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A Discrete Conceptualization of a Volterra Series Model for Surface Runoff

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The Volterra series rainfall-runoff model used to be regarded as a nonlinear black box with no correspondence between the physical prototype and the model parameters. However, as was outlined in earlier contributions by Napiórkowski and coauthors (J. J. Napiórkowski, 1978, 1983; J. J. Napiórkowski and W. G. Strupczewski, 1979, 1981; J. J. Napiórkowski and P. O'Kane, 1984), the output signal of a cascade of identical nonlinear reservoirs can be approximately expressed in the form of a Volterra series model. This offers obvious advantages in the process of identification due to the parsimony of parameters. The conceptualization of the Volterra series model performed in the present study is based on a state space formulation within the discrete framework justified in view of the inherently discrete problems that we are forced to deal with in practice. The methodology presented is tested on data from the catchment of the Cache River in southern Illinois.

1. INTRODUCTION

The technique of Volterra series introduced to hydrology by *Amorocho and Orlob* [1961] and analyzed thereafter by several researchers holds an established position among nonlinear rainfall-runoff models. The general formulation of the Volterra series model is

$$y(t) = \sum_{i=1}^{N} \left\| \int_0^t \right\|_i h_i(\tau_1, \cdots, \tau_i) \prod_{k=1}^{i} x(t-\tau_k)\, d\tau_k \tag{1}$$

where x is the input signal, h is the kernel function, $\|\int\|_i$ is an $i$ple integral, and Π is a product. In hydrologic applications the value of N is usually taken as 2, instead of being infinite as in the theory of series.

From the very beginning of its hydrologic applications the model was conceived to be of the black box type; i.e., it could be regarded as a direct extension of the convolution integral technique accompanying the concept of the instantaneous unit hydrograph. The late Professor Amorocho, the pioneer in the field of Volterra series hydrologic modeling, wrote in 1973 that "at this point no correspondence can be assumed ... to exist between the components of the polynomial system (i.e., Volterra series model, comment added) and any of the physical elements of the prototype" [*Amorocho*, 1973]. This attitude was very characteristic of the first two decades of hydrologic applications of the Volterra series model. No provision was made for use of the physical system characteristics in the process of identification of kernels of the integrals contributing to the Volterra series.

The Volterra series model was never conceptualized, and typically the number of parameters necessary for the kernel determination was large. Approximation of kernel functions with the help of spline functions requires a very great number of parameters, depending on the number of elementary spline functions considered and on the order of spline functions. That is why the only spline approximation used in hydrologic Volterra series identification was the zero-order method (i.e., piecewise constant elementary functions), also called the method of ordinates. The approximation of the kernel functions with the help of orthogonal polynomials (e.g., of Laguerre or Meixner type) also requires numerous parameters. These methods decompose a higher-order kernel of the black box Volterra series model as

$$h_i(\tau_1, \cdots, \tau_i) = \sum_{j_1=1}^{N_1} \cdots \sum_{j_i=1}^{N_i} a_{j_1 \cdots j_i} h_{j_1 \cdots j_i}(\tau_1, \cdots, \tau_i) \tag{2}$$

or

$$h_i(\tau_1, \cdots, \tau_i) = \sum_{j_1=1}^{N_1} \cdots \sum_{j_i=1}^{N_i} a_{j_1 \cdots j_i} h_{j_1}(\tau_1) \cdots h_{j_i}(\tau_i) \tag{3}$$

where $\{h_{j_1 \cdots j_i}\}$ is a set of spline functions, and $\{h_{j_k}\}$ is a set of orthogonal functions. The details of derivation of (2) and (3) can be found in the work by *Diskin and Boneh* [1973] and *Amorocho* [1973], respectively.

The first attempt to attribute some conceptual meaning to the black box kernel of the Volterra series model used in hydrology was due to *Diskin and Boneh* [1972]. The structure of their second-order kernel, however, was not directly related to any physical model. It was merely a hypothetical example, for the sake of illustration, of properties that the theoretical second-order kernel of a conservative system should possess.

Napiórkowski and coauthors [*Napiórkowski*, 1978, 1983; *Napiórkowski and Strupczewski*, 1979, 1981; *Napiórkowski and O'Kane*, 1984] aimed to establish a relationship between the nonlinear conceptual models in the state space framework and the Volterra series model. This important step toward the conceptualization of the kernels of the Volterra series models of order higher than one results in a significant reduction in the number of parameters. This follows earlier developments in unit hydrograph conceptualization, where instead of a great number of parameters of the original black box models (via ordinates, splines, or orthogonal functions resolution) a few conceptual parameters remained to be identified.

The novelty of the present contribution is the conceptualization of the Volterra series via a state space approach within the discrete framework. This is considered necessary, as in the course of mathematical modeling one always deals with discrete signals either at the stage of measurements or at the stage of numerical analysis.

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2. Nonlinear Differential Model and its Relation to Volterra Series

Assume that the hydrologic system considered can be approximately represented by means of the wide class of mathematical models consisting of a series combination of equal nonlinear dynamic structures with differentiable (n times) outflow laws. An established example of a structure embraced by the above formulation is a cascade of equal nonlinear reservoirs with power outflow law [*Napiórkowski and Strupczewski*, 1979].

Assume that the system is represented by the following set of equations.

Law of conservation of mass:
continuity in the ith element

$$\dot{S}_i(t) = I_i(t) - Q_i(t) \qquad i = 1, \ldots, N$$
$$S_i(0) = S_{0i} \tag{4}$$

Law of conservation of mass:
linkage between $(i+1)$th and ith elements

$$I_{i+1}(t) = Q_i(t) \qquad i = 1, \ldots, N-1 \tag{5}$$

Conceptual output equation (outflow law)

$$Q_i(t) = f[S_i(t)] \qquad i = 1, \ldots, N \tag{6}$$

where I_i is the inflow to the ith element, Q_i is the outflow from the ith element, and S_i is the storage in the ith element. The above set of equations can be easily formulated in the state space framework as follows.

State equations

$$\begin{aligned} \dot{S}_1(t) &= x(t) - f[S_1(t)] \\ \dot{S}_2(t) &= f[S_1(t)] - f[S_2(t)] \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{S}_N(t) &= f[S_{N-1}(t)] - f[S_N(t)] \\ \mathbf{S}(0) &= \mathbf{S}_0 \end{aligned} \tag{7}$$

Output equation

$$y(t) = f[S_N(t)] \tag{8}$$

where $x(t) = I_1(t)$ is the input signal, and $y(t) = Q_N(t)$ is the output signal.

If the form of the differentiable function f is not known, the above set of equations cannot be solved by a direct numerical method. One of the ways of solution of the problem is given in the present paper.

The findings presented in this paragraph borrow heavily from *Napiórkowski* [1978, 1983], *Napiórkowski and O'Kane* [1984], and *Napiórkowski and Strupczewski* [1979, 1981]. The state equation (7) can be regarded as a nonlinear operator P mapping the space of inflows (input signals) $\{x\}$ into the space of storages (state variables) $\{\mathbf{S}\}$:

$$\mathbf{S}(t) = [Px](t) \tag{9}$$

Denote now the trajectories of inflow, storage, and outflow for the steady state conditions as x_0, S_{0i}, and $y_0 = f(S_{0i}) = x_0$, respectively. Development for unsteady flow reference conditions given in the work by *Napiórkowski and Strupczewski* [1979] yields more complicated results and will not be followed herein. Let the inflow be perturbed around the reference level as

$$x(t) = x_0 + \Delta x(t) \tag{10}$$

Then the state and the output trajectories would differ from $\mathbf{S}_0$ and y_0 by $\Delta\mathbf{S}(t)$ and $\Delta y(t)$, respectively. These deviations can be expressed in the following form:

$$\Delta\mathbf{S}(t) = \mathbf{S}(t) - \mathbf{S}_0 = \delta\mathbf{S}(t) + \delta^2\mathbf{S}(t) + \cdots \tag{11}$$

$$\Delta y(t) = y(t) - y_0 = \delta y(t) + \delta^2 y(t) + \cdots \tag{12}$$

The first and second terms on the right-hand side of either of (11) or (12) represent the linear variation (δ) and the quadratic variation (δ^2) (or first and second Frechet differential of the operator P; compare *Dieudonné* [1969]), respectively.

Assume that the outflow law given by (6) can be developed in the Taylor series

$$f[S_i(t)] = f(S_{0i}) + a\Delta S_i(t) + b[\Delta S_i(t)]^2 + \cdots \tag{13}$$

where

$$a = \left.\frac{df}{dS_i}\right|_{S_{0i}} \qquad b = 0.5 \left.\frac{d^2 f}{dS_i^{\,2}}\right|_{S_{0i}}$$

Accordingly, equations for linear and quadratic increments from (11) and (12) can be obtained by inserting (11)–(13) into (7) and neglecting all variations of orders higher than 1 or 2. As we do not consider changes in the initial conditions of (7), the initial conditions for variations are zero.

2.1. *Linear Approximation*

Substituting (11)–(13) into (7) and limiting them to the first-order variations yields the following set of equations:

$$\begin{aligned} \delta\dot{\mathbf{S}}(t) &= a\boldsymbol{\varphi}\delta\mathbf{S}(t) + [1, 0, \cdots, 0]^T \Delta x(t) \\ \delta y(t) &= a\delta S_N(t) \\ \delta\mathbf{S}(0) &= \mathbf{0} \end{aligned} \tag{14}$$

where

$$\boldsymbol{\varphi} = \begin{bmatrix} -1, & 0, & \cdots, & 0 \\ 1, & -1, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & -1 \end{bmatrix} \tag{15}$$

Using the convenient formulation originating from dynamic systems theory one can write the set (14) in the form

$$\begin{aligned} \delta\dot{\mathbf{S}}(t) &= \mathbf{A}\delta\mathbf{S}(t) + \mathbf{B}\Delta x(t) \\ \delta y(t) &= \mathbf{C}\delta\mathbf{S}(t) \\ \delta\mathbf{S}(t_0) &= \Delta_0 \end{aligned} \tag{16}$$

with the solution for the stationary case

$$\delta\mathbf{S}(t) = \phi(t - t_0)\delta\mathbf{S}(t_0) + \int_{t_0}^{t} \phi(t - \tau)\mathbf{B}\Delta x(\tau)\, d\tau \tag{17}$$

where

$$\begin{aligned} \mathbf{A} &= a\boldsymbol{\varphi} \\ \mathbf{B} &= [1, 0, \cdots, 0]^T \\ \mathbf{C} &= [0, \cdots, 0, 1] \\ t_0 &= 0 \qquad \Delta_0 = 0 \end{aligned} \tag{18}$$

and $\phi(t) = \exp{(\mathbf{A}t)}$ is the transition matrix that can be effectively determined as follows:

$$[\exp(\mathbf{A}t)]_{ij} = (at)^{i-j}\exp(-at)/(i-j)! \qquad i \geq j$$
$$[\exp(\mathbf{A}t)]_{ij} = 0 \qquad j < i \tag{19}$$

Inserting (18) into (17) one obtains

$$\delta\mathbf{S}(t) = \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\Delta x(\tau)\,d\tau \tag{20}$$

2.2. *Quadratic Approximation*

Substitution of (11)–(13) into (7) and limiting them to the second-order variations yields the following set of equations:

$$\delta^2\dot{\mathbf{S}}(t) = a\boldsymbol{\varphi}\delta^2\mathbf{S}(t) + b\boldsymbol{\varphi}[\delta\mathbf{S}(t)]^2$$
$$\delta^2 y(t) = a\delta^2 S_N(t) + b[\delta S_N(t)]^2 \tag{21}$$
$$\delta^2\mathbf{S}(0) = \mathbf{0}$$

The solution of (21) can be found as for (14). Note that the argument of the forcing function for (21) is the solution of (14) and that the transition matrix is common for both (14) and (21).

One arrives at structures analogous to the first two terms of the Volterra series:

$$\delta y(t) = \int_0^t h_1(\tau)\Delta x(t-\tau)\,d\tau \tag{22}$$

$$\delta^2 y(t) = \int_0^t\int_0^t h_2(\tau_1, \tau_2)\Delta x(t-\tau_1)\Delta x(t-\tau_2)\,d\tau_1\,d\tau_2 \tag{23}$$

where

$$h_1(t) = aH_N(t) \tag{24}$$

$$h_2(t_1, t_2) = b\left\{H_N(t_1)\sum_{i=1}^{N} H_i(t_2) + H_N(t_2)\sum_{i=1}^{N} H_i(t_1)\right\} - bH_N[\max(t_1, t_2)] \tag{25}$$

N is an integral number, a and b are parameters resulting from the outflow-storage relation, and

$$H_i(t) = \frac{(at)^{i-1}}{(i-1)!}\exp(-at) \tag{26}$$

is the dimensionless first-order kernel, the widely used Nash cascade form of the instantaneous unit hydrograph [*Nash*, 1959].

The linear and quadratic increments can be determined either from (22) and (23) or directly from the state space formulation (equations (14) and (21)). From the computational viewpoint the latter method is simpler.

The use of the two-term Volterra series model based on a cascade of nonlinear reservoirs is subject to certain restrictions on the parameters, which are needed to ensure copositivity of the model (a positive response to a positive input). The sufficient copositivity condition is related to the amplitude of the input signal (compare J. J. Napiórkowski, unpublished manuscript, 1986) rather than to the total volume of the input signal, as was suggested by *Diskin et al.* [1984].

3. DISCRETE FRAMEWORK

The discrete formulation of the problem is approached as follows. Instead of a simple mechanical discretization of the kernel, its continuous form is used. The discrete set of outputs is obtained via analytic solution of the continuous problem (compare R. J. Budzianowski and Z. W. Kundzewicz, unpublished manuscript, 1984). This can be regarded as advantageous in comparison both to the prior assumption of a discrete model and to the direct approach, i.e., discretization at the stage of programming and numerical computations. The "via solution" method (R. J. Budzianowski and Z. W. Kundzewicz, unpublished manuscript, 1984) gives better insight into the identification process and compares favorably to the other methods with respect to the necessary computational effort.

Consider equations (14) and (21) stated in discrete framework. The data are given in discrete time instants and the system response needs to be known in discrete time instants. Assume that the time instants of interest belong to the following set:

$$t_k = t_0 + k\Delta T \qquad k = 0, 1, \cdots, n \tag{27}$$

Let the input increment $\Delta x(t)$ be given as a train of rectangular pulses, in accordance with the rainfall measurement. Then the model response (17) can be also used, giving the result

$$\delta\mathbf{S}[(k+1)\Delta T] = \exp(\mathbf{A}\Delta T)\delta\mathbf{S}(k\Delta T) + \int_{k\Delta T}^{(k+1)\Delta T} \exp\{\mathbf{A}[(k+1)\Delta T - \tau]\}\mathbf{B}\,d\tau\Delta x(k\Delta T) \tag{28}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by (18).

Now, denote for brevity $F(t_k) = F(k)$, where F is a dummy notation of a function. Then the model equations can be written as

$$\delta\mathbf{S}(k+1) = \tilde{\mathbf{A}}\delta\mathbf{S}(k) + \tilde{\mathbf{B}}\Delta x(k) \tag{29}$$

$$y(k) = \mathbf{C}\delta\mathbf{S}(k) \tag{30}$$

where the matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are

$$\tilde{\mathbf{A}} = \exp(\mathbf{A}\Delta T) \tag{31}$$

$$\tilde{B}(i) = \left[1 - \exp(-a\Delta T)\sum_{k=0}^{i-1}(a\Delta T)^k/k!\right]\Big/ a \tag{32}$$

and the vector $\mathbf{C}$ is defined by (18). Derivation of (32) is given in the appendix.

Similarly, for the quadratic increments

$$\delta^2\mathbf{S}(k+1) = \exp(\mathbf{A}\Delta T)\delta^2\mathbf{S}(k) + b\int_{t_k}^{t_{k+1}} \exp[\mathbf{A}(t_{k+1} - \tau)]\boldsymbol{\varphi}[\delta'\mathbf{S}(\tau)]^2\,d\tau \tag{33}$$

where $\delta'\mathbf{S}(t)$ is the extension for continuous arguments of the variation $\delta\mathbf{S}(k)$ given for discrete arguments. This is achieved via linear interpolation of $\delta\mathbf{S}(t)$ between the discrete instants, where $\delta\mathbf{S}(t)$ is explicitly given.

After some manipulations (see the appendix), the final equation for the quadratic increment of storage can be developed.

$$\begin{aligned}\delta^2\mathbf{S}(k+1) = {} & \tilde{\mathbf{A}}\delta\mathbf{S}(k) + b\tilde{\mathbf{B}}_1[\delta\mathbf{S}(k)]^2/(\Delta T)^2 \\ & + 2b(\Delta T\tilde{\mathbf{B}}_2 - \tilde{\mathbf{B}}_1)\delta\mathbf{S}(k)\delta\mathbf{S}(k+1)/(\Delta T)^2 \\ & + b[\tilde{\mathbf{B}}_1 - 2\Delta T\tilde{\mathbf{B}}_2 + (\Delta T)^2\tilde{\mathbf{B}}_3][\delta\mathbf{S}(k+1)]^2/(\Delta T)^2\end{aligned} \tag{34}$$

The final formulae for $\tilde{B}_1$, $\tilde{B}_2$, and $\tilde{B}_3$ are developed in the appendix.

When the storage increments are known, one can easily find

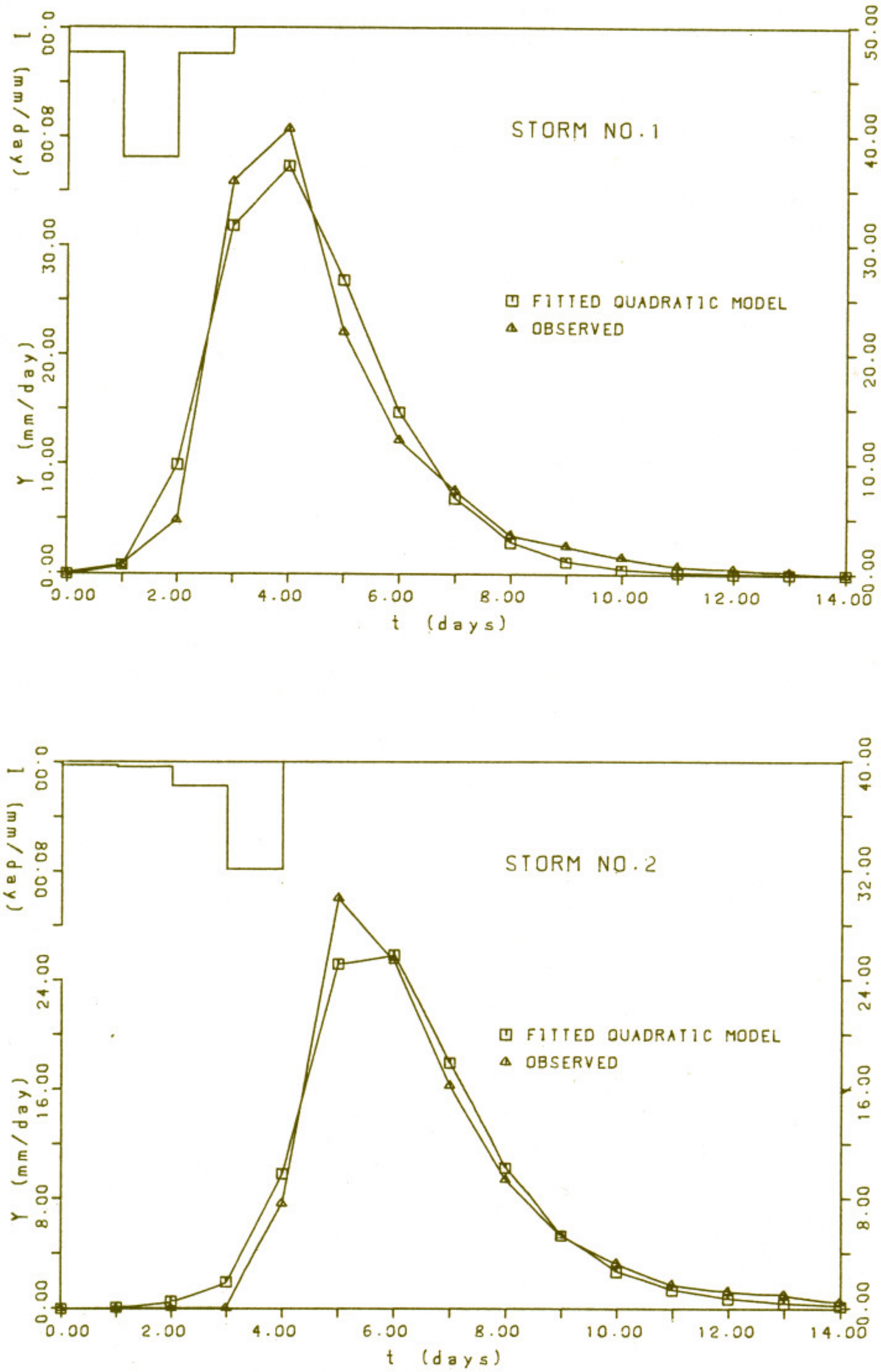

Fig. 1. Simulation with the help of the quadratic rainfall-runoff model (eight storm events).

the corresponding linear and quadratic increments of outflow

$$\delta y(k) = a\delta S_N(k) \tag{35}$$

$$\delta^2 y(k) = a\delta^2 S_N(k) + b[\delta S_N(k)]^2 \tag{36}$$

The check of choice of the time discretization step can be made by comparing the integral of the second-order increment to zero [*Diskin and Boneh*, 1972]. If this value differs considerably from zero, the linear approximation of the increment $\mathbf{S}(t)$ is not sufficient; that is, denser discretization is required.

4. Parameter Identification

The significant advantage of the conceptual Volterra series model is parsimony in the number of parameters. In the present study that pertains to the initially relaxed case (zero initial conditions in equations (4) and (7)) there are three parameters: a and b pertain to the Taylor series resolution of outflow law, and N is the number of nonlinear conceptual elements in series.

The parameters of the model are typically identified via

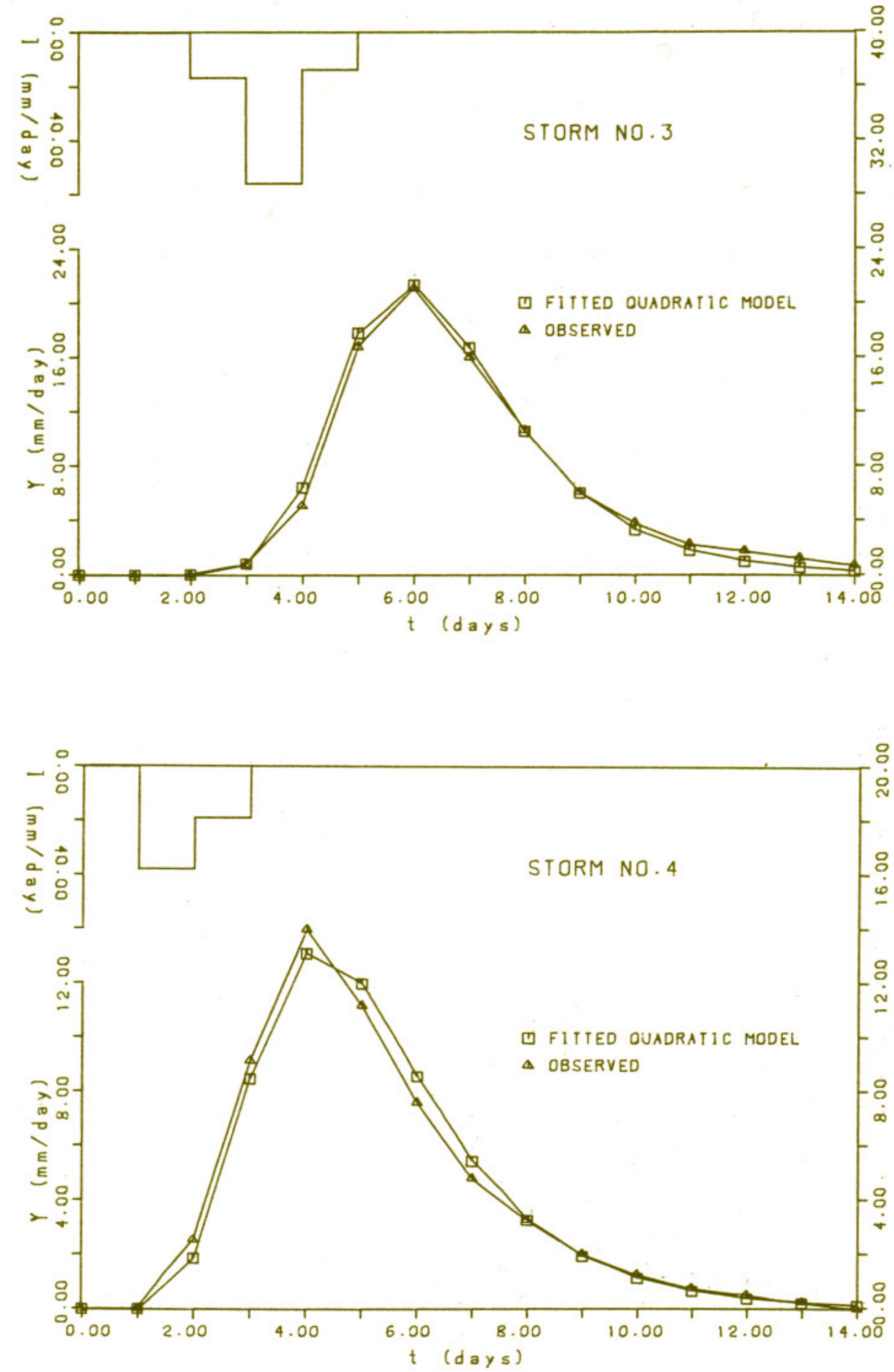

Fig. 1. (continued)

analysis of the discrete input and output data. The optimization criterion to be minimized is

$$\min J(N, a, b) = \min \sum_{k=1}^{m} [y_{\text{obs}}(k) - y_{\text{mod}}(k)]^2 \tag{37}$$

where

$$y_{\text{mod}}(k) = \delta(k) + \delta^2 y(k) \tag{38}$$

The initial values of the parameters a and N related to the linear portion of the model were obtained by means of the moment matching method [*Nash*, 1959]. The real initial value of N resulting from the moment matching was subsequently converted to integer N^* according to the relation

$$N^* = \text{ent}(N + 0.5)$$

where $n \stackrel{df}{=} \text{ent}(x)$ is the largest integer less than or equal to the real value of x. The initial real value of a is then modified in order to compensate for the effect of rounding N to the integer value in such a way that the first two moments remain constant.

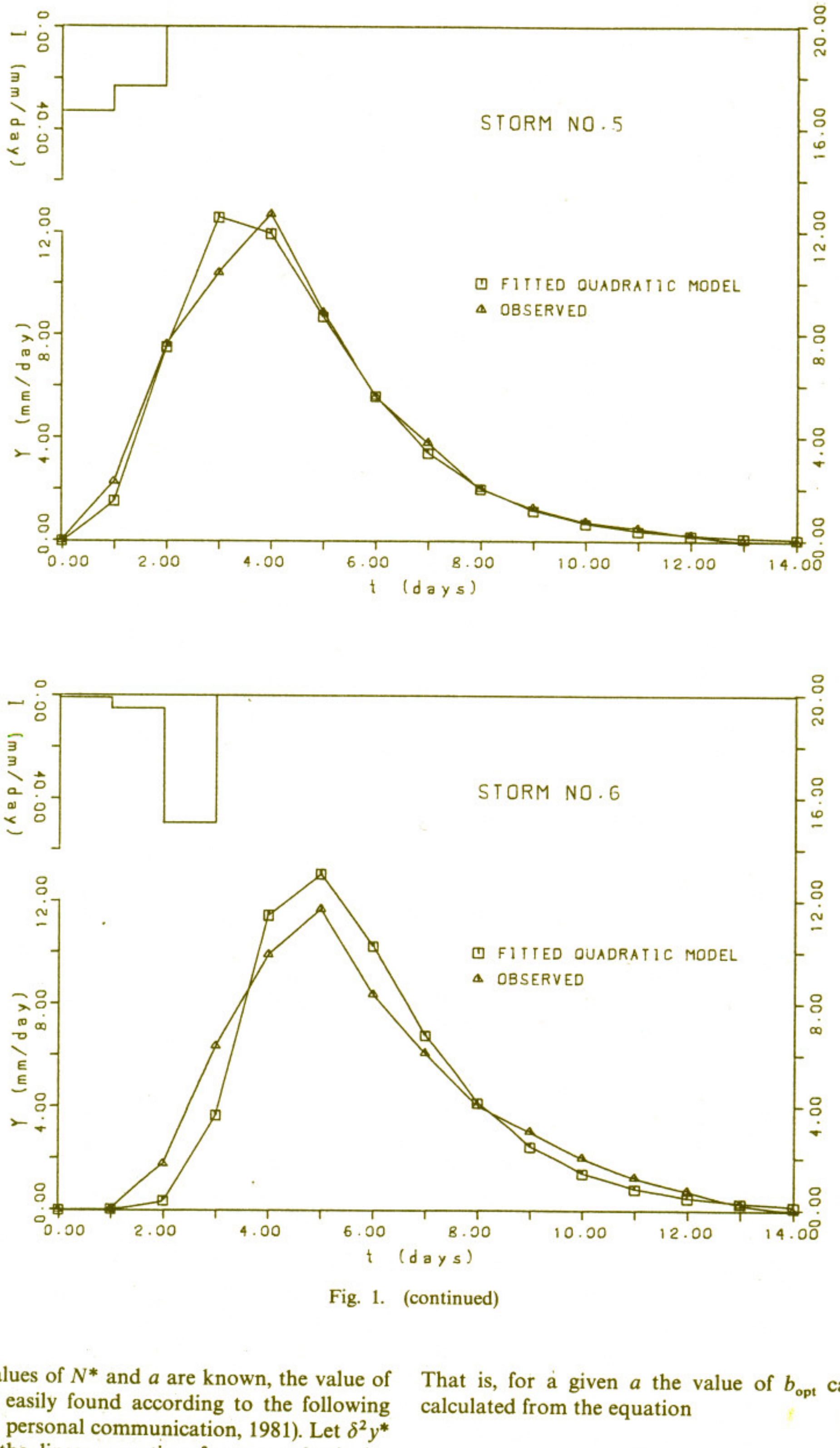

Fig. 1. (continued)

Once the initial values of N^* and a are known, the value of parameter b can be easily found according to the following reasoning (A. Boneh, personal communication, 1981). Let $\delta^2 y^*$ be the solution of the linear equation for a quadratic increment for $b = 1$. Then, due to the linearity the following equation holds:

$$\delta^2 y(k) = b\delta^2 y^*(k) \tag{39}$$

The necessary condition of optimality is

$$\frac{\partial J}{\partial b} = 2 \sum_{k=1}^{m} [y_{\text{obs}}(k) - \delta y(k) - b\delta^2 y^*(k)]\delta^2 y^*(k) = 0 \tag{40}$$

That is, for a given a the value of b_{opt} can be most easily calculated from the equation

$$b_{\text{opt}} = \frac{\sum_{k=1}^{m} [y_{\text{obs}}(k) - \delta y(k)]\delta^2 y^*(k)}{\sum_{k=1}^{m} [\delta^2 y^*(k)]^2} \tag{41}$$

Now the minimization of the criterion function (37) with respect to the parameter a can be performed. Subsequently, a new value of b_{opt} is found from (41) that corresponds to the

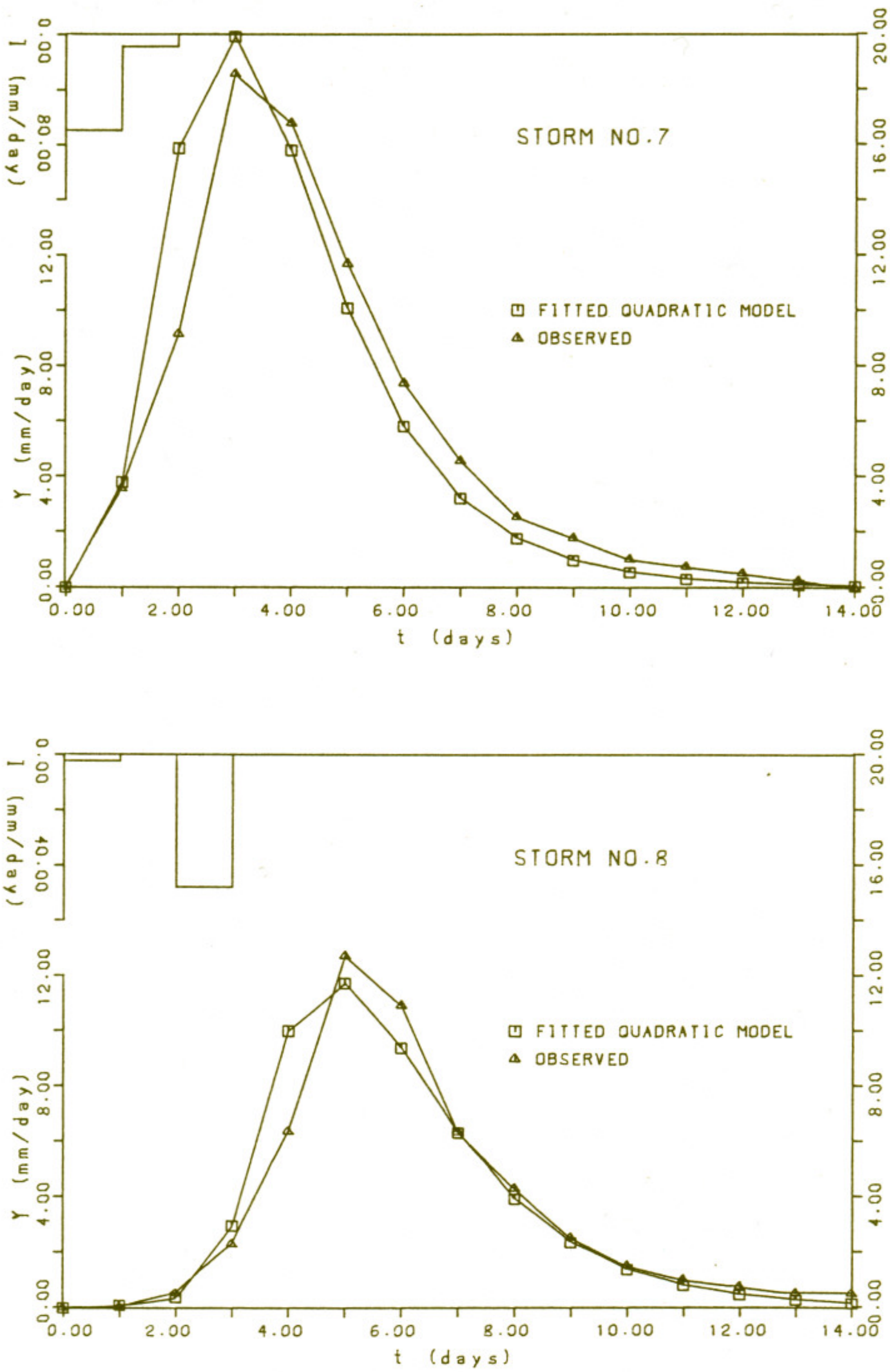

Fig. 1. (continued)

new value of a. After having reached the optimum values of the parameters a and b, the trial calculations of two other values of the integer N, that is, $N^* + 1$ and $N^* - 1$, are made. In our experience so far this simple identification approach has proven effective, since the initial value of parameter a, determined by the moment matching technique, was close to the optimal value, and possible local minima were not active in the optimization process.

It should be underlined that optimization with respect to only one variable is sufficient; that is, the problem is computationally simple.

5. Numerical Example

The methodology presented was tested with rainfall-runoff data from the catchment of the Cache River at Forman in southern Illinois used by *Diskin and Boneh* [1973]. The catchment of the 630-km^2 area has mild slopes and well-developed drainage network. The available data pertain to eight rainfall-runoff events observed in the years 1935–1951 and allow comparison with other approaches to be performed.

At first, the data on all eight events available were used both for identification and verification. Then, the events marked 1–3 were used for identification and the remaining events for verification. The results of these two cases were practically identical. Results of either of these two approaches are shown in Figure 1. The values of model parameters obtained via simplified identification as described in paragraph 4 above read $N = 3$, $a = 0.75$ (day^{-1}), and $b = 6.84$ (day^{-1} mm^{-1}). As the same set of data were used by *Diskin and Boneh* [1973, 1984], it seems useful to compare the results. This can be done in several aspects.

As regards the accuracy of simulation, the results obtained by *Diskin and Boneh* [1973] were better than in the present approach. Values of the criterion function (37) were 53 and 233. However, the number of parameters used by the present authors was much lower (3 versus 68) than in the work by Diskin and Boneh. Moreover, the consistency of observed and simulated outflows does not mean that consistency in system operators has been also achieved. An example shown in the work by *Napiórkowski and Strupczewski* [1984] illustrates that similar outflows can be obtained with entirely different system operators. The ill conditioning of the mathematical problems tackled in the work by Diskin and Boneh did not let them use part of available data (e.g., three as in the present paper) for identification and part for verification purposes. The input series in the cases marked in Figure 1 as 1 to 3 was too short for the great number of necessary parameters to be identified.

6. Concluding Remarks

The method of identification of kernels of Volterra series model via conceptualization is advantageous in comparison to the classical methods of identification in terms of trade-off between the amount of computations required and accuracy achieved. This is supported by the comparison of the methods of kernel identification presented in the work by *Napiórkowski* [1983]. The objective function measuring the root-mean-square departure of the modeled runoff from the measured runoff was better for the three-parameter Volterra series conceptualization than for the fourteen-parameter black box convolution model identified via ordinates. Similar results were obtained by *Diskin et al.* [1984], who concluded that "If this result turns out to be correct also for other sets of data, it may be taken as evidence that the second order model, based on a cascade of nonlinear reservoirs, is indeed a suitable tool for representing the conversion of rainfall excess into direct surface runoff." The conceptualization makes the Volterra series easily tractable on a microcomputer.

The results obtained prove also that the model of cascade of identical nonlinear dynamic systems (e.g., uniformly nonlinear reservoirs) lends itself to applications to both flood routing [e.g., *Napiórkowski and Strupczewski*, 1981, 1984] and to rainfall-runoff modeling [*Napiórkowski*, 1983; *Diskin et al.*, 1984]. That is, in accordance with the applications of nonlinear reservoirs in the rainfall-runoff modeling reported in the hydrology literature, the conceptual nonlinear differential equations (nonlinear state model) used in this research could be also of good value apart from the Volterra framework.

One can also conclude that the methodology presented should enable other methods of conceptualization to be used. That is, other physically significant nonlinear models (e.g., in the case of open channel flow, models of hydrodynamic origin) can be linked with the Volterra series kernels. The prerequisite condition that the model be given in the state space framework has been achieved in several references (e.g., via modal analysis in the work by *Dooge et al.* [1983] and via discretization, in the work by *Muzik* [1975]; *Szöllösi-Nagy* [1982]). Since the series of references by Napiórkowski and coauthors mentioned herein is not widely known, the following statement seems appropriate as the final conclusion. The door to physically based conceptual Volterra series model is declared open.

Appendix

Linear Increments: Derivation of (32)

Combination of (28) and (29) gives

$$\tilde{B} = \int_{t_k}^{t_{k+1}} \exp\,[\mathbf{A}(t_{k+1} - \tau)]\mathbf{B}\,d\tau = \int_0^{\Delta T} \exp\,[\mathbf{A}(\Delta T - \tau)]\mathbf{B}\,d\tau = \int_0^{\Delta T} \boldsymbol{\phi}(\tau)\mathbf{B}\,d\tau \qquad \text{(A1)}$$

where $\boldsymbol{\phi}(t)$ is the transition matrix given by (19). Thus

$$\int_0^{\Delta T} \phi_{ij}(\tau)\,d\tau = \left[1 - \exp\,(-a\Delta T)\sum_{k=0}^{i-j} (a\Delta T)^k/k!\right]\Big/ a \qquad \text{(A2)}$$

Inserting (A2) in (A1) one obtains (32).

Quadratic Increments: Derivation of Equations for $\tilde{B}_1$, $\tilde{B}_2$, and $\tilde{B}_3$ in (34)

Assume that the element $\delta'\mathbf{S}(\tau)$ in (33) results from linear interpolation; that is,

$$\begin{aligned}\delta'\mathbf{S}(\tau) &\cong \delta\mathbf{S}(k) + [\delta\mathbf{S}(k+1) - \delta\mathbf{S}(k)](\tau - k\Delta T)/\Delta T \\ &= \{[(k+1)\Delta T - \tau]\delta S(k) + (\tau - k\Delta T)\delta S(k+1)\}/\Delta T \qquad \text{(A3)}\end{aligned}$$

Inserting (A3) in (33) one obtains

$$\begin{aligned}&\delta^2\mathbf{S}(k+1) \\ &= \exp\,(\mathbf{A}\Delta T)\delta^2\mathbf{S}(k) \\ &\quad + b\int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}[\tau\delta\mathbf{S}(k) + (\Delta T - \tau)\delta\mathbf{S}(k+1)]^2\,d\tau/(\Delta T)^2 \\ &= \exp\,(\mathbf{A}\Delta T)\delta^2\mathbf{S}(k) + b\int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}\tau^2\,d\tau[\delta\mathbf{S}(k)]^2/(\Delta T)^2 \\ &\quad + 2b\int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}(\Delta T\tau - \tau^2)\,d\tau[\delta\mathbf{S}(k)\delta\mathbf{S}(k+1)]/(\Delta T)^2 \\ &\quad + b\int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}[\tau^2 - 2\Delta T\tau + (\Delta T)^2]\,d\tau[\delta\mathbf{S}(k+1)]^2/(\Delta T)^2\end{aligned} \qquad \text{(A4)}$$

Now auxiliary matrix variables $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$, and $\tilde{\mathbf{B}}_3$ are introduced:

$$\begin{aligned}\tilde{\mathbf{B}}_1 &= \int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}\tau^2\,d\tau \\ \tilde{\mathbf{B}}_2 &= \int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}\tau\,d\tau \qquad \text{(A5)} \\ \tilde{\mathbf{B}}_3 &= \int_0^{\Delta T} \exp\,(\mathbf{A}\tau)\boldsymbol{\varphi}\,d\tau\end{aligned}$$

Since for $i \geq j$,

$$\int_0^{\Delta T} (a\tau)^{i-j} \exp(-a\tau)\tau^2/(i-j)!\, d\tau$$
$$= \frac{(i-j+2)(i-j+1)}{a^2} \int_0^{\Delta T} \frac{(a\tau)^{i-j+2}}{(i-j+2)!} e^{-a\tau}\, d\tau$$
$$= a^{-2}(i-j+2)(i-j+1)\tilde{B}(i-j+3) \qquad \text{(A6)}$$

$$\int_0^{\Delta T} (a\tau)^{i-j} \exp(-a\tau)\tau/(i-j)!\, d\tau$$
$$= \frac{(i-j+1)}{a} \int_0^{\Delta T} \frac{(a\tau)^{i-j+1}}{(i-j+1)!} e^{-a\tau}\, d\tau$$
$$= a^{-1}(i-j+1)\tilde{B}(i-j+2) \qquad \text{(A7)}$$

where the vector **B** developed in this appendix was given by (32).

The elements of the matrices $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$, and $\tilde{\mathbf{B}}_3$ for $i \geq j$ can be determined as follows:

$$\tilde{B}_1(i, j) = -a^{-2}(i-j+2)(i-j+1)\tilde{B}(i-j+3)$$
$$+ a^{-2}(i-j+1)(i-j)\tilde{B}(i-j+2) \qquad \text{(A8)}$$

$$\tilde{B}_2(i, j) = -a^{-1}(i-j+1)\tilde{B}(i-j+2)$$
$$+ a^{-1}(i-j)\tilde{B}(i-j+1) \qquad \text{(A9)}$$

$$\tilde{B}_3(i, j) = -\tilde{B}(i-j+1) + \operatorname{sign}(i-j)\tilde{B}(i-j) \qquad \text{(A10)}$$

where

$$\operatorname{sign}(x) \overset{df}{=} +1 \qquad x > 0$$
$$\operatorname{sign}(x) \overset{df}{=} 0 \qquad x = 0$$
$$\operatorname{sign}(x) \overset{df}{=} -1 \qquad x < 0$$

For $i < j$ the elements of the matrices $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$, and $\tilde{\mathbf{B}}_3$ are equal to zero.

After having inserted the (A8)–(A10) in (A5) and then in turn in (A4) one gets the final solution (34) for the second variation of storage.

NOTATION

a, b model parameters.
A, B, C matrices (continuous model).
$\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2, \tilde{\mathbf{B}}_3$ matrices (discrete model).
f outflow law.
h_i ith-order kernel.
H dimensionless first-order kernel.
I_i inflow to ith reservoir.
J objective function.
m number of input-output pairs.
n number of Volterra series terms.
N number of reservoirs.
Q_i outflow from ith reservoir.
P nonlinear operator.
t time.
ΔT time increment.
x input.
Δx input increment.
y output.
Δy output increment.
δ linear variation.
δ^2 quadratic variation.
$\boldsymbol{\phi}$ transition matrix.
$\boldsymbol{\varphi}$ model matrix.

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