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Analytical solution of channel flow model with downstream control*

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Abstract The effect of the downstream boundary condition on the flow in a channel reach is explored through the analysis of the linearized St Venant equations. It is found that there are two effects of the inclusion of the downstream boundary condition: the direct upstream transmission of the downstream boundary condition and the generalization of the upstream response from a single term to an infinite series. The relative effects of upstream and downstream conditions on the water level at any intermediate point in the reach are evaluated.

Solution analytique d'un modèle d'écoulement dans un canal avec contrôle à l'aval

Résumé On recherche dans cet article l'effet de la condition limite aval sur le débit dans le bief d'un canal par une analyse des équations linearisées de St Venant. On trouve qu'il y a deux conséquences de l'inclusion de la condition — limite en aval — une transmission directe en amont de la condition limite aval, et aussi la réponse en amont est generalisée en partant d'un seule terme à une série infinie. Les effets relatifs des conditions amont et aval sur le niveau de l'eau a n'importe quel point intermédiaire du bief sont aussi évalués.

NOTATION

$A(x,t)$	area of flow
A_0	area of flow at reference conditions
$A'(x,t)$	deviation of $A(x,t)$ from A_0
$\bar{A}(x,s)$	Laplace transform of $A'(x,t)$
$A_u(t)$	boundary value of $A'(x,t)$ at $x = 0$
$\bar{A}_u(s)$	Laplace transform of $A_u(t)$
$A_d(t)$	boundary value of $A'(x,t)$ at $x = L$
$\bar{A}_d(s)$	Laplace transform of $A_d(t)$
$C_1(s)$, $C_2(s)$	unknown functions in transform solution

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F_0	Froude number at reference conditions
$I_1[\ ]$	modified Bessel function
L	total length of channel
L'	dimensionless channel length
$Q(x,t)$	rate of flow
Q_0	rate of flow at reference conditions
$Q'(x,t)$	deviation of $Q(x,t)$ from Q_0
$S_f(x,t)$	friction slope
S_0	bottom slope of channel
T	width of channel at water surface
T_0	value of T at reference conditions
T_n	nth term in series
$U[\ ]$	unit step function
$\forall[\ ]$	volume due to individual term in series
a	channel parameter [equation (12a)]
b	channel parameter [equation (12b)]
c	channel parameter [equation (12c)]
c_1	downstream dynamic celerity [equation (23d)]
c_1'	dimensionless celerity [equation (29d)]
c_2	upstream dynamic celerity [equation (23e)]
c_2'	dimensionless celerity [equation (29e)]
c_k	kinematic celerity [equation (7)]
d	channel parameter [equation (12d)]
e	channel parameter [equation (12e)]
f	channel parameter [equation (12f)]
g	acceleration due to gravity
h	channel parameter [equation (25c)]
h'	dimensionless value of h [equation (29i)]
$h_u(x,t)$	response to impulse at upstream boundary
$h_u(x,s)$	Laplace transform of $h_u(x,t)$
$h_d(x,t)$	response to impulse at downstream boundary
$h_d(x,s)$	Laplace transform of $h_d(x,t)$
$h_u^1(x,t)$	head of the impulse response $h_u(x,t)$
$h_u^2(x,t)$	body of the impulse response $h_u(x,t)$
$h_d^1(x,t)$	head of the impulse response $h_d(x,t)$
$h_d^2(x,t)$	body of the impulse response $h_d(x,t)$
m	velocity ratio [equation (6)]
n	number of reflection cycles
$sh[\ ]$	hyperbolic sine
t	elapsed time
t'	dimensionless value of t [equation (28b)]
t_0	period of reflection cycle [equation (23f)]
t_0'	dimensionless value of t_0 [equation (29f)]
v_0	average velocity at reference conditions
x	distance from upstream end
x'	dimensionless value of x [equation (28b)]
$\langle y_0 \rangle$	hydraulic mean depth at reference conditions

α_1	channel parameter [equation (23a)]
α_1'	dimensionless value of α_1 [equation (29a)]
α_2	channel parameter [equation (23b)]
α_2'	dimensionless value of α_2 [equation (29b)]
α_3	channel parameter [equation (23c)]
α_3'	dimensionless value of α_3 [equation (29c)]
β_1	channel parameter [equation (25a)]
β_1'	dimensionless value of β_1 [equation (29g)]
β_2	channel parameter [equation (25b)]
β_2'	dimensionless value of β_2 [equation (29h)]
λ_1, λ_2	roots of characteristic equation [equation (12)]
$\delta(\ )$	Dirac delta function

INTRODUCTION

The hydraulic formulation for unsteady flow in open channels requires two boundary conditions and in the case of tranquil flow (i.e. Froude number less than unity) one of these is at the downstream end of the channel. Most hydrological methods of flood routing are formulated either in terms of an upstream boundary condition only or by assuming a steady discharge-level relationship at some downstream section. The present paper examines analytically the solution of the linearized St Venant equations so that the relative effect of the upstream and downstream conditions at any intermediate point can be compared.

The two-point boundary problem in which both an upstream and a downstream boundary conditions are taken into account has not, as far as the authors are aware, been reported in the hydrological literature. The effect of downstream control was examined for the simplified diffusion analogy model only (Dooge *et al.*, 1983; Dooge & Napiorkowski, 1984).

LINEARIZED ST VENANT EQUATIONS

The one-dimensional equation of continuity for unsteady flow in an open channel is given by:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{1}$$

where $Q(x,t)$ is the discharge, $A(x,t)$ is the cross-sectional area, x is the distance from the upstream boundary and t is the elapsed time.

If the assumption is made that only acceleration in the direction of motion needs to be taken into account then the equation for the conservation of linear momentum in this direction can be written in terms of the same variables (Dooge *et al.*, 1982) as:

$$g\frac{A}{T}(1 - F^2)\frac{\partial A}{\partial x} + \frac{2Q}{A}\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} = gA\,(S_0 - S_f) \tag{2}$$

where T is the width of the channel at the water surface, S_0 is the bottom slope and F is the Froude number. The friction slope depends on the type of friction law assumed, the shape of the cross section, the flow at the section and the area of flow. In this discussion the friction slope S_f is taken in the completely general form and may be written as:

$$S_f = S_f(A,\ Q,\ \text{shape},\ \text{roughness}) \tag{3}$$

The problem of unsteady open channel flow involves the solution of the above set of nonlinear hyperbolic equations subject to given initial conditions and two appropriate boundary conditions. No analytical solution is available and equations (1) and (2) must be solved by some method of numerical approximation or by some simplification of the nonlinear momentum equations.

Hence, to determine analytically the sensitivity of the solution to the downstream boundary condition, the St Venant equations are simplified by considering the first order variations from a steady-state trajectory. Then we obtain a linear approximation to the solution of the problem. For any given channel, the relative error due to linearization varies with the inflow hydrograph being routed and with the choice of reference conditions for the linearization.

To compute the linearized second-order equation we make use of:

(a) expansion of nonlinear terms in equation (2) in a Taylor series around the uniform steady state (Q_0, A_0) and limitation of this expansion to the first order increments $Q'(x,t), A'(x,t)$;

(b) making the necessary substitution to eliminate the dependent variable $Q'(x,t)$ leaving a single dependent variable $A'(x,t)$.

The resulting equation (Dooge & Napiorkowski, 1984):

$$(1 - F_0^2)\frac{gA_0}{T_0}\frac{\partial^2 A'}{\partial x^2} - \frac{2Q_0}{A_0}\cdot\frac{\partial^2 A'}{\partial x\,\partial t} - \frac{\partial^2 A}{\partial t^2} = gA_0\left[-\frac{\partial S_f}{\partial A}\cdot\frac{\partial A'}{\partial x} + \frac{\partial S_f}{\partial Q}\cdot\frac{\partial A'}{\partial t}\right] \tag{4}$$

is a second order partial differential equation for the perturbation $A'(x,t)$ from the steady uniform reference area A_0. The form of boundary conditions for this equation can be derived from an accurate continuous record of water level.

The variation of the friction slope in equation (3) with discharge at the reference condition for all frictional formula for rough turbulent flow could be taken as:

$$\frac{\partial S_f}{\partial Q} = 2\cdot\frac{S_0}{Q_0} \tag{5}$$

For convenience we may define a parameter m as the ratio of the kinematic wave speed to the average velocity of flow:

$$m = \frac{c_k}{Q_0/A_0} \tag{6}$$

where c_k is the kinematic wave speed as given by Lighthill & Whitham (1955):

$$c_k = \frac{\mathrm{d}Q}{\mathrm{d}A} = -\frac{\partial S_f}{\partial A} / \frac{\partial S_f}{\partial Q} \tag{7}$$

The parameter m is a function of the shape of channel and of the area of flow. For a wide rectangular channel with Chézy friction, m is always equal to 3/2, and with Manning friction always equal to 5/3. For shapes of channel other than wide rectangular, m will take on different values.

BOUNDARY CONDITIONS FOR LINEARIZED ST VENANT EQUATION

The equation to be solved is hyperbolic in form. Accordingly there are two real characteristics defined by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c_{1,2} = \frac{Q_0}{A_0} \pm \sqrt{[(gA_0)/T_0]} \tag{8}$$

along which the discontinuities in the derivatives of the solution will propagate. For a Froude number less than unity, the secondary characteristic direction involving the negative root will be in an upstream direction and the flow within the range of influence of the condition at the downstream boundary will be affected by that boundary condition.

The problem of unsteady flow in rivers and canals can be classified on the basis of the nature of the boundary conditions. In problems of flood routing, the aim is to predict the hydrograph of level or flow at the downstream end of the channel when given the hydrograph of level or flow at the upstream end. In estuarine hydraulics, the aim is to predict levels or velocities at various points in the channel given the variation of water level at the downstream end. In either case the problem can only adequately be posed and adequately solved if both an upstream and a downstream boundary condition are specified. By studying the linearized St Venant equations for a finite channel reach with a properly defined boundary condition at each end, we can provide a basis for analysis of the errors in the solution due to inadequate specification of one of the boundary conditions.

LAPLACE TRANSFORM SOLUTION FOR FINITE CHANNEL REACH

We will consider the basic case in which $A'(x,t)$ will be prescribed both at the upstream boundary $x = 0$ and at the downstream boundary $x = L$. The use of other boundary conditions does not introduce any new principle. The problem is to solve equation (4) subject to the double initial condition:

$$A'(x,0) = 0 \quad \text{and} \quad \frac{\partial A'}{\partial t} = 0 \text{ at } t = 0 \tag{9a}$$

and subject to the boundary conditions:

$$A'(0, t) = A_u(t) \tag{9b}$$

$$A'(L, t) = A_d(t) \tag{9c}$$

The solution can be sought in terms of the Laplace transform.

Equation (4) when transformed to the Laplace transform domain becomes:

$$(1 - F_0^2)\frac{gA_0}{T_0}\frac{d^2\bar{A}}{dx^2} - \left[\frac{2Q_0}{A_0}s - \frac{\partial S_f}{\partial A}gA_0\right]\frac{d\bar{A}}{dx} - \left[s^2 + gA_0\frac{\partial S_f}{\partial Q}s\right]\bar{A} = 0 \tag{10}$$

where $\bar{A}(x,s)$ is the Laplace transform of $A'(x,t)$. Equation (10) is a second-order homogeneous ordinary equation, so the solution can be written in the general form:

$$\bar{A}(x,s) = c_1(s).\exp[\lambda_1(s)x] + c_2(s).\exp[\lambda_2(s)x] \tag{11}$$

where λ_1 and λ_2 are the roots of the characteristic equation for equation (10) and are given by:

$$\lambda_{1,2} = es + f \pm \sqrt{(as^2 + bs + c)} \tag{12}$$

where the parameters a, b, c, e, and f are functions of the channel parameters, viz.:

$$a = 1/[g\langle y_0\rangle(1 - F_0^2)^2] \tag{12a}$$

$$b = 2S_0[1 + (m - 1)F_0^2]/[V_0\langle y_0\rangle (1 - F_0^2)^2] \tag{12b}$$

$$c = (mS_0/\langle y_0\rangle)^2/(1 - F_0^2)^2 \tag{12c}$$

$$d = b^2/4 - ac \tag{12d}$$

$$e = 1/[v_0(1 - F_0^2)^2] \tag{12e}$$

$$f = mS_0/[<y_0>(1 - F_0^2)] \tag{12f}$$

where $<y_0> = A_0/T_0$ is the hydraulic mean depth and $v_0 = Q_0/A_0$ is the average velocity for the steady-state trajectory of uniform flow around which perturbations are considered.

The functions, $c_1(s)$ and $c_2(s)$, in equation (11) can be determined from the boundary conditions. For $x = 0$ we get:

$$\bar{A}_u(s) = c_1(s) + c_2(s) \tag{13}$$

and for the downstream boundary condition at $x = L$:

$$\bar{A}_d(s) = c_1(s)\exp(\lambda_1 L) + c_2(s)\exp(\lambda_2 L) \tag{14}$$

After having solved equations (13) and (14) for the unknown functions c_1 and c_2, one can write $\bar{A}(x,s)$ in terms of $h_u(x,s)$ and $h_d(x,s)$ which may be defined as the Laplace transform of the response of the channel reach to a delta function input at the upstream end and the downstream end respectively.

Accordingly we write:

$$\bar{A}(x,s) = h_u(x,s)\bar{A}_u(s) + h_d(x,s)\bar{A}_d(s) \tag{15}$$

The linear channel response to an upstream input h_u in equation (15) is given by (Dooge & Napiorkowski, 1987):

$$h_u(x,s) = \exp[(es + f)x]\,\frac{sh[(L - x)\sqrt{(as^2 + bs + c)}]}{sh[L\sqrt{(as^2 + bs + c)}]} \tag{16}$$

The Laplace transform of the linear channel response to a downstream input h_d is given by:

$$h_d(x,s) = \exp[-es + f)\ (L - x)]\,\frac{sh[x\sqrt{(as^2 + bs + c)}]}{sh[L\sqrt{(as^2 + bs + c)}]} \tag{17}$$

where the parameters a, b, c, e and f are given by equation (12) in which it will be noted that $f = \sqrt{(c)}$.

The original function $A'(x,t)$ in the time domain is determined from the corresponding boundary conditions through the relationship:

$$A'(x,t) = h_u(x,t) * A_u(t) + h_d(x,t) * A_d(t) \tag{18}$$

The explicit formulations for the transfer functions $h_u(x,t)$ and $h_d(x,t)$ in the time domain have been obtained by Dooge & Napiorkowski (1987). This is accomplished by writing the reciprocal form of the denominators in equations (16) and (17) as convergent infinite series and then inverting term by term to the time domain. In the case of the response to an upstream delta function input we can thus write equation (16) as:

$$h_u(x,s) = \Sigma_0^\infty \exp[(es+f)x - (2nL+x)\sqrt{(as^2+bs+c)}]$$
$$- \Sigma_1^\infty \exp[(es+f)x - (2nL-x)\sqrt{(as^2+bs+c)}] \qquad (19)$$

The response to a downstream delta function input as given by equation (17) can similarly be shown to be:

$$h_d(x,s) = \Sigma_0^\infty \exp[-(es+f)\ (L-x) - (2nL+L-x)\sqrt{(as^2+bs+c)}]$$
$$- \Sigma_0^\infty \exp[-(es+f)\ (L-x) - (2nL+L+x)\sqrt{(as^2+bs+c)}] \qquad (20)$$

The inversion of these expressions to the time domain is dealt with in the next section.

TRANSFER FUNCTIONS IN THE TIME DOMAIN

The transfer function due to an upstream input has been found to have two distinct parts so that we can write:

$$h_u(x,t) = h_u^{\prime}(x,t) + h_u^2(x,t) \qquad (21)$$

This is an extension of the result found for a semi-infinite wide rectangular channel by Dooge & Harley (1967). The first part of the solution, which may be termed the head of the wave, is given by (Dooge & Napiorkowski, 1987):

$$h_u^{\prime}(x,t) = \Sigma_0^\infty [\exp(-2nL\alpha_1 - \alpha_2 x)\cdot\delta(t - nt_0 - x/c_1)]$$
$$- \Sigma_1^\infty [\exp(-2n\mathrm{L}\alpha_1 + \alpha_3 x)\cdot\delta(t - nt_0 - x/c_2)] \qquad (22)$$

where

$$\alpha_1 = b/2\sqrt{(a)} \qquad (23a)$$

$$\alpha_2 = \alpha_1 - f \qquad (23b)$$

$$\alpha_3 = \alpha_1 + f \qquad (23c)$$

$$c_1 = v_0 + \sqrt{(g\langle y_0\rangle)} \qquad (23d)$$

$$c_2 = v_0 - \sqrt{(g\langle y_0\rangle)} \tag{23e}$$

$$t_0 = L/c_1 - L/c_2 \tag{23f}$$

The behaviour of the first part of the solution is shown on Fig. 1. It can be

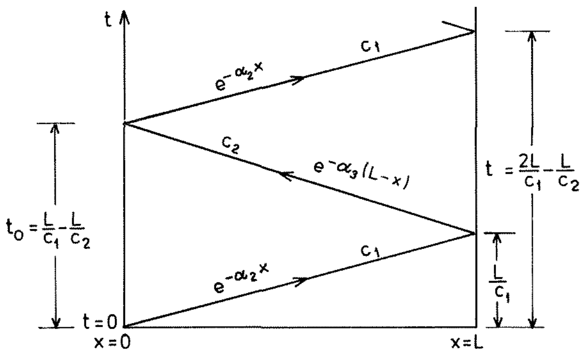


Fig. 1 Reflection of the head of the wave indicating the speed of travel, interval between reflections, and the rate of volume decrease.

seen that the head of the wave moves downstream at the dynamic speed c_1 in the form of a delta function of exponentially declining volume proportional to $\exp(-\alpha_2 x)$. At $x = L$ the delta function is reflected with inversion of sign and then is propagated upstream at the speed c_2 and with a heavier damping factor $\exp[-\alpha_3(L - x)]$, then is reflected again at $x = 0$ to move in a downstream direction and so on until the volume of the head of the wave becomes negligible.

The second part of the upstream response, which may be termed the body of the wave, is:

$$\begin{aligned} h_u^2(x,t) = {} & \textstyle\sum_0^\infty \exp(-\beta_1 t + \beta_2 x)\ (h/c_1 - h/c_2)\ (2nL + x)\cdot \\ & \frac{I_1\{2h\sqrt{[(t - nt_0 - x/c_1)\ (t + nt_0 - x/c_2)]}\}U[t - nt_0 - x/c_1]}{\sqrt{[(t - nt_0 - x/c_1)\ (t + nt_0 - x/c_2)]}} \\ & - \textstyle\sum_1^\infty \exp(-\beta_1 t + \beta_2 x)\ (h/c_1 - h/c_2)\ (2nL + x)\cdot \\ & \frac{I_1\{2h\sqrt{[(t - nt_0 - x/c_2)\ (t + nt_0 - x/c_1)]}\}U[t - nt_0 - x/c_2]}{\sqrt{[(t - nt_0 - x/c_2)\ (t + nt_0 - x/c_1)]}} \end{aligned} \tag{24}$$

where $I_1\{\ \}$ is a modified Bessel function of the first kind, $U[t]$ is a unit step function, and the remaining parameters are given by:

$$\beta_1 = b/2a \tag{25a}$$

$$\beta_2 = f - be/2a \tag{25b}$$

$$h = \sqrt{(d)}/2a \tag{25c}$$

As in the case of the head of the wave, the body of the wave is subject to successive reflection at both the downstream and upstream boundaries but moves and dissipates more slowly than the head of the wave.

For the downstream transfer function the head of the wave is given by:

$$\begin{aligned} h'_d(x,t) = {} & \Sigma_0^\infty \exp[-2nL\alpha_1 - \alpha_3(L-x)] \cdot \delta[t - nt_0 + (L-x)/c_2] \\ & - \Sigma_0^\infty \exp[-(2n+1)L\alpha_1 - fL - \alpha_2 x]\delta[t - nt_0 + L/c_2 - x/c_1] \end{aligned} \tag{26}$$

and is subject to reflection at the two ends of the reach as in the case of $h'_u(x,t)$. The body of the wave is given by

$$\begin{aligned} h_d^2(x,t) = {} & \Sigma_0^\infty \exp[-\beta_1 t - \beta_2(L-x)]\,(h/c_1 - h/c_2)\,[2nL + (L-x)] \\ & \cdot \frac{I_1\{2h\sqrt{[(t - nt_0 + L/c_2 - x/c_2)\,(t + nt_0 + L/c_1 - x/c_1)]}\}U[t - nt_0 + L/c_2 - x/c_2]}{\sqrt{[(t - nt_0 + L/c_2 - x/c_2)\,(t - nt_0 + L/c_1 - x/c_1)]}} \\ & - \Sigma_0^\infty \exp[-\beta_1 t - \beta_2(L-x)] \cdot (h/c_1 - h/c_2)\,[2(n+1)L + x] \\ & \cdot \frac{I_1\{2h\sqrt{[(t + nt_0 - x/c_2 + L/c_1)\,(t - nt_0 + L/c_2 - x/c_1)]}\}U[t - nt_0 + L/c_2 - x/c_1]}{\sqrt{[(t + nt_0 - x/c_2 + L/c_1)\,(t - nt_0 + L/c_2 - x/c_1)]}} \end{aligned} \tag{27}$$

Considering that the modified Bessel function is itself represented by an infinite series, the solution is in the form of a double infinite series which seems too complicated for practical application in river flow forecasting. However, due to heavy damping, only the first few terms of the two transfer functions would normally be required, and the polynomial approximation of the first order modified Bessel function (Abramowitz & Stegun, 1965) is sufficiently accurate and can be easily calculated.

VOLUMES OF IMPULSE RESPONSES

The water level at any intermediate point in the reach is determined by the upstream boundary condition $A_u(t)$ and the downstream boundary condition $A_d(t)$ in accordance with equation (15). It is clear that if the value of $A_d(t)$ is very much larger than $A_u(t)$ then this boundary condition will have a dominant influence on conditions throughout most of the reach. In many cases, however, the two boundary conditions are of the same order of magnitude. Accordingly it is instructive to compare the relation of magnitudes of the impulse responses $h_u(x,t)$ and $h_d(x,t)$.

For the purpose of illustration the flow in a broad rectangular channel with Chézy friction (m = 1.5) is considered. It is convenient to analyse the problem and to evaluate the results in terms of dimensionless independent variables defined with the help of the bottom slope S_0, the depth $\langle y_0 \rangle$, and the velocity v_0, for the steady uniform reference conditions about which perturbations are taken. Thus we can write:

$$x' = xS_0/\langle y_0 \rangle \tag{28a}$$

$$t' = t\, v_0\, S_0/\langle y_0 \rangle \tag{28b}$$

Hence, the dimensionless parameters of the transfer functions due to upstream and downstream inputs are given respectively by:

$$\alpha_1' = (1 + F_0/2)/F_0(1 - F_0^2) \tag{29a}$$

$$\alpha_2' = (1 - F_0/2)/F_0(1 + F_0) \tag{29b}$$

$$\alpha_3' = (1 + F_0/2)/F_0(1 - F_0) \tag{29c}$$

$$c_1' = 1 + \frac{1}{F_0} \tag{29d}$$

$$c_2' = 1 - \frac{1}{F_0} \tag{29e}$$

$$t_0' = L'/c_1' - L'/c_2' \tag{29f}$$

$$\beta_1' = \tfrac{1}{2} + \frac{1}{F_0^2} \tag{29g}$$

$$\beta_2' = \tfrac{1}{2} \tag{29h}$$

$$h' = \sqrt{[(1 - F_0^2)\,(4 - F_0^2)]}/4F_0^2 \tag{29i}$$

The position of the downstream boundary condition has two effects on the value at a fixed point x' in a reach, one via the upstream impulse response given by equations (22) and (24) and the second via the downstream impulse response given by equations (26) and (27).

The question arises of the number of terms required to represent the total transfer function in each case to an accuracy sufficient for practical purposes. One way of tackling this problem is to estimate the contribution of each term to the volume of the total response. The volume under any function of time can be determined by evaluating the Laplace transform with respect to time for the value of the transfer parameter $s = 0$. Applying this to the response to an upstream input as given by equation (16) we obtain for the volume $\forall$:

$$\forall[h_u(x,t)] = \frac{1 - \exp[-2f(L - x)]}{1 - \exp[-2fL]} \tag{30}$$

which gives the expected value of unity for $x = 0$ and zero for $x = L$. Similarly, from equation (17) we obtain for the volume of the response to a downstream input:

$$\forall[h_d(x,t)] = \frac{\exp[-2f(L - x)] - \exp[-2fL]}{1 - \exp[-2fL]} \tag{31}$$

which has the expected value of zero at $x = 0$ and of unity at $x = L$. For any intermediate point x between $x = 0$ and $x = L$ we can combine equations (30) and (31) to obtain:

$$\forall[h_u(x,t)] + \forall[h_d(x,t)] = 1 \tag{32}$$

which is not an obvious result.

If the volume of inflow is different at the upstream and downstream boundaries we can write the contribution of the upstream volume $\forall_u(0)$ to the volume of the hydrograph at the point x as:

$$\forall_u(x) = \forall_u(0) \cdot \frac{1 - \exp[-2f(L - x)]}{1 - \exp[-2fL]} \tag{33}$$

and the contribution of the downstream volume $\forall_d(L)$ to the volume of the hydrograph at the same point x as:

$$\forall_d(x) = \forall_d(L). \frac{\exp[-2f(L - x)] - \exp[-2fL]}{1 - \exp[-2fL]} \tag{34}$$

Adding these two contributions and rearranging terms we get for the total volume of the hydrograph at the point x in the reach:

$$\forall(x) = \forall_u(0) - [\forall_u(0) - \forall_d(L)] \left[\frac{\exp(2fx) - 1}{\exp(2fL) - 1}\right] \tag{35}$$

which enables us to evaluate the transition of the hydrograph volume from $\forall_u(0)$ at $x = 0$ to $\forall_d(L)$ at $x = L$.

RATE OF CONVERGENCE OF SERIES SOLUTION

The contribution of the individual terms in the two series in equation (19) can be evaluated in the same way. For the first series representing downstream propagation arising from the upstream input, the volume due to the nth term is:

$$\forall[T_n] = \exp(-f \cdot 2nL) = [\exp(-2fL)]^n \tag{36}$$

The first term corresponds to $n = 0$ and thus contributes unit volume to the hydrograph. The ratio of the contribution of successive terms is given by:

$$\frac{\forall[T_{n+1}]}{\forall[T_n]} = \exp(-2fL) \tag{37}$$

and the rate of convergence depends on the value of:

$$2fL = (2m\ S_0L)/[\langle y_0\rangle(1 - F_0^2)] \tag{38}$$

Values of the ratio of successive terms for the case of $m = 1.5$ (wide rectangular channel and Chézy friction) for various values of Froude number F_0 and dimensionless length $L' = S_0L/\langle y_0\rangle$ are shown in Table 1. It is clear

Table 1 *Damping factor exp(−2fL)*

L'	$F_0 = 0.2$	$F_0 = 0.5$	$F_0 = 0.8$
0.1	0.73	0.67	0.43
1.0	0.04	0.02	2.4×10^{-4}
5.0	1.7×10^{-7}	2.1×10^{-9}	8.2×10^{-19}
10.0	2.8×10^{-14}	4.2×10^{-18}	6.6×10^{-37}

that the convergence is very rapid except for small values of the Froude number and of the dimensionless length. Substitution of typical values of S_0, L and $\langle y_0\rangle$ in the expression for dimensionless length will confirm that small values of the dimensionless length correspond to such short lengths of channel as to be of little practical interest in flood routing.

The contribution of the terms in the second series in equation (19), representing upstream propagation due to the reflection at the downstream boundary of the effect of upstream input, can be similarly analysed. In this case the volume of the individual terms is given by:

$$\forall[T_n] = \exp(2fx)\ [\exp(-2fL)]^n \tag{39}$$

Consequently the contribution of the first term of this series is given by:

$$\forall[T_1] = \exp[-2f(L - x)] \tag{40}$$

and the ratio of successive terms is again given by:

$$\frac{\forall[T_{n+1}]}{\forall[T_n]} = \exp(-2fL) \tag{41}$$

as for the first series.

In a similar fashion the contribution of the terms of the first series in equation (20) representing upstream propagation of the effects of the input at the downstream end is given by:

$$\forall[T_n] = \exp[-2f(L - x)] \cdot [\exp(-2fL)]^n \tag{42}$$

so that the first term of this series is given by:

$$\forall[T_1] = \exp[-2f(L - x)] \tag{43}$$

and the ratio of the successive terms is the same as for the series in equation (19) i.e.:

$$\frac{\forall[T_{n+1}]}{\forall[T_n]} = \exp(-2fL) \tag{44}$$

Finally, the second series in equation (20), which represents the downstream propagation of the effects of the downstream input when reflected at the upstream boundary gives:

$$\forall[T_n] = \exp[-2f(L - x)] \; [\exp(-2fL)]^n \tag{45}$$

and the ratio of the successive terms is again given by:

$$\frac{\forall[T_{n+1}]}{\forall[T_n]} = \exp(-2fL) \tag{46}$$

Thus the rate of convergence is the same in all four series involved in the two-point boundary version of the linearized St Venant equation.

It is clear from Table 1 that for $S_0L/\langle y_0\rangle$ greater than unity only the first two terms both in equation (22) and in equation (24) are needed to represent the response $h_u(x,t)$ to a high degree of accuracy. Similarly for the response function $h_d(x,t)$ only the first term both in equation (26) and in equation (27) is required.

EFFECT OF DOWNSTREAM BOUNDARY

Figures 2 and 3 illustrate the effect of the position of the downstream control on the shape of the body of the wave for $F = 0.2$, and $F = 0.8$ respectively for two values of the length factor ($x' = 1$ i.e. a short channel, and $x' = 20$ i.e. a long channel) and the following locations of the downstream boundary: (a) $(L' - x') = 0.1$, (b) $(L' - x') = 0.3$, (c) $(L' - x') = 0.5$, (d) $(L' - x') = 1.0$, (e) $(L' - x') = \infty$, as indicated on Figs 2 and 3. The behaviour shown in Fig. 2 can be explained in terms of successive reflections shown in Fig. 1. For values of t' less than $L'/c_1' - (L' - x')/c_2'$ only the first term in the first sum in equation (22) and the first term in the first sum in equation (24) differ from zero. For values of t' greater than

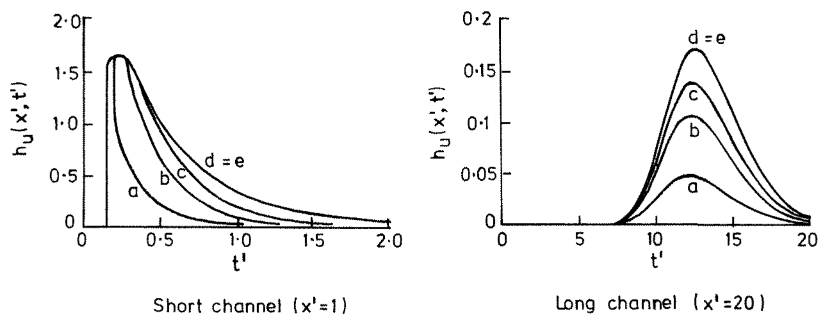


Fig. 2 Hydrograph of channel response at point x' due to an impulse at the upstream end for five different locations L' of the downstream boundary of zero change: (a) $L' - x' = 0.1$, (b) $L' - x' = 0.3$, (c) $L' - x' = 0.5$, (d) $L' - x' = 1.0$, (e) $L' - x'$ = infinity, for a reference value of the Froude number $F_0 = 0.2$.

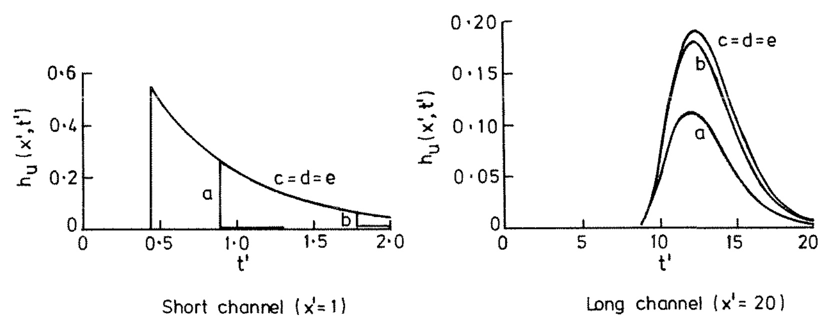


Fig. 3 Hydrograph of channel response at point x' due to an impulse at the upstream end for five different locations L' of the downstream boundary of zero change: (a) $L' - x' = 0.1$, (b) $L' - x' = 0.3$, (c) $L' - x' = 0.5$, (d) $L' - x' = 1.0$, (e) $L' - x'$ = infinity, for a reference value of the Froude number $F_0 = 0.8$.

$L'/c_1' - (L' - x')/c_2'$ the first term in the second sums in both equations (22) and (24) comes into play because of the first reflection of the wave by the zero downstream boundary condition at $t' = L'/c_1'$. For values of t' greater than $(L' + c_1')/\ c_1' - L'/c_2'$ the second term in the first sums in both equations (22) and (24) becomes effective because of the reflection by the upstream boundary condition at $t' = L'/c_1' - L'/c_2'$. Each reflection brings a new term into effect at a time appropriate to the position in the channel. Even Fig. 2 for a short channel shows only one reflection and thus confirms that only the leading terms in the infinite series are significant.

It is clear from Figs 2 and 3 that the well-known impulse response for the case of a semi-infinite channel (Deymie, 1935; Lighthill & Whitham, 1955; Dooge & Harley, 1967):

$$h_u^1(x,t) = \exp(-\alpha_2 x)\ \delta(t - x/c_1) \tag{47}$$

$$h_u^2(x,t) = \exp(-\beta_1 t + \beta_2 x) h \left[\frac{x}{c_1} - \frac{x}{c_2}\right] \frac{I_1\{2h\sqrt{[(t - x/c_1)(t - x/c_2)]}\}}{\sqrt{[(t - x/c_1)(t - x/c_2)]}} \cdot U[t - x/c_1] \tag{48}$$

which corresponds to use of only the first term in equations (22) and (24) is adequate for flow prediction if $(L' - x') > 1$ for $F_0 = 0.2$ and if $(L' - x') > 0.5$ for $F_0 = 0.8$. These results give practical guidance for the appropriate location of the downstream control relative to the channel reach of interest.

Due to heavy damping of the upstream wave, for $x' > 1$ the impulse response for a downstream input can be approximated by the impulse response for a semi-infinite channel, i.e. only the first term in the first sum both in equation (26) and in equation (27) is required.

The shapes of the body of the wave of the downstream response for $F_0 = 0.2$, $F_0 = 0.8$ and for locations of the downstream boundary $(L' - x') = 0.1$, $(L' - x') = 0.3$ and $(L' - x') = 0.5$ are shown in Fig. 4. If the point of interest is closer to the downstream boundary the head of the wave plays a more important role in the transformation of the downstream boundary condition than in the transformation of the upstream boundary condition. The comparison of the area under the curve of the body of the wave with the corresponding area under the head of the wave is presented in Table 2.

As an illustration of the effect of the transmission of an error or a change in the value of $A_d(t)$ at the point $(L' - x')$, the change in water level due to a constant downstream boundary $A_d(t) = 1$ is calculated. The backwater curve is shown in Fig. 5(a) for $F_0 = 0.2$ and $F_0 = 0.8$. It is clear from Fig. 5(a) that the backwater effect is effective only for $(L' - x') < 1.2$ for $F_0 = 0.2$ and for $(L' - x') < 0.5$ for $F_0 = 0.8$. The corresponding effect of a change in the value of $A_u(t)$ at the point $x' = 0$ is shown in Fig. 5(b) where it can be seen that the effect is substantial except close to the downstream boundary.

The above analysis has been possible only because of the linearization of

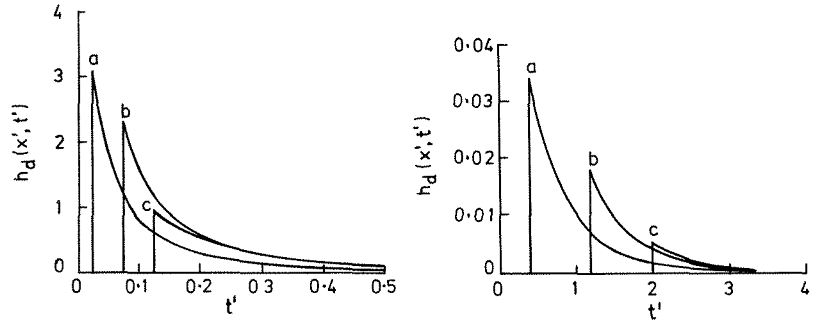


Fig. 4 *Hydrograph of channel response at point x' due to an impulse at the downstream end of the channel L' for three different relative locations: (a) $L' - x' = 0.1$, (b) $L' - x' = 0.3$, (c) $L' - x' = 0.5$, and zero upstream input.*

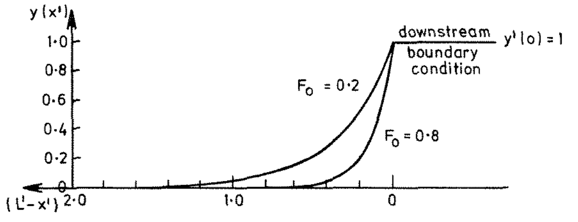
Table 2 *Comparison of the area under the curve of the head of the wave with the corresponding area under the curve of the body of the wave for downstream impulse response*

F_0	$L' - x'$	$\int_0^\infty h'_d(x,t)dt$	$\int_0^\infty h^2_d(x,t)dt$
0.2	0.1	0.50	0.22
	0.3	0.13	0.27
	0.5	0.03	0.18
0.5	0.1	0.61	0.06
	0.3	0.22	0.08
	0.5	0.08	0.05
0.8	0.1	0.42	0.02
	0.3	0.07	0.01
	0.5	0.01	0.00

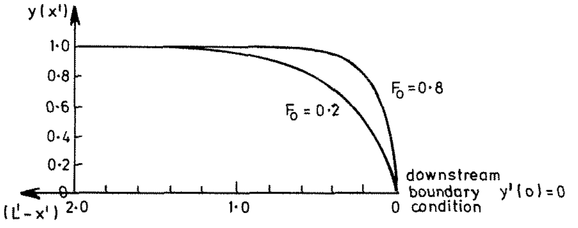
the basic St Venant equations. It is to be expected that the linear analysis can provide a satisfactory qualitative picture of the effect of the downstream boundary condition in the nonlinear case and a good first approximation to the quantitative effects. The errors due to linearization can only be estimated for a particular channel and particular prescribed boundary conditions.

CONCLUSIONS

The effect of the downstream boundary condition on unsteady flow in channels is explored through an analysis of the linearized St Venant



(a) Effect of shift in downstream condition



(b) Effect of shift in upstream condition

Fig. 5 (a) Backwater profile due to a constant unit excess of downstream level: (b) Backwater profile due to a constant unit excess of upstream over downstream level.

equations. The effects of downstream and upstream boundary conditions at an intermediate point in the reach are evaluated. The analytical results obtained are applicable to any shape of cross section and any type of friction law. The quantitative effects of the two boundary conditions are compared for the case of a wide rectangular channel with Chézy friction. It is anticipated that the relative effects would be similar for other shapes of channel and other friction formulae.

It is found that:

(a) For the case where there is both an upstream and a downstream boundary condition the unsteady wave motion produced by each of the boundary conditions will be successively reflected at each end of the channel reach and thus will require representation by an infinite series.

(b) Allowance for a downstream boundary condition thus has a double effect since it produces the reflection of the movement due to the upstream input as well as a direct effect on the channel reach of the downstream boundary condition.

(c) The reflection of the two sets of wave motion at the opposite end of the channel from the point of generation results in representing each of the linear channel responses by two infinite series, one representing each direction of propagation.

(d) The total volume of the contributions to the impulse response due to a delta function input at the upstream boundary and a delta function input at the downstream boundary sum to unity for any point in the channel reach.

(e) The rate of convergence in the two infinite series characterizing the response to an upstream input and the two infinite series characterizing the response to a downstream input is the same in all four cases. Typical values of the common ratio $\exp(-2fL)$ are shown in Table 1.

(f) Except for low values of the Froude number and of the dimensionless channel length, the common convergence factor is such that only the leading term in each of the two series representing the effect of the upstream input and the leading term in the first of the series representing the response to a downstream input need to be taken into account.

(g) For a Froude number of $F_0 = 0.2$ any error in the downstream boundary condition will decrease as shown on Fig. 5(a) and will reduce to 5% of its original value at $(L' - x') = 0.95$ and to 1% of its original value at $(L' - x') = 1.5$. For the higher Froude number of 0.8, the effect of an error in the downstream boundary condition dies out more rapidly, reducing to 5% at $(L' - x') = 0.38$ and to 1% at $(L' - x') = 0.58$.

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Analytical solution of channel flow with downstream control

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Erratum

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published in the previous issue of *Hydrological Sciences Journal* (vol. 33, no. 3, June 1988)

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$$h_d(x,s) = \exp[-(es + f)\ (L - x)]\ \frac{sh[x\ \surd(as^2 + bs + c)]}{sh[L\ \surd(as^2 + bs + c)]}$$

Page 277: equation (24) should read:

$$h_u(x,t) = \Sigma_0^\infty \exp(-\beta_1 t + \beta_2 x)\ (h/c_1 - h/c_2)\ (2nL + x)\cdot$$

$$\frac{I_1\{2h\ \surd[(t - nt_0 - x/c_1)\ (t + nt_0 - x/c_2)]\}}{\surd[(t - nt_0 - x/c_1)\ (t + nt_0 - x/c_2)]}\ U(t - nt_0 - x/c_1)$$

$$- \Sigma_1^\infty \exp(-\beta_1 t + \beta_2 x)\ (h/c_1 - h/c_2)\ (2nL - x)\cdot$$

$$\frac{I_1\{2h\ \surd[(t - nt_0 - x/c_2)\ (t + nt_0 - x/c_1)]\}}{\surd[(t - nt_0 - x/c_2)\ (t + nt_0 - x/c_1)]}\ U[t - nt_0 - x/c_2]$$

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