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A NEW NON-LINEAR CONCEPTUAL MODEL OF FLOOD WAVES

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ABSTRACT

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This paper reports the successful simulation of St. Vénant's nonlinear distributed model of flood waves in open channels using a much simpler nonlinear lumped conceptual model. The simpler model is composed of a cascade of equal nonlinear storage elements preceded by an element of pure delay. The simpler model depends on four parameters only. The first two terms of the Taylor expansion of the state trajectory are shown to be equivalent to the first two terms of a Volterra convolution series. The accuracy of this quadratic approximation is shown by an example.

1. INTRODUCTION

The problem of synthesis or simulation in systems hydrology is the quest for a model which will convert a known input to a known output within certain limits of accuracy. It involves the selection of a model and the testing of the operation of this model by analysis (Dooge, 1973, pp. 10—12). The "known input" in this study is an assumed flood wave at a measuring station on a river. The "known output" is the corresponding flood wave at a station further downstream as given by a numerical solution of the non-linear St. Vénant model.

A relatively simple non-linear conceptual model is chosen. The parameters of the model are selected so as to convert the "known input" into an output which is as close as possible to the "known output". A second input is chosen and the model is tested by comparing the outputs from the conceptual model and the St. Vénant model. Such a test can be repeated as often as desired, in order to establish the accuracy with which the conceptual model simulates the more complex St. Vénant model.

If the conceptual model can be made to match the output from the St. Vénant equations for a variety of inputs, then a considerable simplification would be achieved. The advantages to be gained are:

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(a) Less field-measurement, computation and storage are required for computerised flood routing.

(b) Statistically efficient algorithms for model identification and parameter estimation can be used.

(c) The techniques of control theory can also be applied easily.

(d) The parsimony of only four lumped parameters suggests that statistically significant relationships might be found between them and broad-scale channel characteristics.

2. THE SECOND-ORDER STATE MODEL (SOSM)

The uniform channel is modelled by a linear channel and a cascade of nonlinear reservoirs. The linear channel (a pure translatory system) reflects the time of propagation of a perturbation along the positive characteristic. The cascade of nonlinear reservoirs is responsible for the attenuation of the system response.

Let $Q_i(t)$ be the inflow, and $Q_{i+1}(t)$ the outflow for the ith reach. Then the changes in the storage of each reservoir can be derived by solving simultaneously the continuity equation:

$$\dot{S}_i(t) = Q_i(t) - Q_{i+1}(t) \tag{1a}$$

and the outflow equation:

$$Q_{i+1}(t) = f[S_i(t)] \tag{1b}$$

under the initial condition S_{i_0}. The dot on S indicates a first-order derivative.

The function $f[S_i(t)]$ is unknown. We assume only that it is differentiable as many times as may be required for $S_i(t) \geqslant 0$. Substituting eq. 1b into eq. 1a and taking into account the delay T_0 in the input $Q_1(t)$ gives the following conceptual model for an open channel:

$$\left.\begin{aligned} \dot{S}_1(t) &= -f[S_1(t)] + Q_1(t - T_0) \\ \dot{S}_2(t) &= -f[S_2(t)] + f[S_1(t)] \\ &\vdots \\ \dot{S}_n(t) &= -f[S_n(t)] + f[S_{n-1}(t)] \end{aligned}\right| \tag{2a}$$

$$y(t) = f[S_n(t)] \tag{2b}$$

subject to the initial condition $S(0)$, and any given inflow:

$$Q_1(\tau), \ \tau \in [-T_0, 0]$$

We now derive the first two terms of the Taylor expansion of the state trajectory $y(t)$ and $S(t)$.

2.1. The Taylor expansion of a trajectory

The vector differential equation (2) can be considered as the definition of a nonlinear operator P mapping a space of inflows into a space of corresponding outflows. In order to determine how P operates for a given inflow hydrograph $Q_1(t)$, we must solve the set of eq. 2 together with its initial condition.

Let us denote by Q_0 the solution of equation set (2) corresponding to a steady state, and let:

$$y(t) = [PQ_1](t - T_0) \tag{3}$$

be a solution corresponding to a given inflow hydrograph $Q_1(t)$. The change of the trajectory from the steady state, Q_0 to $y(t)$, can be determined by means of a Taylor series expansion for operators (Wierzbicki, 1977) about Q_0:

$$y(t) - Q_0 = [PQ_0, \Delta Q_1](t - T_0) + 0.5[PQ_0, \Delta Q_1^2](t - T_0) + \tilde{y}(t) \tag{4a}$$

where $[PQ_0, \Delta Q_1](t - T_0) = \delta y(t)$, is the linear part of the outflow increment, namely, the first-order Fréchet differential of the operator P; $0.5[PQ_0, \Delta Q_1^2](t - T_0) = \delta^2 y(t)$, is the quadratic part of the outflow increment, the second-order Fréchet differential of the operator P; and $\tilde{y}(t)$ is a function of time, and is the error of approximation in the Taylor series.

So, the change in the inflow from Q_0 to $Q_1(t)$ by an increment $\Delta Q(t)$, implies the trajectory change $\Delta y(t)$ from Q_0 to $y(t)$:

$$y(t) = Q_0 + \Delta y(t) \tag{5}$$

This change in the trajectory of the output from the SOSM model can be divided into a linear part, a quadratic part and a residual error:

$$\Delta y(t) = \delta y(t) + \delta^2 y(t) + \tilde{y}(t) \tag{4b}$$

Lower-case deltas are used to make this distinction. In order to compute the linear and quadratic parts of the increment $\Delta y(t)$ we make use of a second Taylor expansion, namely an expansion of the storage—outflow relation (1b) about the steady-state storage S_{i_0} defined by:

$$Q_0 = f(S_{i_0}) \tag{6a}$$

Hence:

$$f[S_i(t)] - Q_0 = a\Delta S_i(t) + b[\Delta S_i(t)]^2 + \tilde{Q}_i(t) \tag{6b}$$

where

$$a = \partial f/\partial S_i \quad \text{and} \quad b = 0.5(\partial^2 f/\partial S_i^2) \tag{7}$$

$\Delta S_i(t) = S_i(t) - S_{i_0}$ is the change-of-storage trajectory due to a change in the inflow from Q_0 to $Q_1(t)$, namely the increment $\Delta Q_1(t)$. $\tilde{Q}_i(t)$ is the error term in Taylor series for the flow increment between each reservoir.

Furthermore, we can divide $\Delta S_i(t)$ into linear and quadratic parts:

$$\Delta S_i(t) = \delta S_i(t) + \delta^2 S_i(t) + \tilde{S}_i(t) \tag{8}$$

in exactly the same way as in expression (4b). Note that $\tilde{Q}_i(t) \neq f[\tilde{S}_i(t)]$ where $\tilde{S}_i(t)$ is the error term in the Taylor series for the storage in each reservoir $S_i(t)$.

We now derive the differential equations for the linear and quadratic parts of $\Delta y(t)$ and $\Delta S_i(t)$.

2.2. The function $\delta y(t)$ — The first part of the Taylor expansion

Substituting eqs. 6—8 into eq. 2 and limiting them to the first-order increments, yields the following set of equations:

$$\left.\begin{aligned} \delta\dot{S}_1(t) &= -a\delta S_1(t) + \Delta Q_1(t - T_0) \\ \delta\dot{S}_2(t) &= -a\delta S_2(t) + a\delta S_1(t) \\ &\vdots \\ \delta\dot{S}_n(t) &= -a\delta S_n(t) + a\delta S_{n-1}(t) \end{aligned}\right| \tag{9a}$$

$$\delta y(t) = a\delta S_n(t) \tag{9b}$$

$\delta S(0) = 0$ is the initial condition for this set of equations since we are not concerned in this study with increments in the initial condition for eq. 2, the starting point of this development.

The details of this linearisation are illustrated in the Appendix.

Eq. 9a can be expressed in a matrix notation as follows

$$\left.\begin{aligned} \delta\dot{S}(t) &= a\phi\delta S(t) + [1,0,\ldots,0]^{\mathrm{T}}\Delta Q_1(t - T_0) \\ \delta S(0) &= 0 \end{aligned}\right| \tag{10}$$

where

$$\phi = \begin{bmatrix} -1, & 0, & 0,\ldots, & 0 \\ 1, & -1, & 0,\ldots, & 0 \\ 0, & 1, & -1,\ldots, & 0 \\ \vdots & & & \\ 0, & 0, & 0,\ldots, & -1 \end{bmatrix} \tag{11}$$

2.3. The function $\delta^2 y(t)$ — The second part of the Taylor expansion

Substitution of eqs. 6—8 into eq. 2, and limiting them to the second-order increments, yields the following set of equations:

$$\left.\begin{aligned} \delta^2\dot{S}(t) &= a\phi\delta^2 S(t) + b\phi[\delta S(t)]^2 \\ \delta^2 S(0) &= 0 \end{aligned}\right| \tag{12a}$$

$$\delta^2 y(t) = a\delta^2 S_n(t) + b[\delta S_n(t)]^2 \tag{12b}$$

This is also illustrated in the Appendix. Note that the argument of the forcing function for eq. 12a is the solution of eq. 10, namely the function $\delta S(t)$, and that ϕ is common to both eqs. 10 and 12.

Having determined the functions $\delta S(t)$, $\delta^2 S(t)$, the third-order increment of the outflow trajectory can also be obtained in a similar way by expansion of the set (2) up to third-order increments.

The linear channel and the first and second terms of the Taylor expansion of the non-linear cascade form the model SOSM of flow about a steady state in an open channel.

2.4. The relationship between the SOSM model and the Volterra series for the cascade

2.4.1. The linear approximation. The solution of the linear set of eqs. 9a is (Athans and Falb, 1969):

$$\delta S(t) = \int_0^t \varphi(t-\xi)[1,0,\ldots,0]^{\mathrm{T}} \Delta Q_1(\xi - T_0)\mathrm{d}\xi \tag{13}$$

where $\varphi(t)$ is the transition matrix.

Hence, we must derive the transition matrix $\varphi(t)$ for the equations describing the first part of the SOSM. The easiest way to achieve this in the case of a stationary system is by using the Laplace transform. Hence:

$$\varphi(t) = \exp[a\phi t] = \mathscr{L}^{-1}[\{pI - a\phi\}^{-1}]$$

where p is a complex variable; I is the identity matrix, and ϕ is defined by eq. 11.

The state-transition matrix derived in this way for eq. 10 is of the form (Napiórkowski, 1978):

$$\varphi_{ij}(t) = \left\{\begin{array}{ll} [(at)^{i-j}/(i-j)!]\exp(-at), & \text{for } i \geqslant j \\ 0, & \text{for } j > i \end{array}\right\} i,j = 1,\ldots,n \tag{14}$$

Having the solution for the state-transition matrix we conclude that the linear increment of storage in ith reservoir can be determined according to the formula:

$$\delta S_i(t) = \int_0^t H_{1i}(\tau)\Delta Q_1(t-\tau-T_0)\mathrm{d}\tau \tag{15a}$$

where

$$H_{1i}(t) = \varphi_{i1}(t)$$

Substituting eq. 15a into eq. 9b for $i = n$ yields the following equation for the first term of the Taylor expansion of outflow:

$$\delta y(t) = \int_0^t h_1(\tau)\Delta Q_1(t-\tau-T_0)\mathrm{d}\tau \tag{15b}$$

where $h_1(t) = aH_{1n}(t)$ is the well-known transfer function for a cascade of linear reservoirs (Nash, 1957).

2.4.2. The quadratic approximation. The solution of the linear set of eqs. 12a describing the second Taylor term of the storage trajectory is:

$$\delta^2 S(t) = \int_0^t \varphi(t-\xi)b\phi[\delta S(\xi)]^2\mathrm{d}\xi \tag{16}$$

where $\varphi(t)$ is the transition matrix for eq. 12a. Note that the transition matrices for eq. 12a and for eq. 9a are the same and are given by eq. 14.

Substituting eq. 15a into eq. 16 and using a double change in the order of integration, after considerable manipulation yields (Napiórkowski, 1978):

$$\delta^2 S_i(t) = \int_0^t\int_0^t H_{2i}(\tau,\sigma)\Delta Q_1(t-\tau-T_0)\Delta Q_1(t-\sigma-T_0)\mathrm{d}\tau\mathrm{d}\sigma \tag{17a}$$

where

$$H_{2i}(\tau,\sigma) = \frac{b}{a}\left\{H_{1i}(\tau)\sum_{k=1}^{i} H_{1k}(\sigma) + H_{1i}(\sigma)\sum_{k=1}^{i-1} H_{1k}(\tau) - H_{1i}[\max(\tau,\sigma)]\right\} \tag{18a}$$

Substituting eq. 17a into eq. 12b for $i = n$ yields the following equation for the second Taylor term of the outflow trajectory:

$$\delta^2 y(t) = \int_0^t\int_0^t h_2(\tau,\sigma)\Delta Q_1(t-\tau-T_0)\Delta Q_1(t-\sigma-T_0)\mathrm{d}\tau\mathrm{d}\sigma \tag{19b}$$

where

$$\begin{aligned} h_2(\tau,\sigma) &= aH_{2n}(\tau,\sigma) + bH_{1n}(\tau)H_{1n}(\sigma) \\ &= b\left\{H_{1n}(\tau)\sum_{k=1}^{n} H_{1k}(\sigma) + H_{1n}(\sigma)\sum_{k=1}^{n} H_{1k}(\tau) - H_{1n}[\max(\tau,\sigma)]\right\} \end{aligned} \tag{18b}$$

It was proved by Napiórkowski (1978) that the second-order kernel defined by eq. 18b meets the conditions specified by Diskin and Boneh (1972) for a conservative inflow—outflow system described by the Volterra series.

So, the first and second Taylor terms of the outflow trajectory can be regarded as the first and second terms of the Volterra series expansion. The complete proof that series (6) corresponds to the Volterra series, and that the condition for convergence depends on the magnitude of the increase in the inflow, can be found in Napiórkowski (1978) and Napiórkowski and Strupczewski (1979, 1981).

3. THE IDENTIFICATION PROBLEM FOR THE SOSM MODEL

The problem to be solved is how to find the best estimates of the parameters of the SOSM model in the sense of minimum mean square error using records of the inflow increment $\Delta Q_1(t)$ and the outflow increment $\Delta Q_2(t)$ which have been observed in a finite time interval T. The problem of fitting the SOSM model can be represented as in Fig. 1. We are looking for the parameter estimates which minimize the objective function:

$$J(a, b, n, T_0) = \int_0^T [\Delta Q_2(t) - \delta y(t) - \delta^2 y(t)]^2 dt \tag{19}$$

The Wolfe variable metric optimization method with a modification due to Wierzbicki (1977) was used in solving this problem, since we can determine the gradient of the objective function (19) with respect to a, b and T_0. n was restricted to the integers and was subjected to a direct search. See Fig. 2 for the structure of the complete method.

It is computationally more efficient to calculate both the approximations and the gradient from the state-space representation of the model, rather than by using the double integrals in the corresponding Volterra series. See the Appendix for a comparison and also Napiórkowski et al. (1983).

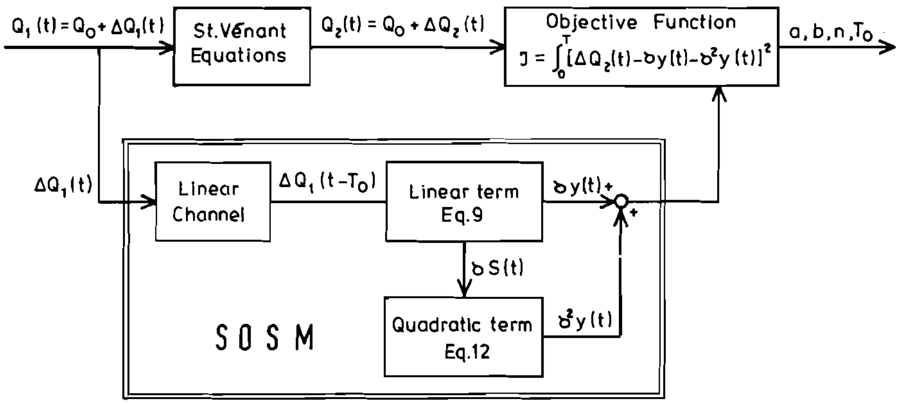

Fig. 1. The identification of the SOSM describing the flow deviations from a steady state.

3.1. *The gradient with respect to a*

The gradient with respect to a is:

$$\frac{\partial}{\partial a} J = \int_0^T 2[\Delta Q_2(t) - \delta y(t) - \delta^2 y(t)]\,[-\delta y_a(t) - \delta^2 y_a(t)]\,dt \tag{20}$$

where the subscript denotes derivatives with respect to a. From eqs. 9b and 12b we have respectively:

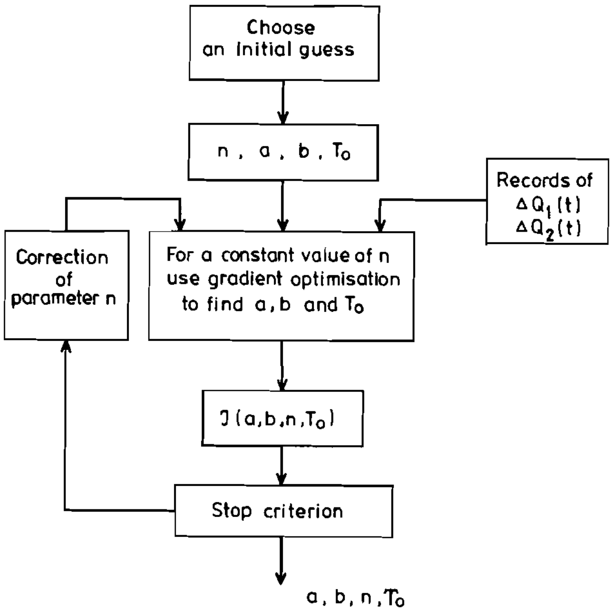

Fig. 2. The algorithm for the identification problem of the SOSM.

$$\delta y_a(t) = \delta S_n(t) + a\delta S_{na}(t) \tag{21}$$

$$\delta^2 y_a(t) = \delta^2 S_n(t) + \delta^2 S_{na}(t) + 2b\delta S_n(t)\delta S_{na}(t) \tag{22}$$

Hence, in order to calculate the derivatives of the first- and second-order increments, $\delta y_a(t)$ and $\delta^2 y_a(t)$, we must first calculate $\delta S_a(t)$ and $\delta^2 S_a(t)$.

From eqs. 9a and 12a we get:

$$\left.\begin{aligned} \delta\dot{S}_a(t) &= a\phi\delta S_a(t) + \phi\delta S(t) \\ \delta S_a(0) &= 0 \end{aligned}\right| \quad (23)$$

and

$$\left.\begin{aligned} \delta^2\dot{S}_a(t) &= a\phi\delta^2 S_a(t) + \phi\delta^2 S(t) + 2b\phi\delta S(t)\delta S_a(t) \\ \delta^2 S_a(0) &= 0 \end{aligned}\right| \quad (24)$$

where the forcing function for eq. 23 is known from the solution of eq. 9a, and the forcing function for eq. 24 is known after solving eqs. 9a, 12a and 23.

3.2. The gradient with respect to b

The gradient with respect to b is:

$$\frac{\partial}{\partial b} J = \int_0^T 2[\Delta Q_2(t) - \delta y(t) - \delta^2 y(t)][-\delta y_b(t) - \delta^2 y_b(t)]\,dt \tag{25}$$

The derivative of $\delta y(t)$ with respect to b is equal to 0, since $\delta y(t)$ does not depend on b. To calculate $\delta^2 y_b(t)$ we make use of eq. 12b:

$$\delta^2 y_b(t) = a\delta^2 S_{nb}(t) + [\delta S_n(t)]^2 \tag{26}$$

First, we must calculate $\delta^2 S_b(t)$ in order to find $\delta^2 y_b(t)$. From eq. 12a:

$$\left.\begin{aligned} \delta^2 \dot{S}_b(t) &= a\phi\delta^2 S_b(t) + \phi[\delta S(t)]^2 \\ \delta^2 S_b(0) &= 0 \end{aligned}\right| \tag{27}$$

By comparing eqs. 27 and 12a we conclude that:

$$b\delta^2 S_b(t) = \delta^2 S(t) \tag{28}$$

and from eq. 26 that:

$$b\delta^2 y_b(t) = \delta^2 y(t) \tag{29}$$

3.3. The gradient with respect to T_0

The gradient with respect to T_0 is:

$$\frac{\partial}{\partial T_0} J = \int_0^T 2[\Delta Q_2(t) - \delta y(t) - \delta^2 y(t)][-\delta y_{T_0}(t) - \delta^2 y_{T_0}(t)]\,dt \tag{30}$$

From eq. 6 and directly from the analytical solutions for linear and quadratic-order increments $\delta y(t)$ and $\delta^2 y(t)$ (see eqs. 15b and 19b) one can see that:

$$\frac{\partial}{\partial T_0}[\delta y(t) + \delta^2 y(t)] = -\frac{\partial}{\partial t}[\delta y(t) + \delta^2 y(t)] \tag{31}$$

due to pure delay in the inflow of the SOSM.

3.4. Initial parameter values

All gradient optimization techniques require initial values of the parameters. A cascade of n identical linear reservoirs (i.e. $b = 0$) preceded by a pure delay of magnitude T_0 is taken as a first approximation of the SOSM. The parameters T_0, a and n for this cascade are calculated using the moment or cumulant technique (Nash, 1959; Dooge, 1973; Dooge and O'Kane, 1977). For uniform channels, the first three cumulants for the linearised St. Vénant equations are (Dooge, 1980):

$$K_1 = x/mu_0 \quad (32a)$$

$$K_2 = 2(mr)^{-1}[1-(m-1)^2F_o^2](\bar{y}_0/S_0x)(x/mu_0)^2 \quad (32b)$$

$$K_3 = 12(mr)^{-2}[1-(m-1)^2F_o^2][1+(m-1)F_o^2](\bar{y}_0/S_0x)^2(x/mu_0)^3 \quad (32c)$$

where x is the distance along the channel; $u_0 = Q_0/A_0$ is the reference velocity through the area A; $\bar{y} = A_0/B_0$ is the reference value of the hydraulic mean depth; B_0 is the width of the channel at the water surface; m is the number which represents the ratio of the kinematic wave celerity to the reference velocity $m = (dQ/dA)/(Q/A)$; F_o is the Froude number defined by $F_o = Q_0^2B_0/gA^3$; r depends on friction and shape $r = Q_0(\partial S_f/\partial Q)/S_0$; and S_f and S_0 are the friction and bed slopes, respectively.

For the cascade of linear reservoirs with delay the first three cumulants are:

$$K_1 = n/a + T_0, \qquad K_2 = n/a^2 \qquad \text{and} \qquad K_3 = 2n/a^3$$

where n is the number of reservoirs in the cascade and the parameter a is defined in eq. 7.

By equating the first three cumulants of the linearised St. Vênant equations to the first three cumulants of the cascade of linear reservoirs with delay we find the following expressions which relate the parameters of the more complex model (x, m, r, u_0, $\bar{y}_0$, F_o, S_0) to the parameters of the simpler model (a, b, n, T_0). This provides the initial values for these parameters in the SOSM model $a = 2K_2/K_3$, $b = 0$, $n = a^2K_2$, $T_0 = K_1 - n/a$.

4. RESULTS OF NUMERICAL EXPERIMENTS

We present below an example which illustrates the applicability of the suggested method. The objective is to solve the problem of identifying the four parameters a, b, T_0 and n of the model which describes the flow deviations from a steady state in a rectangular prismatic channel. The transformation of the flow is modelled by the St. Vénant equations. The steady flow in the channel itself is characterized by the following parameters: flow $Q_0 = 200\,\text{m}^3\,\text{s}^{-1}$, velocity $u_0 = 1\,\text{m}\,\text{s}^{-1}$, depth of flow $\bar{y}_0 = 2\,\text{m}$, width $B_0 = 100\,\text{m}$, slope $S_0 = 0.000248$, Chézy coefficient $C = 44.9$. Calculations were carried out for a reach length of $x = 40\,\text{km}$. The initial values of the parameters were calculated from eq. 32 and found to be $a = 0.184 \cdot 10^{-3}\,\text{s}^{-1}$, $b = 0$, $n = 3$, $T_0 = 9552\,\text{s}$.

In the numerical experiment the parameters a, b, n and T_0 were identified for an inflow increment ΔQ_1 in the form of a rectangular pulse function:

$$\Delta Q_1(t) = \begin{cases} 200\,\text{m}^3\,\text{s}^{-1}, & \text{for} \quad t\ [0, 6000\,\text{s}] \\ 0, & \text{for} \quad t > 6000\,\text{s} \end{cases}$$

We control the ill-conditioned nature of this identification problem by choosing an input which contains almost all important frequencies. Such an input will reveal almost all aspects of the system response.

Rapid convergence was found in the cases which were studied due to superlinear convergence of the Wolfe method, and the results obtained are $a = 0.1437 \cdot 10^{-3}\,s^{-1}$, $b = 0.4812 \cdot 10^{-10}\,s^{-1}\,m^{-3}$, $n = 3$, $T_0 = 7228$. The optimized fit of the SOSM to the St. Vénant equation is shown in Fig. 3.

The accuracy with which the SOSM model simulates the model of St. Vénant was examined using a typical input increment taken from Ponce and Theurer (1982):

$$\Delta Q_1^*(t) = Q_0 \exp[-(t - T_p)/(T_g - T_p)]\,(t/T_p)^{T_p/(T_g - T_p)}$$

$$Q_0 = 200\,m^3\,s^{-1}, \qquad T_g = 4000\,s, \qquad T_p = 2000\,s$$

which is much smoother than the pulse function which was used during model identification. The comparison of outputs is shown in Fig. 4 and was found to be satisfactory. The second term of the Taylor expansion is seen to produce a marked improvement in the conceptual model when compared with its linearised version, namely, a lagged cascade of n identical linear reservoirs.

Using the same values of a and b but doubling the values of n and T_0 a prediction was made at $x = 80$ km using the smooth input. This is shown in Fig. 5 where it is compared with the solution of the St. Vénant equation for the same smooth input. Clearly, the result is in accordance with our expectations.

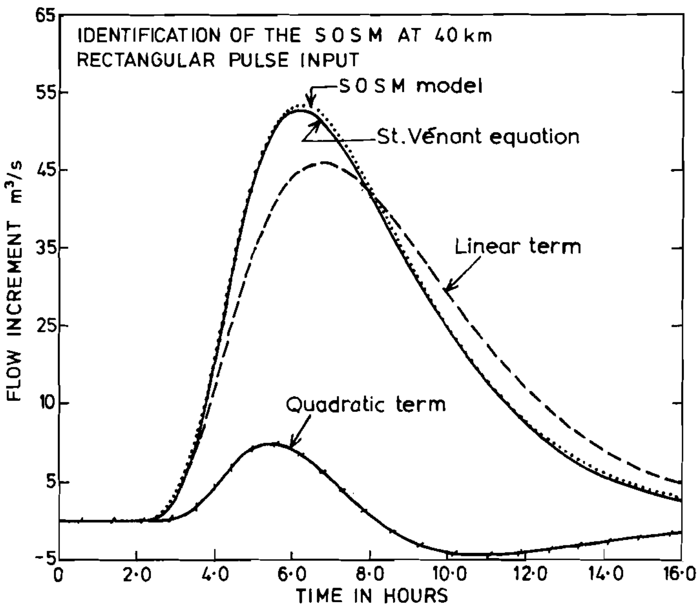

Fig. 3. Comparison of SOSM with the model of St. Vénant at 40 km using a rectangular pulse input.

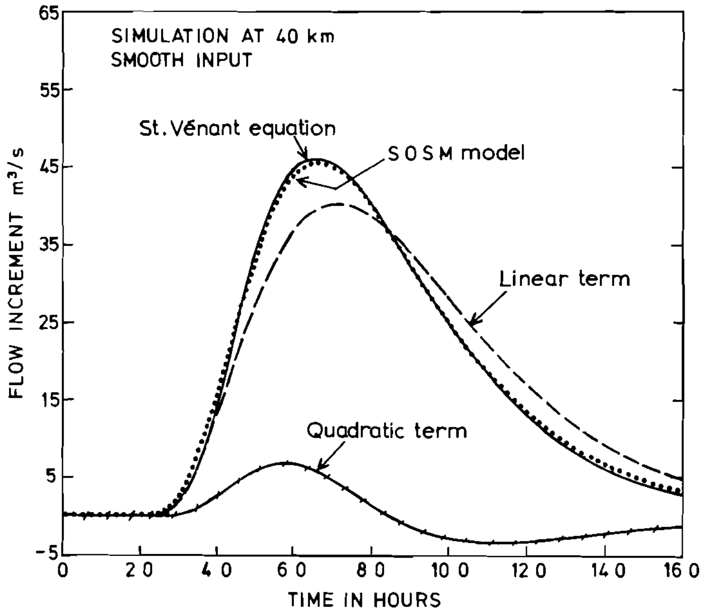

Fig. 4. Simulation of St. Vénant model at 40 km using a smooth input.

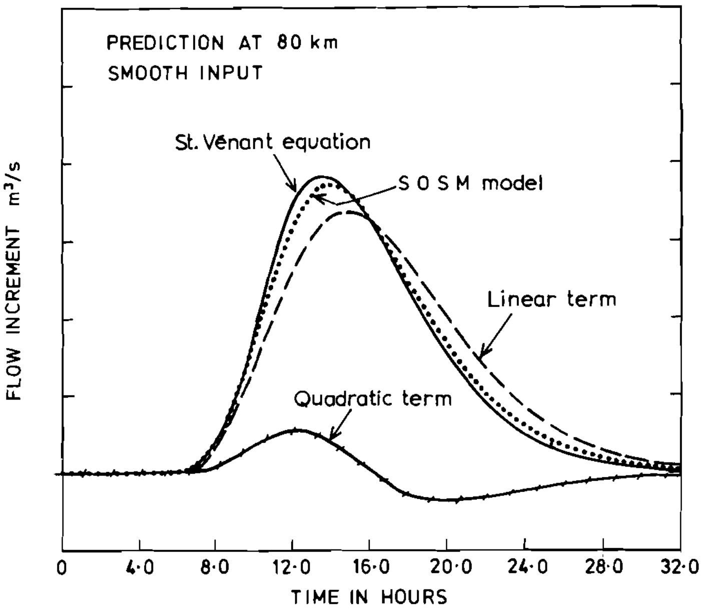

Fig. 5. Prediction of the output from St. Vénant model at 80 km using the same smooth input.

APPENDIX

Consider a single reach channel. The following nonlinear first-order differential equation applies:

$$\dot{S}(t) = -f[S(t)] + Q(t-T_0) \quad \text{and} \quad S(0) = S_0 \tag{A-1a}$$

$$y(t) = f[S(t)] \tag{A-1b}$$

The change in inflow from a steady flow Q_0 to $Q(t)$ such as:

$$Q(t) - Q_0 = \Delta Q(t) \tag{A-2}$$

implies the change of outflow and storage trajectories:

$$y(t) - Q_0 = \Delta y(t) = \delta y(t) + \delta^2 y(t) + \tilde{y}(t) \tag{A-3a}$$

$$S(t) - S_0 = \Delta S(t) = \delta S(t) + \delta^2 S(t) + \tilde{S}(t) \tag{A-3b}$$

We now compute the first (δ) and second (δ^2) terms of a Taylor expansion of eq. A-1 for the increments defined by eqs. A-2 and A-3:

$$\delta\dot{S}(t) + \delta^2\dot{S}(t) + \dot{\tilde{S}}(t) = -f(S_0) - a[\delta S(t) + \delta^2 S(t) + \tilde{S}(t)]$$
$$- b[\delta S(t) + \delta^2 S(t) + \tilde{S}(t)]^2 - \tilde{Q}(t) + Q_0 + 1 \cdot \Delta Q(t - T_0) \tag{A-4a}$$

$$Q_0 + \delta y(t) + \delta^2 y(t) + \tilde{y}(t) = f(S_0) + a[\delta S(t) + \delta^2 S(t) + \tilde{S}(t)]$$
$$+ b[\delta S(t) + \delta^2 S(t) + \tilde{S}(t)]^2 + \tilde{Q}(t) \tag{A-4b}$$

where $a = \partial f/\partial S$, $b = 0.5\,\partial^2 f/\partial S^2$.

The first term of the Taylor expansion

By subtracting from eq. A-4 the terms corresponding to the steady state:

$$Q_0 = f(S_0) \tag{A-5}$$

and by limiting eq. A-4 to the first-order terms we have:

$$\delta\dot{S}(t) = -a\delta S(t) + \Delta Q(t - T_0) \quad \text{and} \quad \delta S(t) = 0 \tag{A-6a}$$

$$\delta y(t) = a\delta S(t) \tag{A-6b}$$

Eqs. A-6 have the following solution:

$$\delta S(t) = \int_0^t \exp[-a(t-\lambda)]\,\Delta Q(\lambda - T_0)\,\mathrm{d}\lambda \tag{A-7}$$

$$\delta y(t) = \int_0^t a\exp[-a(t-\lambda)]\,\Delta Q(\lambda - T_0)\,\mathrm{d}\lambda \tag{A-8}$$

where $\exp(-at)$ is the transition matrix for eq. A-6a.

The kernels of the transformations (A-7) and (A-8) are respectively:

$$H_1(t) = \exp(-at) \quad \text{and} \quad h_1(t) = a\exp(-at) \qquad \text{(A-9), (A-10)}$$

Eqs. A-9 and A-10 describe the first-order kernels of the Volterra series for the relations inflow—storage and inflow—outflow.

The second term of the Taylor expansion

By subtracting eq. A-6 (the first-order terms) and eq. A-5 (the terms corresponding to steady state) from eq. A-4 we get:

$$\delta^2\dot{S}(t) = -a\delta^2 S(t) - b[\delta S(t)]^2 \quad \text{and} \quad \delta^2 S(0) = 0 \qquad \text{(A-11a)}$$

$$\delta^2 y(t) = a\delta^2 S(t) + b[\delta S(t)]^2 \qquad \text{(A-11b)}$$

Eq. A-11a has the following solution:

$$\begin{aligned}\delta^2 S(t) &= \int_0^t \exp[-a(t-\xi)]\,(-b)\,[\delta S(\xi)]^2 d\xi \\ &= \int_0^t -b\exp[-a(t-\xi)] \int_0^\xi\int_0^\xi \exp[-a(2\xi-\lambda_1-\lambda_2)] \\ &\quad \times \Delta Q(\lambda_1 - T_0)\Delta Q(\lambda_2 - T_0) d\lambda_1 d\lambda_2 d\xi \qquad \text{(A-12)}\end{aligned}$$

The double change in the order of integration results in:

$$\begin{aligned}\delta^2 S(t) &= \int_0^t\int_0^t -b\exp[-a(t-\lambda_1-\lambda_2)] \int_0^t 1(\xi-\lambda_1)1(\xi-\lambda_2)\exp(-a\xi)d\xi \\ &\quad \times \Delta Q(\lambda_1 - T_0)\Delta Q(\lambda_2 - T_0) d\lambda_1 d\lambda_2 \qquad \text{(A-13)}\end{aligned}$$

where $1(t)$ is the unit step function.

Because:

$$\begin{aligned}&\int_0^t 1(\xi-\lambda_1)1(\xi-\lambda_2)\exp(-a\xi)d\xi \\ &= \int_{\max(\lambda_1,\lambda_2)}^t \exp(-a\xi)d\xi = -[\exp(-at) - \exp\{-a\max(\lambda_1,\lambda_2)\}]/a\end{aligned}$$

the second term of the Taylor expansion of the trajectory is described by:

$$\begin{aligned}\delta^2 S(t) &= \int_0^t\int_0^t \frac{b}{a}(\exp[-a(2t-\lambda_1-\lambda_2)] - \exp[-a\{t-\min(\lambda_1,\lambda_2)\}]) \\ &\quad \times \Delta Q(\lambda_1 - T_0)\Delta Q(\lambda_2 - T_0) d\lambda_1 d\lambda_2 \qquad \text{(A-14)}\end{aligned}$$

After substitution of the variables $\tau = t - \lambda_1$; $\sigma = t - \lambda_2$, the kernel of that transformation can be described as:

$$H_2(\tau,\sigma) = (b/a)[\exp\{-a(\tau+\sigma)\} - \exp[-a\max(\tau,\sigma)\}] \qquad \text{(A-15)}$$

From eq. A-11b one can see that:

$$\delta^2 y(t) = \int_0^t \int_0^t h_2(\tau, \sigma)\Delta Q(t-\tau-T_0)\Delta Q(t-\sigma-T_0)\mathrm{d}\tau\mathrm{d}\sigma \tag{A-16}$$

where

$$h_2(\tau, \sigma) = aH_2(\tau, \sigma) + bH_1(\tau)H_1(\sigma)$$
$$= b\,[2\exp\{-a(\tau+\sigma)\} - \exp[-a\max(\tau, \sigma)] \tag{A-17}$$

Eqs. A-15 and A-17 describe the second-order kernels of the Volterra series for the relations inflow—storage and inflow—outflow, respectively.

Derivatives with respect to model parameters

The derivatives of $\delta y(t)$ and $\delta^2 y(t)$ with respect to parameters a, b and T_0 can be calculated directly from eqs. A-8 and A-16, or can also be sought as solutions of differential equations obtained by differentiating eqs. A-6 and A-11.

As an example a derivative of $\delta y(t)$ with respect to a will be calculated below. From eq. A-8:

$$\frac{\partial}{\partial a}\delta y(t) = \int_0^t \frac{\partial}{\partial a} h_1(\tau)\Delta Q(t-\tau-T_0)\mathrm{d}\tau$$
$$= \int_0^t [\exp(-a\tau) - a\tau\exp(-a\tau)]\,\Delta Q(t-\tau-T_0)\mathrm{d}\tau \tag{A-18}$$

From eq. A-6:

$$\delta\dot{S}_a(t) = -a\delta S_a(t) - \delta S(t) \quad \text{and} \quad \delta S_a(0) = 0 \tag{A-19a}$$

$$\delta y_a(y) = a\delta S_a(t) + \delta S(t) \tag{A-19b}$$

The solution of eq. A-19a is:

$$\delta S_a(t) = -\int_0^t \exp[-a(t-\xi)]\,\delta S(\xi)\mathrm{d}\xi \tag{A-20}$$

Inserting eq. A-7 into eq. A-20 for $\delta S(\xi)$:

$$\delta S_a(t) = -\int_0^t \exp[-a(t-\xi)] \int_0^\xi \exp[-a(\xi-\lambda)]\,\Delta Q(\lambda-T_0)\mathrm{d}\lambda\mathrm{d}\xi$$

Integration with respect to ξ gives:

$$\delta S_a(t) = -\int_0^t (t-\lambda)\exp\left[-a(t-\lambda)\right]\Delta Q(\lambda - T_0)\mathrm{d}\lambda$$
$$= -\int_0^t \tau \exp(-a\tau)\Delta Q(t-\tau-T_0)\mathrm{d}\tau \qquad \text{(A-21)}$$

Inserting eqs. A-7 and A-21 into eq. A-19b gives a solution identical to that obtained by the direct method.

Note that the second method which uses a state-space representation reduces the time of calculation in the case of $\delta^2 S_a(t)$. It is computationally much more efficient to solve the linear differential equation than to calculate double integrals.

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