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DISCRETE THIRD-ORDER VOLTERRA MODEL OF SURFACE RUNOFF SYSTEMS

By

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ABSTRACT

In the paper a conceptual third-order discrete Volterra model is fitted to distributed nonlinear surface runoff system. This fitting is carried out for several records of storms. The sufficient convergence and copositivity conditions of the model are discussed. An example illustrating the applicability of the Volterra series to modelling of the Cache River Catchment is presented.

INTRODUCTION

When modelling surface runoff for natural catchments with the help of hydrodynamic methods one requires a detailed topographical survey and assessment of roughness parameters. In order to avoid this difficulty the modellers use the external approach via conceptual and black box models. The method discussed in the paper combines the black box analysis with the conceptual model approach. The nonlinear behaviour of the system is desribed by a model in the form of third-order approximation of a cascade of nonlinear reservoirs. Such a model is equivalent to the three first terms of the Volterra series (Napiórkowski, 1983).

In the course of mathematical modelling of surface runoff systems, one deals with discrete signals at the stage of measurements. In the paper the discrete set of outputs is obtained via analytical solution of the continuous problem. This approach compares favourably to simple mechanical discretization of the kernels with respect to accuracy and necessary computational effort.

CONTINUOUS THIRD-ORDER VOLTERRA MODEL

The findings presented in this section borrow heavily from Napiórkowski (1983). The surface runoff system is represented as a cascade of equal nonlinear reservoirs in which each reservoir is responsible for a part of the attenuation of the system response. This lumped dynamic model can be represented by a set of ordinary differential equations

$$\dot{S}_1(t) = - f[S_1(t)] + I(t)$$
$$\dot{S}_2(t) = - f[S_2(t)] + f[S_1(t)]$$
$$\dots\dots\dots\dots\dots\dots \qquad (1a)$$
$$\dot{S}_n(t) = - f[S_n(t)] + f[S_{n-1}(t)]$$

$$y(t) = f[S_n(t)] \qquad (1b)$$

where n is the number of reservoirs, I(t) is the effective rainfall, $S_i(t)$ is the storage of i-th reservoir, f( ) represents the outflow-storage relation of the individual reservoirs, y(t) is the output from the model. The function f( ) is not prescribed. One assumes only that it is differentiable for $S_i \geq 0$ as many times as required. In this study we are concerned with initially relaxed systems, hence the initial condition is S(0)=0.

The solution of the differential Eq.(1) can be given in the form of Taylor series. Accordingly $S_i(t)$ and y(t) are divided into linear, quadratic and cubic parts and a residual error

$$S_i(t) = \delta S_i(t) + \delta^2 S_i(t) + \delta^3 S_i(t) + \dots \qquad (2)$$
$$y(t) = \delta y(t) + \delta^2 y(t) + \delta^3 y(t) + \dots \qquad (3)$$

In order to compute the linear (δ), quadratic (δ^2) and cubic (δ^3) components of y(t) and $S_i(t)$ we make use of expansion of the outflow-storage relation

$$f[S_i(t)] = a\, S_i(t) + b[S_i(t)]^2 + c[S_i(t)]^3 + \dots \qquad (4)$$

where

$$a = \frac{df}{dS_i} \qquad b = \frac{1}{2}\frac{d^2 f}{dS_i^2} \qquad c = \frac{1}{6}\frac{d^3 f}{dS_i^3} \qquad (5)$$

The linear, quadratic and cubic components from Eqs.(2,3,4,5) should fulfil the following set of equations:

<u>The Linear Components</u>

$$\delta\dot{S}(t) = a\,\phi\,\delta S(t) + [1,0,\ \ldots\ ,0]^T\ I(t) \qquad (6a)$$

$$\delta y(t) = a\,\delta S_n(t) \qquad (6b)$$

<u>The Quadratic Components</u>

$$\delta^2\dot{S}(t) = a\,\phi\,\delta^2 S(t) + b\,\phi\,[\delta S(t)]^2 \qquad (7a)$$

$$\delta^2 y(t) = a\,\delta^2 S_n(t) + b\,[\delta S_n(t)]^2 \qquad (7b)$$

<u>The Cubic Components</u>

$$\delta^3\dot{S}(t) = a\phi\delta^3 S(t) + 2b\phi\delta S(t)\,\delta^2 S(t) + c\,\phi[\delta S(t)]^3 \qquad (8a)$$

$$\delta^3 y(t) = a\,\delta^3 S_n(t) + 2b\,\delta S_n(t)\,\delta^2 S_n(t) + c\,[\delta S_n(t)]^3 \qquad (8b)$$

It was shown by Napiórkowski (1983), that the linear, quadratic and cubic components described by Eqs.(6,7,8) are equivalent respectively to the first second and third terms of the Volterra series

$$\delta y(t) = \int_0^t h_1(r)\ I(t-r)\ dr \qquad (9)$$

$$\delta^2 y(t) = \int_0^t\!\!\int_0^t h_2(r_1,r_2)\ I(t-r_1)\,I(t-r_2)\,dr_1 dr_2 \qquad (10)$$

$$\delta^3 y(t) = \int_0^t\!\!\int_0^t\!\!\int_0^t h_3(r_1,r_2,r_3)\,I(t-r_1)\,I(t-r_2)\,I(t-r_3)\,dr_1 dr_2 dr_3 \qquad (11)$$

where kernels $h_1(r)$, $h_2(r_1,r_2)$ and $h_3(r_1,r_2,r_3)$ can be derived analytically from Eqs.(6, 7, 8) where the matrix ϕ is given by

$$\phi(i,j) = \begin{cases} -1 & \text{for } i-j=0 \\ 1 & \text{for } i-j=1 \\ & \text{otherwise} \end{cases} \quad (12)$$

The first, second and third terms described by differential equations (6,7,8) or by integral equations (9,10,11) form the continuous third-order Volterra model. However, it is computationally more efficient to calculate approximations from the state-space representation of the model, rather than by using triple integrals.

From the computational point of view it is convenient to denote the solution of Eq.(7) for b=1 by $y^2(t)$, the solution of Eq.(8) for b=1 and c=0 by $y^3(t)$, and the solution of Eq.(8) for b=0 and c=1 by $y^4(t)$. Then due to linearity of Eqs.(7,8) the following relations are fulfiled

$$\delta y(t) = y^1(t) \quad (13a)$$

$$\delta^2 y(t) = b\, y^2(t) \quad (13b)$$

$$\delta^3 y(t) = b^2\, y^3(t) + c\, y^4(t) \quad (13c)$$

The functions $y^1(t)$, $y^2(t)$, $y^3(t)$ and $y^4(t)$ depend on the parameters a and n, do not depend on the parameters b and c, and are governed by the following state transition equations

<u>Linear Term</u>

$$\dot{S}^1(t) = a\phi S^1(t) + [1,0, \dots ,0]^T\, I(t) \quad (14a)$$

$$y^1(t) = a\, S^1_n(t) \quad (14b)$$

<u>Quadratic Term</u>

$$\dot{S}^2(t) = a\phi S^2(t) + \phi\,[S^1(t)]^2 \quad (15a)$$

$$y^2(t) = a\, S^2_n(t) + [S^1_n(t)]^2 \quad (15b)$$

<u>Cubic-b Term</u>

$$\dot{S}^3(t) = a\phi S^3(t) + 2\phi S^1(t) S^2(t) \quad (16a)$$

$$y^3(t) = a\, S^3_n(t) + 2\, S^1_n(t) S^2_n(t) \quad (16b)$$

Cubic-c Term

$$\dot{S}^4(t) = a\phi S^4(t) + \phi[S^1(t)]^3 \qquad (17a)$$

$$y^4(t) = a\, S_n^4(t) + [S_n^1(t)]^3 \qquad (17b)$$

Note that:

1) cubic-b term is that part of cubic component which results from the forsing by the product of the linear and quadratic terms, and that cubic-c term results from the forsing by the cube of the linear term. This decomposition and subsequent superposition involves no assumption or approximation.

2) the transition matrix for Eq.(14) is the same as for Eqs.(15,16,17) and is given by

$$\exp(a\phi t) = \begin{cases} \exp(-at)(at)^{i-j}/(i-j)! \;;\; i \geq j \\ 0 \quad \text{for } j < i \end{cases} \qquad (18)$$

$$i,j=1,\ldots,n$$

3) the input of effective precipitation $I(t)$ occurs only in Eq.(14). Consequently the addition of the components $y^2(t)$, $y^3(t)$ and $y^4(t)$ effects only the distribution of the predicted runoff and the total volume of each of these components is zero. It is illustrated in Fig.1 where functions $y^1(t)$, $y^2(t)$, $y^3(t)$ and $y^4(t)$ are plotted for the case $n=3$, $a=1$ and

$$I(t) = \begin{cases} 1 & \text{for } t \in (0,1) \\ 0 & \text{otherwise} \end{cases} \qquad (19)$$

DISCRETE THIRD-ORDER VOLTERRA MODEL

The discrete set of outputs is obtained via analytic solution of the continuous problem described by Eqs.(14-17). This is advantageous in comparison to prior assumption of discrete model (e.i. simple mechanical discretization of the kernels in Eqs.(9,10,11)). This approach was used for the case of two-term Volterra model by Napiórkowski and Kundzewicz (1985).

Consider the Eqs.(14-17) expressed in discrete framework. The data are given in discrete time instants and the system response needs to be known in discrete time instants. Assume that the time instants of interest belong to the following set

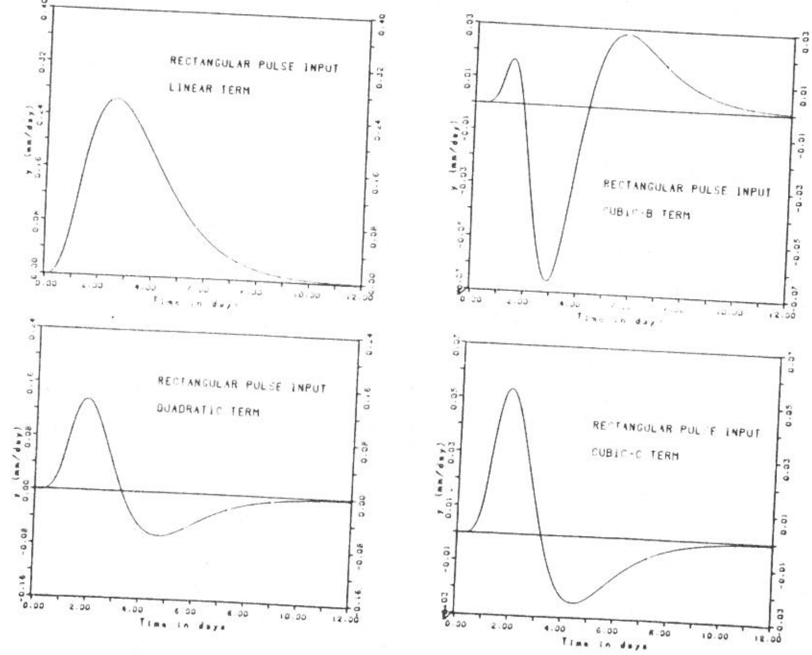


Figure 1. The trajectories $y^1(t)$, $y^2(t)$, $y^3(t)$ and $y^4(t)$ for the case of n=3, a=1 and rectangular pulse input.

$$t_k = k\ T \quad \text{where } k=0,\ldots,N \qquad (19)$$

Assume for brevity the notation $f(t_k)=f(k)$ and let the input $I(t)$ be given as a train of rectangular pulses, in accordance with the rainfall measurement.

Then the model response in discrete time instants can be calculated from Eqs.(14-17) as follows.

The Linear Approximation

From Eq.(14) (Wierzbicki,1977) the linear model response reads

$$S^1(k+1)=e^{a\phi T}\ S^1(k)\ +\int_{t_k}^{t_{k+1}} \exp\left[a\phi(t_{k+1}-r)\right]\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}dr\ I(k) \qquad (20)$$

That is, in the handy notation

$$S^1(k+1) = \tilde{A}\, S^1(k) + \tilde{B}\, I(k) \tag{21a}$$

$$y^1(k) = a\, S_n^1(k) \tag{21b}$$

where

$$\tilde{A} = \exp(a\phi T) \tag{22}$$

$$\tilde{B} = \int_0^T \exp[a\phi(T-r)]\, dr \tag{23}$$

The Quadratic Approximation

Similarly, from Eq.(15) one can see that the quadratic component is

$$S^2(k+1) = e^{a\phi T}\, S^2(k) + \int_{t_k}^{t_{k+1}} \exp[a\phi(t_{k+1}-r)]\, \phi[\tilde{S}^1(r)]^2 dr \tag{24}$$

where $\tilde{S}^1(t)$ is the extension for continuous arguments of the component $S^1(k)$ given for discrete arguments. This is achieved by linear interpolation of $S^1(\mathbf{k})$ between the discrete instants, where $S^1(\mathbf{k})$ is explicitly given

$$S^1(r) = [S^1(k)r + S^1(k+1)(T-r)]/T \tag{25}$$

Inserting Eq.(25) into Eq.(24) one gets the final equation for the quadratic component

$$S^2(k+1) = \tilde{A}\, S^2(k) + B_1\, [S^1(k)]^2 + B_2\, 2S^1(k)S^1(k+1) + B_3\, [S^1(k+1)]^2 \tag{26a}$$

$$y^2(k) = a\, S_n^2(k) + [S_n^1(k)]^2 \tag{26b}$$

The final formulae for B_1, B_2 and B_3 are given in the Appendix.

The Cubic-b Approximation

The cubic-b response is obtained with the help of Eq.(16)

$$S^3(k+1)=e^{a\phi T}\, S^3(k)+\int_{t_k}^{t_{k+1}} \exp[a\phi(t_{k+1}-r)]\,\phi\, 2S^1(r)S^2(r)\,dr \tag{27}$$

Using the linear interpolation of $S^1(t)$ and $S^2(t)$ between the discrete instants one gets

$$\begin{aligned} S^3(k+1) = {} & \tilde{A}\, S^3(k) + 2\, B_1\, [S^1(k)\, S^2(k)] \\ & + 2\, B_2\, [S^1(k)S^2(k+1) + S^1(k+1)S^2(k)] \\ & + 2\, B_3\, [S^1(k+1)\, S^2(k+1)] \end{aligned} \tag{28a}$$

$$y^3(k) = a\, S^3_n(k) + 2\, S^1_n(k)\, S^2_n(k) \tag{28b}$$

where the matrices B_1, B_2 and B_3 are the same as for Eq.(26a).

The Cubic-c Approximation

Finally, the cubic-c response is determined from Eq.(17)

$$S^4(k+1)= e^{a\phi T}\, S^4(k)+\int_{t_k}^{t_{k+1}} \exp[a\phi(t_{k+1}-r)]\,\phi[S^1(r)]^3\,dr \tag{29}$$

The linear interpolation of $S^1(t)$ results in the vector state transition equation

$$\begin{aligned} S^4(k+1) = {} & \tilde{A}\, S^4(k) + C_1 [S^1(k)]^3 + C_2\, S^1(k)[S^1(k+1)]^2 \\ & + C_3 [S^1(k)]^2\, S^1(k+1) + C_4 [S^1(k+1)]^3 \end{aligned} \tag{30a}$$

$$y^4(k) = a\, S^4_n(k) + [S^1(k)]^3 \tag{30b}$$

The final formulae for C_1, C_2, C_3, and C_4 are given in the Appendix.

Eqs.(21,26,28,30) form the discrete third-order Volterra model of surface runoff systems.

The check of choice of the time discretization step (T) can be made by comparing the integral of the second and third order increments to zero (Diskin and Boneh, 1972). If these values differ considerably from zero, the linear approximation of the increment $S^1(t)$ in Eqs.(24,27,29) and of the increment $S^2(t)$ in Eq.(27) is not sufficient, that is denser discretization is required.

COPOSITIVITY VS. CONVERGENCE OF THE VOLTERRA MODEL

Any infinite Volterra series has a range of convergence $I(t) \leq M$. Inside this range the error of approximation of tha dynamics of the system is inversely proportional to the number of terms in the integral series. In order to identify the dynamics of the system one has to truncate the infinite series. In that case one requires copositivity of the model (a positive output response to a positive input). Joint analysis of copositivity and convergence of Volterra model has not been referred to in professional literature. Expressed in general terms the problem is complex, but nevertheless it must be considered. To explain the difference between the copositivity condition and convergence condition a simple two-term Volterra model based on a single reservoir is discussed below in details.

Consider the model

$$\dot{S}(t) = -a\,S(t) - b\,S^2(t) + I(t)$$
$$S(0) = 0 \qquad (31a)$$

$$y(t) = a\,S(t) + b\,S^2(t) \qquad (31b)$$

and the input signal

$$I(t) = \begin{cases} I & \text{for} \quad t \leq T \\ 0 & \text{for} \quad t > T \end{cases} \qquad (32)$$

The solution of Eq.(31) can be expressed as an infinite integral series following the method presented in this paper. It has been shown by Napiórkowski and Strupczewski (1979) that the differential Eq.(31) for a, b > 0 has a finite solution when $0 = I(t) < \infty$ but the infinite Volterra series yields a finite solution when

$$I \le 0.25\, a^2/b \tag{33}$$

To derive the physically based copositivity condition we make use of Eqs.(6,7). Denoting

$$v=e^{at};\quad z=e^{-a(t-T)};\quad r=e^{aT};\quad p=2b/a^2 \tag{34}$$

and taking into account Eq.(32) one gets the two-term Volterra model response in the form

$$\delta y(t) + \delta^2 y(t) =$$

$$\begin{cases} y_1(v) = \dfrac{I}{v^2}\left(v^2 + p + pv\ln v - pv - v\right) & \text{for } t \le T \quad (35a) \\[2ex] y_2(z) = I\left\{z^2 p(1-r)^2 + z\left[1 - \dfrac{1}{r} + \dfrac{p}{r}(1 + \ln r - r)\right]\right\} & \text{for } t > T \quad (35b) \end{cases}$$

where

$$v \in (1,\ e^{aT}) \quad \text{and} \quad z \in (0,1)$$

One can check, that for $v \ge 1$ the function on the right hand side of Eq.(35a) is greater than zero. Hence, the copositivity requirement is met when $y_2(z) \ge 0$ for z (0,1). This is fulfilled when

$$\left.\frac{dy_2}{dz}\right|_{z=0} \ge 0 \tag{36}$$

that is when

$$p \le \frac{r-1}{r - \ln r - 1} \tag{37}$$

Above condition is illustrated in Fig.2. Positive outflows correspond to points below the curve $p=p(r)$ defined by inequality (37). Note, that $p(r)$ is monotonically decreasing function and (see Fig.2)

$$\lim_{r\to\infty} p(r) \longrightarrow 1 \tag{38}$$

Hence, for $p \le 1$, that is for

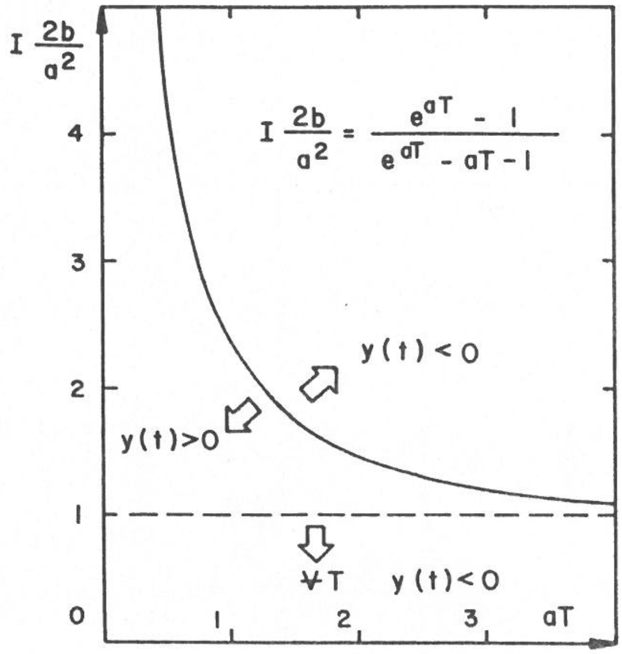


Figure 2. The copositivity condition for a single reservoir

$$I \leq 0.5\ a^2/b \qquad (39)$$

the outflow from the model is positive for any value of T (c.f. Eq.(32)). When $p > 1$, the outflow is positive only for short time interval. The sufficient copositivty condition (39) is twice weaker than the sufficient convergence condition (33).

The main conclusion from the above consideration is that the use of the truncated Volterra series is subject to certain restriction on the input magnitude, which is very difficult to derive for series with more than two terms. It is the recommendation of the present authors that the input magnitude used for simulation should not be greater than the input magnitudes used for identification for the copositivity to be established.

NUMERICAL EXAMPLE

The methodology presented was tested at the rainfall-runoff system for the data from the catchment of the Cache River at Forman in Southern Illinois used by Diskin and Boneh (1973). The catchment of the area equal to 630 km^2 has mild slopes and a well developed drainage network. The data pertain to eight rainfall-runoff events observed in the years 1935-1951.

The problem to be solved was to fit the discrete third-order Volterra model in the sense of least-squares. So we were looking for the parameters minimizing the objective function

$$J(n,a,b,c)=\sum_{i=1}^{M}\sum_{k=1}^{T_i}\left[O_i(k)-y_i^1(k)-by_i^2(k)-b^2y_i^3(k)-cy_i^4(k)\right]^2 \tag{40}$$

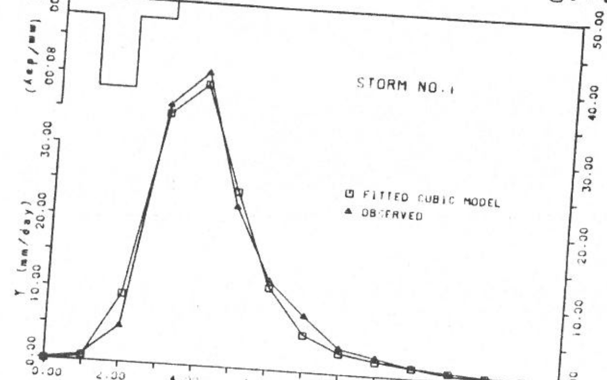
where M is the number of independent records, T_i is the length of the i-th outflow record, $O_i(k)$ is the i-th direct surface runoff and y_i^1, y_i^2, y_i^3, y_i^4 are the solutions of Eqs.(21,26,28,30) for i-th record.

It was shown by Napiórkowski (1983), that the problem of identification can be reduced to optimization with respect to two variables only n and a. For given n and a two other parameters b and c result from the necessary condition for optimum $\partial J/\partial b=0$ and $\partial J/\partial c$. In the mentioned paper a detail algorithm was presented.

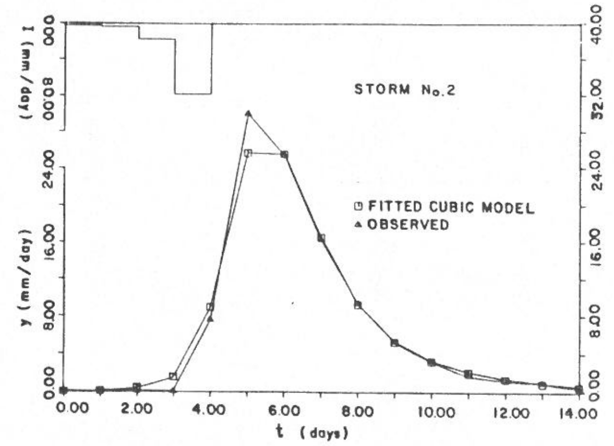
The optimal values of the parameters (n,a,b,c) of the discrete third-order model were found to be

n = 3
a = 0.68 (1/day)
b = 5.6 10-3 (day-1 mm-1)
c = 84.0 10-6 (day-1 mm-2)

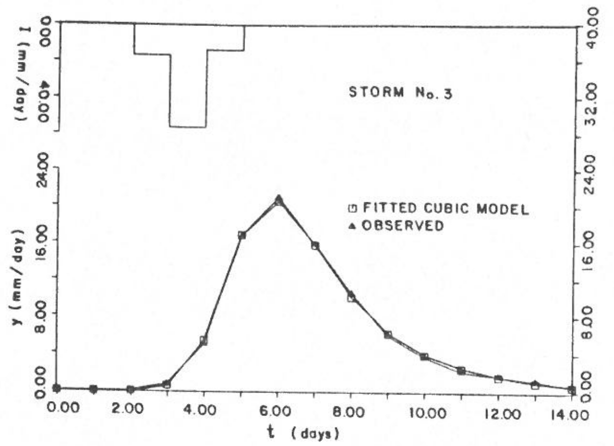
Results of simulation are shown in Fig.3.



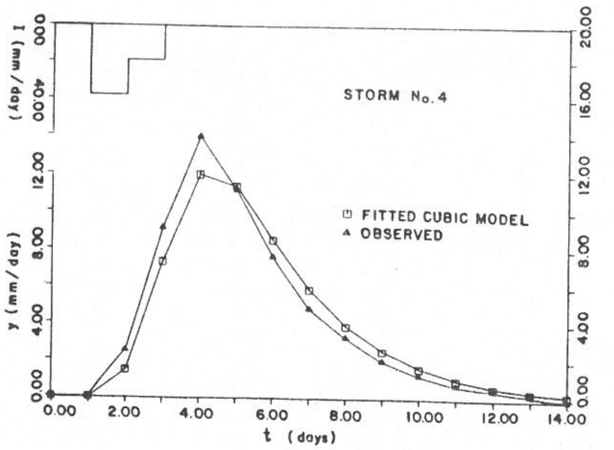
(3a)


STORM No. 2
FITTED CUBIC MODEL
OBSERVED
I (mm/day)
y (mm/day)
t (days)

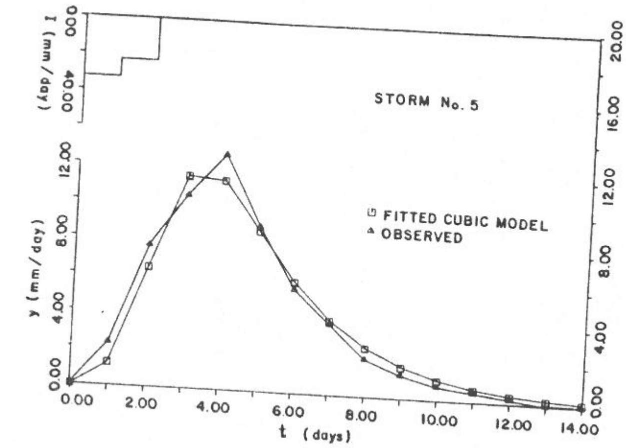
(3b)


STORM No. 3
FITTED CUBIC MODEL
OBSERVED
I (mm/day)
y (mm/day)
t (days)

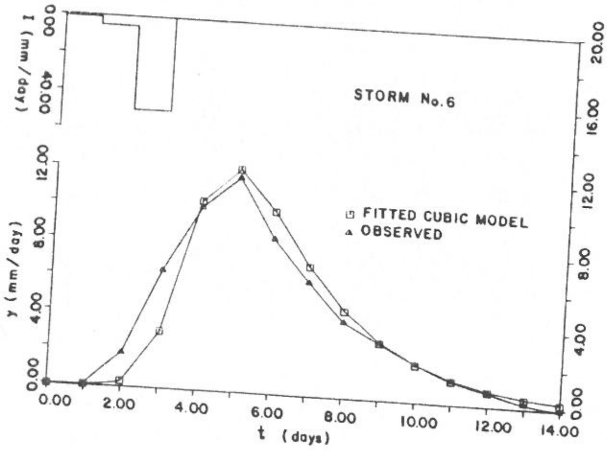
(3c)


STORM No. 4
FITTED CUBIC MODEL
OBSERVED
I (mm/day)
y (mm/day)
t (days)

(3d)


I (mm/day)
STORM No. 5
FITTED CUBIC MODEL
OBSERVED
y (mm/day)
t (days)

(3e)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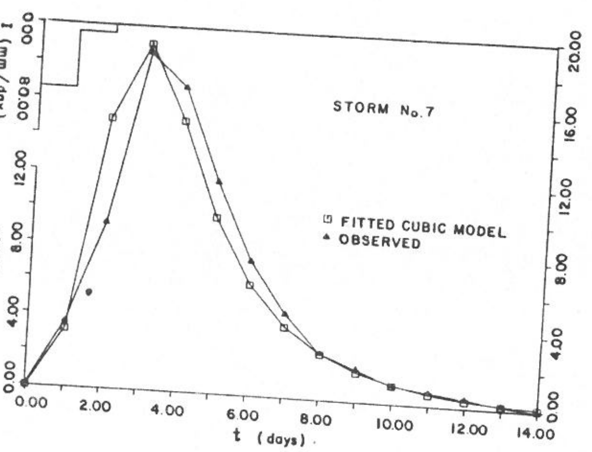
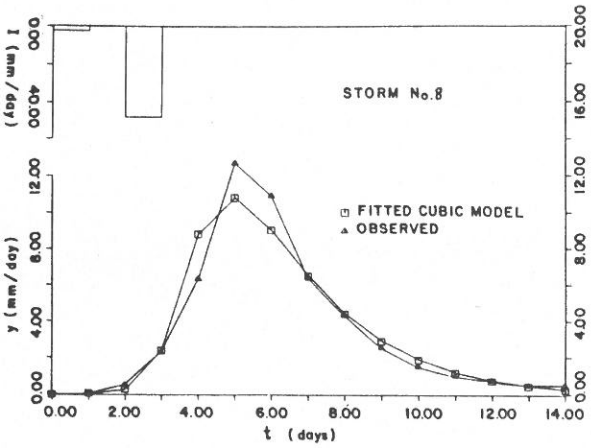
STORM No. 6
FITTED CUBIC MODEL
OBSERVED
y (mm/day)
t (days)

(3f)


I (mm/day)
STORM No. 7
FITTED CUBIC MODEL
OBSERVED
y (mm/day)
t (days)

(3g)

(3e)



(3h)

Figure 3. Comparison of observed runoff and that predicted by the discrete third-order Volterra model

(3f)

CONCLUSIONS

The significant advantage of the Volterra series model based on a cascade of nonlinear reservoirs is the parsimony in the number of parameters. The model used in the present study is characterized by four parameters only; a, b and c that pertain to the Taylor series resolution of outflow law and n - number of nonlinear conceptual elements in series. It ensures that identification problem is well-conditioned and that the solution is robust in presence of error in the input-output measurements. Moreover, the determination of the unknown four parameters can be reduced to optimization with respect to two variables only.

In the paper the discrete set of outputs is obtained via analytic solution of continuous problem which is simple to apply and guarantes better accuracy than methods used so far.

(3g)

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APPENDIX

Denoting

$$F_i = \int_0^T (t/T)^i \exp(a\phi t)\phi \, dt \,, \quad i=0,1,2,3$$

we have

$$B_1 = F_2$$
$$B_2 = F_1 - F_2$$
$$B_3 = F_o - 2F_1 + F_2$$
$$C_1 = F_3$$
$$C_2 = 3\ (F_1 - 2F_2 + F_3)$$
$$C_3 = 3\ (F_2 - F_3)$$
$$C_4 = F_o - 3F_1 + 3F_2 - F_3$$