\left. + \frac{(a_1 t_2)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \frac{(a_1 t_1)^k}{k!} \right] - a_2 e^{-a_1 \max(t_1, t_2)} \frac{[a_1 \max(t_1, t_2)]^{n-1}}{(n-1)!}. \tag{41}$$

The equations (40) and (41) describe the two first kernels of the conceptual nonlinear model. Its structure corresponds to the structure of the Volterra series. Relation (40) describes the transfer function (IUH) for the cascade of linear reservoirs. The latter properties are not discussed here, as they were analyzed in many publications.

**The properties of the second kernel of the Volterra series.** It was proved (see Napiórkowski, 1978) that the second kernel of the Volterra series describing the cascade of nonlinear reservoirs determined analytically and defined by (41) fulfils the following conditions;

1. $h_2(t_1, t_2) = 0$ for either $t_1 < 0$ or $t_2 < 0$,
2. $h_2(t_1, t_2) = 0$ for $t_1 = 0$ or $t_2 = 0$,
3. $|h_2(t_1, t_2)| < M$ for all $t_1$ and $t_2$,
4. $h_2(t_1, t_2) = h_2(t_2, t_1)$ for all $t_1$ and $t_2$,
5. $h_2(t_1, t_2) \to 0$ for either $t_1 \to \infty$ or $t_2 \to \infty$,
6. $\int_0^\infty \int_0^\infty h_2(t_1, t_2) dt_1 dt_2 = 0$,
7. $\int_0^\infty h_2(t, t+c) dt = 0$ for all $c \geqslant 0$.

The above conditions were specified by Diskin and Boneh (1972) in terms of physical laws for the inflow–outflow systems without loss expressed by the two first terms of the Volterra series!

A contour diagrams representing (41) for one, three and five nonlinear reservoirs are given in Figs. 3, 4, 5 respectively.

All three diagrams are plotted in terms of dimensionless variables $a_1 \tau_1$, $a_1 \tau_2$, $h_2(\tau_1, \tau_2) a_2^{-1}$.

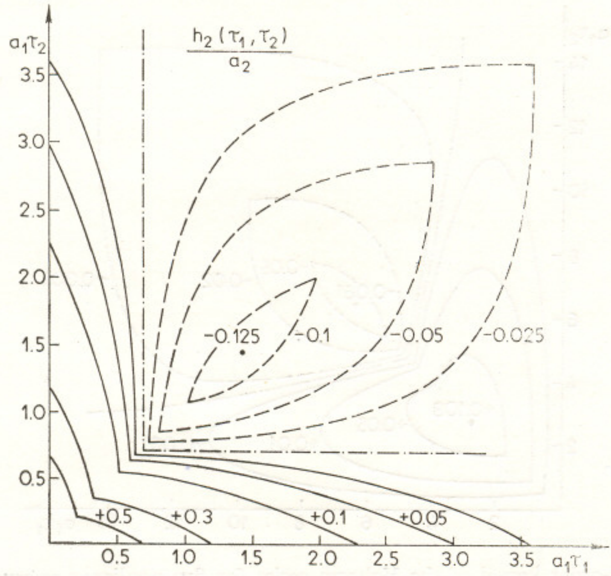


Fig. 3. The second-order kernel of the Volterra series for one nonlinear reservoir

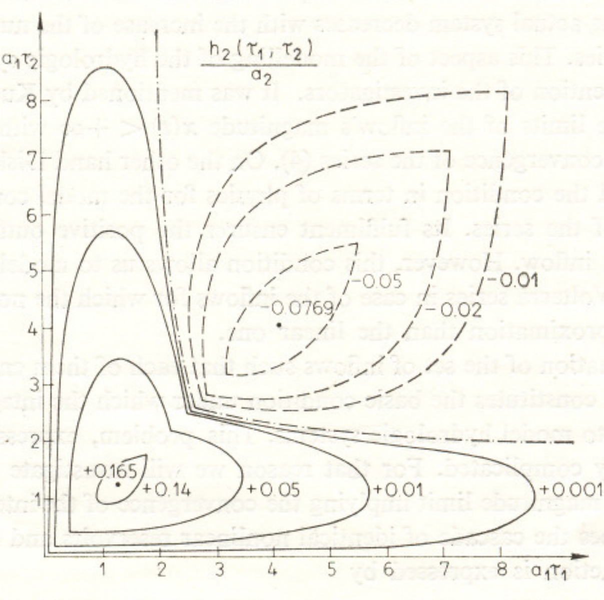


Fig. 4. The second-order kernel of the Volterra series for three nonlinear reservoirs

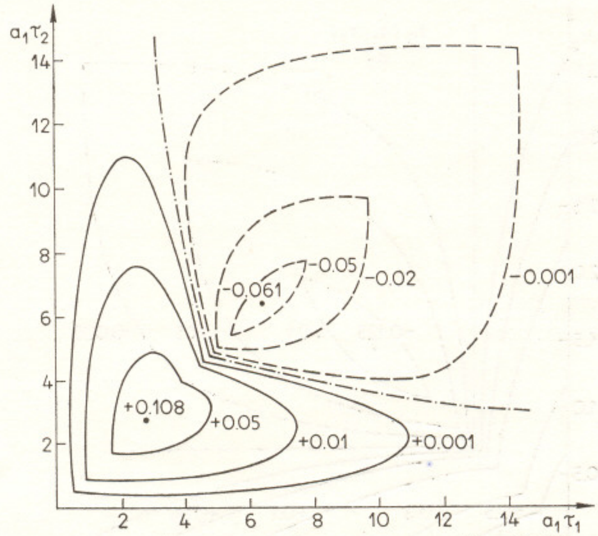


Fig. 5. The second-order kernel of the Volterra series for five nonlinear reservoirs

## 5. THE EFFECT OF THE INFLOW'S MAGNITUDE ON THE CONVERGENCE OF THE VOLTERRA SERIES DESCRIBING THE CASCADE OF NONLINEAR RESERVOIRS

The convergence of the Volterra series implies that the error of the aproximation of dynamics of the actual system decreases with the increase of the number of terms in the integral series. This aspect of the modelling of the hydrologic systems has not so far focused attention of the investigators. It was mentioned by Kuchment (1972) who identifies the limits of the inflow's magnitude $x(t) < +\infty$ with the sufficient condition for the convergence of the series (4). On the other hand Diskin and Boneh (1972) formulated the condition in terms of physics for the model consisting of the two first terms of the series. Its fulfilment ensures the positive outflow, provided there is a positive inflow. However, this condition allows us to model the dynamics by means of the Volterra series in case of the inflows for which the nonlinear model offers a worse aproximation than the linear one.

The determination of the set of inflows such that each of them ensures the convergence solution constitutes the basic condition under which the integral series can be used in order to model hydrologic systems. This problem, expressed in general terms, is extremly complicated. For that reason we will investigate whether there exists an inflow's magnitude limit implying the convergence of the integral series (4). The latter describes the cascade of identical nonlinear reservoirs and the outflow in the retention function is expressed by

$$y_i(t) = a_1 S_i(t) + a_2 S_i^2(t), \quad i = 1, \ldots, n. \tag{42}$$

Consider first the individual nonlinear reservoir (Fig. 6). Its dynamics is described by the following nonlinear first-order differential equation

$$\begin{aligned} \dot{S}(t) &= -a_1 S(t) - a_2 S^2(t) + x(t), \\ S(0) &= 0, \quad a_1 > 0, \quad a_2 > 0. \end{aligned} \tag{43}$$

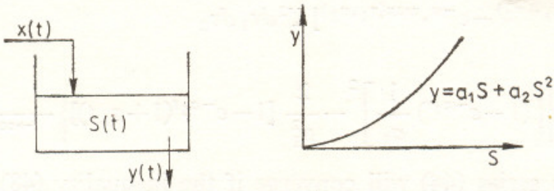


Fig. 6. The individual nonlinear reservoir

The solution of the above equation can be expressed as an infinite integral series following the method presented in this paper. Two first terms are derived by substituting $i = 1$ in (36) and (39)

$$S(t) = \int_0^t e^{-a_1 \tau} x(t-\tau)\, d\tau + \int_0^t \int_0^t \frac{a_2}{a_1} \left[ e^{-a_1(\tau_1+\tau_2)} - e^{-a_1 \max(\tau_1 \tau_2)} \right] \times$$

$$\times x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 + \ldots \tag{44}$$

Consider the steady state of the reservoir with inflow with amplitude $X$. We conclude from (43) that the steady state is described by the following equation

$$0 = -a_1 S - a_2 S^2 + X. \tag{45}$$

The obvious solution of this equation under $S > 0, X > 0$ is

$$S = \frac{-a_1 + \sqrt{a_1^2 + 4a_2}}{2a_2} \tag{46}$$

or the series expansion

$$S = \frac{1}{a_1} X - \frac{a_2}{a_1^3} X^2 + \frac{2a_2^2}{a_1^5} X^3 - \ldots \tag{47}$$

The above series is convergent provided the following inequalities are met

$$\frac{4a_2 X}{a_1^2} \leqslant 1 \quad \text{or} \quad X \leqslant \frac{a_1^2}{4a_2}. \tag{48}$$

Every term of (44) will aproximate, under $x(t) = X$, $t \geqslant 0$ and $t \to \infty$ the corresponding term of (47). For example

$$\int_0^t e^{-a_1\tau} X d\tau = \frac{1}{a_1} X(1-e^{-a_1\tau}) \xrightarrow[t\to\infty]{} \frac{1}{a_1} X,$$

$$\int_0^t \int_0^t \frac{a_2}{a_1} [e^{-a_1(\tau_1+\tau_2)} - e^{-a_1 \max(\tau_1,\tau_2)}] X^2 d\tau_1 d\tau_2$$

$$= \frac{X^2 a_2}{a_1} \left\{ \left[ (1-e^{-a_1 t}) \frac{1}{a_1} \right]^2 - \frac{2}{a_1^2} [1-e^{-a_1 t}(1+a_1 t)] \right\} \xrightarrow[t\to\infty]{} -\frac{a_2}{a_1^3} X^2.$$

So, for $t \to \infty$ series (44) will converge if the inequality (48) is met.

It can easily be shown that every term of series (44) is in absolute magnitude less than, or equal to, the corresponding term in series (47) if

$$\max_{t \geqslant 0} x(t) \leqslant X.$$

This implies

$$\max_{t \geqslant 0} S(t) \leqslant S.$$

We conclude that the sufficient condition for convergence of the Volterra series describing the individual nonlinear reservoir defined by equation (42) is

$$X = \max x(t) \leqslant \frac{a_1^2}{4a_2}.$$

It should be noted that the differential equation

$$\dot{S}(t) + a_1 S(t) + a_2 S^2(t) = x(t) = X = \text{const}$$

has a finite solution for $t \to \infty$ and $0 < X < +\infty$ but the Volterra series yields a finite solution when $X \leqslant a_1^2/4a_2$.

The derived condition for the convergence of Volterra series in case of one reservoir can be generalized on the entire cascade. Equation (43) is relevant to describe the retention state in each reservoir at steady state. In particular it refers to the last reservoir. This implies the conclusion that (48) will also be the condition for the convergence of the series describing the cascade of nonlinear reservoirs. In modelling of the retention of the last reservoir by means of the convergent Volterra series, ($S_n(t)$ has a finite value for all $t \in [0, +\infty)$) and this implies the convergence of the series $a_1 S_n(t) + a_2 S_n^2(t)$. The latter equals to $y(t)$. So we come to the conclusion that the convergence of the series describing the inflow-retention relation implies the convergence of the series describing the inflow-outflow relation. The condition (48) referring to the convergence of the Volterra series is a sufficient condition determined for the most unfavourable case. However, despite the condition of maximum amplitude

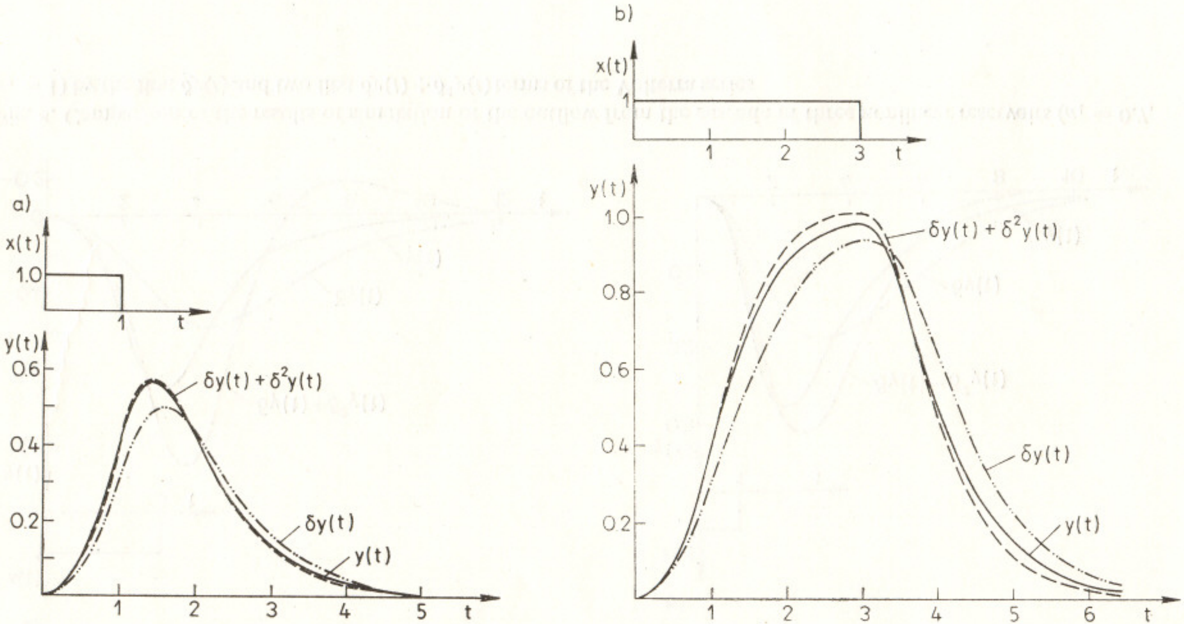


Fig. 7. Comparison of the results of simulation of the outflow from the cascade of three nonlinear reservoirs ($a_1 = 2, a_2 = 1$) by the first $\delta y(t)$ and two first $\delta y(t)+\delta^2 y(t)$ terms of the Volterra series

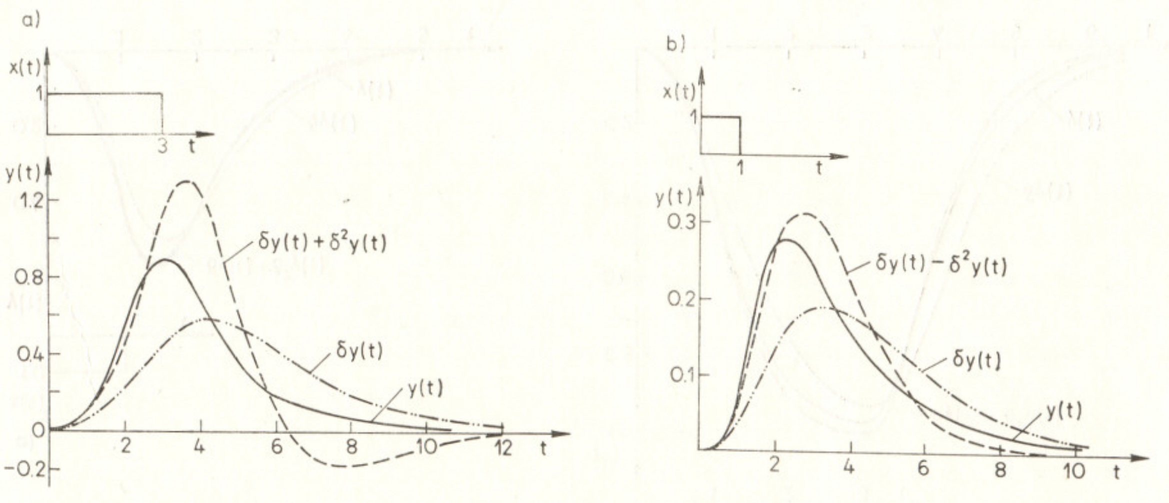


Fig. 8. Comparison of the results of simulation of the outflow from the cascade of three nonlinear reservoirs ($a_1 = 0.7$, $a_2 = 1$) by the first $\delta y(t)$ and two first $\delta y(t) + \delta^2 y(t)$ terms of the Volterra series

is not met, the inflows' hydrographs can exist such that the corresponding series converge. The above remains valid, provided the time in which the input is not met (48) is short enough. This conclusion is confirmed by the results of the numerical experiments. In order to complete our discussion we will try to evaluate the value of inflows ensuring the convergence of the Volterra series in case of the conceptual model (42) simulating the flow's deviation from the steady state in open channels. We conclude from (29), (30) that the condition in question is that the maximum amplitude fulfills the inequalities

$$\max_{t\geqslant 0}\left(Q(t)-Q_0\right)\leqslant\frac{5}{4}\,Q_0.$$

## 6 THE RESULTS OF NUMERICAL EXPERIMENTS

Our considerations in this paper are illustrated in Figs. 7 and 8. They show the effect of the type of input signal and of the model's parameters on the accuracy of the solution. The cascade of three nonlinear reservoirs was investigated. The aproximation of the outflow in retention function for each of them was of form (42) under the assumption of the zero initial conditions. The nonlinear set of differential equations was solved numerically by the fourth-order Runge–Kutta method. The standard IBM procedure was used, according to which the step of integration is chosen automatically for an imposed computational accuracy. The inflows of the type of a rectangular pulse function with an unit amplitude and different duration were taken as input signals. The linear and quadratic aproximations for the analyzed inflows were determined analytically. Fig. 7 presents the results of transformation if the parameters $a_1$, $a_2$ and the inflow's amplitude fulfil the convergence condition (48). The closer is the inflow $x(t)$ to the steady state the higher is the accuracy of the aproximation of the cascade's outflow. That is why the aproximation in Fig. 7a is significantly better than that in Fig. 7b. Fig. 8 shows the rusults of transformation for two inflow cases for which condition (48) is not met. The Volterra series does not converge (Fig. 8a) if the inflow lasts for a long time, but otherwise it does (Fig. 8b).

Dr. Jarosław J. Napiórkowski
Institute of Geophysics
Polish Academy of Sciences
Pasteura 3
02-093 Warsaw, Poland

Doc. Dr. Witold G. Strupczewski
Institute of Geophysics
Polish Academy of Sciences
Pasteura 3
02-093 Warsaw, Poland

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*Ярослав Напюрковски, Витольд Струпчевски*

## АНАЛИТИЧЕСКОЕ ОПРЕДЕЛЕНИЕ ЯДРОВ РЯДА ВОЛЬТЕРРЫ ОПИСУЮЩЕГО КАСКАДУ НЕЛИНЕЙНЫХ РЕЗЕРВУАРОВ

РЕЗЮМЕ

В статьи представлена концепция связания двух основных математических техник использованных к моделированию нелинейных гидрологических систем. Первый метод пользуется нелинейным дифференцяльными уровнениями, второй многолинейным интегральным рядом.

Для каскада нелинейных резервуаров моделирующих сток в открытом русле определено аналитически ядра интегрального ряда Вольтерры на основе нелинейных дифференциальных уровнении. Обсуждено свойства и структуры двух первых ядров. Исследовано проблему сходимости ряда Вольтерры в зависимости от величины входной функции и свойства нелинейности системы.

*Jarosław J. Napiórkowski, Witold G. Strupczewski*

## ANALITYCZNE WYZNACZENIE JĄDER SZEREGU VOLTERRY OPISUJĄCEGO KASKADĘ ZBIORNIKÓW NIELINIOWYCH

STRESZCZENIE

Prezentowana jest koncepcja powiązania dwóch podstawowych aparatów matematycznych wykorzystywanych do modelowania nieliniowych systemów hydrologicznych, opisu za pomocą równań stanu i wieloliniowego szeregu całkowego. Dla kaskady ziborników nieliniowych modelujących przepływ w korycie otwartym wyznaczono analitycznie jądra szeregu całkowego Volterry w oparciu o nieliniowe równanie stanu. Dyskutowane są własności i struktura dwóch pierwszych jąder. Badany est problem zbieżności szeregu Volterry w zależności od wielkości sygnału wejściowego i charakteru jnieliniowości obiektu.