determination on the relative lineanity of a catchment should be made objectively during the calibration of the model; not before the model is used.

In closing the discusser expresses his wish that the authors had had a better understanding of the theory before going to press with this unfortunate paper.

Appendix.—Reference

7. Amorocho, J., and Brandstetter, A., "Determination of Nonlinear Functional Response Functions in Rainfall-Runoff Processes," *Water Resources Research*, Vol. 7, No. 5, Oct., 1971.

Discussion by Jarosław J. Napiórkowski[5] and Aodh Dowley[6]

The "selection method" of Tichonov and Yarsenin (16) provides a simple solution for the ill-conditioned problem of the identification of the kernels of the Volterra series. According to this method the solution is sought from a subset of feasible solutions defined in terms of physical and mathematical characteristics. The proper definition of that subset is of great importance. Any arbitrary assumption may lead to a search for the solution within the wrong class of functions.

Napiórkowski and Strupczewski (12,13,15) analytically obtained the first two kernels of the Volterra series for the simplest quasi-physical nonlinear model, namely the cascade of identical nonlinear reservoirs. For that model, which combines linear static and nonlinear dynamic characteristics, the structure of the kernels was shown to be

$$h_1(s) = aH_n(S) \qquad (14)$$

$$h_2(s,t) = b\left\{ H_N(s) \sum_{i=1}^{N} H_i(t) + H_N(t) \sum_{i=1}^{N} H_i(s) - H_N\,[\max\,(s,t)] \right\} \qquad (15)$$

in which N = number of reservoirs; and a and b = parameters resulting from the outflow-storage relation and

$$H_i(s) = \exp\,(-as)\,\frac{(as)^{i-1}}{(i-1)!} \qquad (16)$$

The kernels in Eqs. 14 and 15 fulfill the conditions specified by Diskin and Boneh (11) for conservative systems.

On the basis of Eqs. 14 and 15 the subset is

$$h_1(s) = \sum_{i=0}^{M} a_i P_i(s) \qquad (17)$$

$$h_2(s,t) = \sum_{i=0}^{M} \sum_{j=0}^{M} a_{ij} P_i(s) P_j(t) + \sum_{i=0}^{M} b_i P\,[\max\,(s,t)] \qquad (18)$$

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in which the $P_i(s)$ = orthonormal polynomials, was recommended (12,14) for identification of closed or open systems with nonlinear dynamic and quasi-linear static behavior.

Having determined the kernels of the Volterra series in an analytical manner, different methods of identification were examined (12). The conclusion was that the arbitrary subset used by the authors may give good results in the case of systems with nonlinear (quadratic) static characteristics but in the case of quasi-closed systems it cannot lead to satisfactory results.

The writers fully agree with the authors as far as the use of the Laguerre orthonormal polynomials (or their discrete analog) is concerned. However, they suggest using them in time-scaled form:

$$P_m(t) = \sqrt{\alpha}\exp\left(\frac{-\alpha t}{2}\right)\sum_{i=0}^{M}(-1)^i\binom{M}{i}\frac{(\alpha t)^i}{i!} \quad \text{(19)}$$

The presence of the exponential damping term in Eq. 19 means that for practical purposes the memory of the model is the time U for which $\int_0^U [P_i(t)]^2 dt \geq 0.99$. One can see the memory of the model has nothing to do with the memory of the system unless a time scaled parameter α is introduced, which is calibrated during identification. Also, the subset used by the authors, the second order kernel, results in damping which is twice as strong as that corresponding to Eq. 18.

Any Volterra series has a range of convergence $|x(t)| < M$. Inside this range the error of approximation of the dynamics of the system is inversely proportional to the number of terms in the integral series. This aspect of modelling of hydrologic systems has not been referred to by the authors. Expressed in general terms the problem is complex, but nevertheless it must be considered and has been solved in the special case of the nonlinear cascade model (12,13,15).

As a practical means of overcoming this problem Boneh and Golan (10), derived a criterion for maximum amplitude of input which when applied ensures a positive output response to a positive input signal.

The functional power series (Eq. 1) was used for the first time in mathematics by Volterra in 1887 (17). Since then, in recognition of Volterra's original contribution, the series (Eq. 1) has been called the Volterra series in many fields of science and engineering (9). For this reason the writers prefer to refer to Eq. 1 as the Volterra series model while acknowledging the fact that it was first applied to hydrologic modelling by Amorocho in 1961 (8).

As indicated by Amorocho (8) a watershed is a time-variable system and consequently the assumption of a time-invariant model may lead to unacceptable predictions of runoff. Nevertheless in the identification process under review the authors have not separated the effective rainfall and the base flow.

Appendix.—References

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Closure by Otto J. Helweg,[7] M. ASCE and Ralph H. Finch[8]

The writers thank Napiorkowski and Dowley for their comments and regret typographical and other errors that concerned Amorocho. The purpose of the original paper was more modest than perhaps assumed by the discussers. The writers merely attempted to make the nonlinear model more "user friendly" for persons with insufficient theoretical background to estimate the several model parameters.

No attempt was made to improve upon the theoretical basis of the model or to rigorously review the theory. The few equations presented were meant to assist a practitioner's understanding of the parameter estimation techniques. These techniques were empirical with no claim to mathematical substantiation other than the statistical tests presented.

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