Hydrological Sciences Journal
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t911751996

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Online publication date: 21 December 2009

To cite this Article KUNDZEWICZ, ZBIGNIEW W. and NAPIÓRKOWSKI, JAROSŁAW J.(1986) 'Nonlinear models of dynamic hydrology / Modèles non linéaires d'hydrologie dynamique', Hydrological Sciences Journal, 31: 2, 163 — 185
To link to this Article: DOI: 10.1080/02626668609491038
URL: http://dx.doi.org/10.1080/02626668609491038

Hydrological Sciences – Journal – des Sciences Hydrologiques, 31, 2, 6/1986

Nonlinear models of dynamic hydrology*

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ABSTRACT Methods of description of the nonlinear effects in dynamic hydrological systems have been surveyed. Particular reference was given to those sorts of non-linear methods which do not require significantly more computational effort in comparison with classical linear models. Original concepts in the field of nonlinear state-space models, Volterra series models and the general framework of multilinear models were tackled in more detail. Illustrative numerical examples of flood routing and rainfall-runoff modelling were presented.

Modèles non linéaires d'hydrologie dynamique
RESUME On a étudié les méthodes de description des effets non linéaires dans les systèmes d'hydrologie dynamique. On s'est référé particulièrement à ces types de méthodes non linéaires qui n'exigent pas une masse de calculs significativement plus importante que dans le cas des modèles linéaires classiques. On a entrepris plus en détail l'études de concepts originaux dans le domaine des modèles non linéaires état-espace, des modèles des séries de Volterra et de la structure générale de modèles multi-linéaires. On a présenté des exemples numériques représentatifs de la mise en modèle de la propagation des crues et des relations pluies-débits.

INTRODUCTION

The geophysical processes contributing to the hydrological cycle are described by the theoretically sound nonlinear partial differential equations of mass and energy transfer. Also, in macroscopic hydrological descriptions, there are inescapable nonlinearities, e.g. thresholds that separate clearly distinct domains of different system behaviour.

In fact, all physical hydrological systems are nonlinear. Even if they are assumed to be linear, this assumption is restricted to within some range of conditions only.

In spite of this, there are numerous examples of successful applications of linear models. They typically pertain to the cases where the accuracy requirements are not critical and practically acceptable results may be obtained by means of linear models.

**Paper presented at the Anglo-Polish Workshop held at Jabłonna, Poland, September 1984. (See report in Hydrological Sciences Journal, vol.30, no.1, p.165).*

The inconvenience in applications of nonlinear models results from the following premises:

(a) the solution of a nonlinear mathematical model is more difficult to obtain than the solution of a linear model;

(b) the solution of a nonlinear model is valid for the particular set of initial and boundary conditions and the input signal used for computations; in order to infer the system behaviour for other conditions, one has to repeat the (possibly tedious) computations for the new conditions.

The hydrodynamic equations describing hydrological processes were developed in nonlinear form in the nineteenth century. As far as hydrological research is concerned, probably the first recognitions of nonlinear behaviour were made in modelling overland flow on natural catchments (Horton, 1938) and on paved surfaces (Izzard, 1944), where a conceptual nonlinear relationship between outflow and storage was used. The first comprehensive and systematic treatment of the nonlinearity aspects of rainfall-runoff modelling was due to Minshall (1960). Based on a large amount of observed data, Minshall demonstrated that the instantaneous unit hydrograph (IUH) depended on the rainfall intensity (higher intensity caused a rise in the value of the peak and a reduction in the time to peak of the IUH).

Contrary to the linear systems approach, there is no unique, compact and general theory of nonlinear dynamic systems. This situation clearly influences the state of the art of hydrological modelling. In the present paper, a systematic investigation of the nonlinearity aspects of modelling hydrological processes is given. Several alternative means of describing system nonlinearity are analysed with regard to both methodological concepts and practical aspects.

LINEAR MODELS

Linear models are frequently used in dynamic hydrology due to their simplicity and low cost. This pertains to all genetic types of linear methods, namely to hydrodynamic, conceptual and black box system models. The formulation of linear models of hydrological systems or processes follows one or other of the following approaches:

Linear partial differential equations

Linear partial differential equations are obtained from the methods of mathematical physics with the help of linearization (e.g. linearization for small increments, or harmonic linearization). An example of this sort of model can be given for the case of open channel flow in the following general form of the second order linear partial differential equation:

$$a\frac{\partial^2 Q}{\partial t^2} + b\frac{\partial^2 Q}{\partial x \partial t} + c\frac{\partial^2 Q}{\partial x^2} + d\frac{\partial Q}{\partial t} + e\frac{\partial Q}{\partial x} = 0 \tag{1}$$

where Q is the flow rate and a,b,c,d,e are parameters depending on the channel characteristics (length of the reach, bottom slope, roughness) and reference levels for linearization (cf. Dooge, 1973; Dooge & Harley, 1967). The linearization is usually performed for small increments and the reference levels pertain to the conditions for a steady and uniform flow situation.

Linear ordinary differential equations

The other linear models used in dynamic hydrology are described by an ordinary differential equation of the form:

$$a_n \frac{d^n y(t)}{dt^n} + \ldots + a_1 \frac{dy(t)}{dt} + a_o y(t) = x(t) \tag{2}$$

where x is the input signal and y is the output signal. The relevant set of initial conditions is:

$$y(0) = y_o, \ \ldots \ , \left. \frac{d^{n-1} y(t)}{dt^{n-1}} \right|_0 = y_{(n-1)0}$$

This general formulation can be decomposed within the state space framework to yield a set of n differential equations of the first order that can be conveniently written in matrix form as:

$$\dot{\underline{y}} = A\,\underline{y} + B\,x \tag{3}$$

where x is the scalar input signal, $\underline{y}$ and B are vectors and A is a matrix. The way of transforming equation (2) into equation (3) can readily be found in numerous textbooks on state space methods or dynamic systems.

Linear conceptual models

Yet another form of description of a hydrological system with the help of an ordinary differential equation is via a conceptual model. In the case of flood routing, the conceptual model used consists of (a) a rigorous continuity equation (a lumped form of a hydrodynamic description of the law of conservation of mass):

$$\dot{S} = x - y \tag{4}$$

where S, x and y are respectively the storage in, the inflow to, and the outflow from a river reach; and (b) a conceptual equation replacing the hydrodynamic equation of conservation of momentum, and that takes a general linear form (Chow & Kulandaiswamy, 1971):

$$S = \Sigma_{m=1}^{M} a_m x^{(m)} + \Sigma_{n=1}^{N} b_n y^{(n)} \tag{5}$$

A special and widely used case of equation (5) reads:

$$S = ax + by \tag{6}$$

Examples of this relationship are the storage equations of a linear reservoir where $a = 0$ or of a Muskingum model, where $a = K\alpha$ and $b = K(1 - \alpha)$ and α is a weighting factor.

Combination of n elementary conceptual models in series yields an n-th order ordinary differential equation describing, for example, the operation of a cascade of linear reservoirs.

Linear integral operators

As an alternative to linear differential equations, models in the form of linear integral operators are frequently used. In the case of a non-anticipating (causal) system the model formulation is:

$$y(t) = \int_0^\infty h(\tau)x(t - \tau)d\tau = y_0(t) + \int_0^t h(\tau)x(t - \tau)d\tau \qquad (7)$$

where h( ) is the impulse response function (kernel function) and $y_0(t)$ is the effect of initial conditions (for $t \leqslant 0$) upon the output signal at the time instant t, that is $y_0(t) = \int_t^\infty h(\tau)x(t - \tau)d\tau$.

For the initially relaxed case, equation (7) takes the convolution form:

$$y(t) = \int_0^t h(\tau)x(t - \tau)d\tau \qquad (8)$$

Under the assumption of finite memory of the system, usually accepted in dynamic hydrology, equation (7) gives:

$$y(t) = \int_0^{\min(t,T)} h(\tau)x(t - \tau)d\tau + \int_{\min(t,T)}^T h(\tau)x(t - \tau)d\tau \qquad (9)$$

whereas equation (8) yields:

$$y(t) = \int_0^{\min(t,T)} h(\tau)x(t - \tau)d\tau \qquad (10)$$

Nonstationarity

The forms of linear models considered above can be extended to cover nonstationarity effects. If the parameters a,b,c,d,e in equation (1), $a_0, a_1, \ldots, a_n$ in equation (2), A and B in equation (3), a_m and b_n in equation (5) or a and b in equation (6) vary with time, the nonstationarity is accounted for and the models remain linear. Similarly, if the kernel function h( ) in equations (7)-(10) depends on two arguments of a temporal nature (i.e. the dummy variable, τ, and the actual time instant, t, for which the output signal is being calculated), the model is linear and nonstationary. However, nonstationary models may require significantly more effort than their stationary counterparts. This pertains particularly to the identification phase of modelling.

FUNDAMENTALS OF NONLINEARITY

All the linear models considered above obey the superposition principle, usually understood as the condition of additivity, and formulated as follows.

A system is called additive if and only if the system operator acting on a sum of input signals yields the same response as the sum of the system operator acting on the individual input signals for all possible input signals x_1 and x_2.

From the above condition it follows that the sense of the superposition principle can be conceived in two ways:

(a) superposition with respect to amplitude of the input signal; and

(b) superposition with respect to the occurrence time of the input signal.

The meaning of both aspects is illustrated in Fig.1. It is obvious

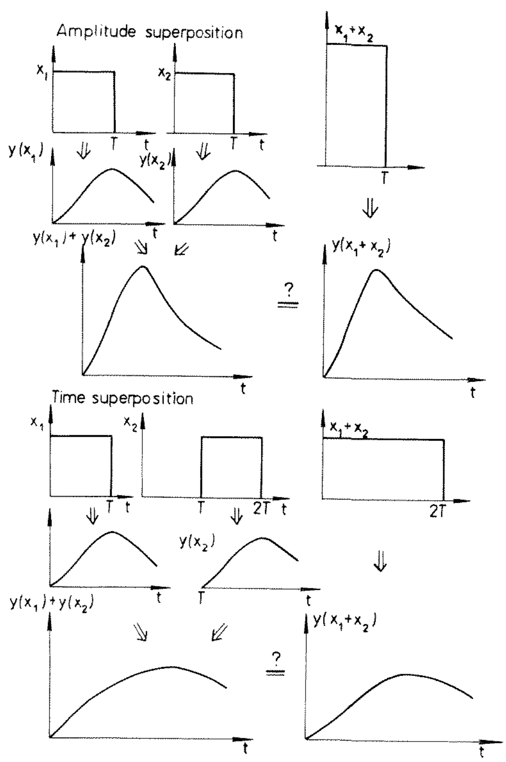

Fig. 1 Two aspects of the superposition principle.

that all linear models fulfil both the amplitude superposition principle and the temporal superposition principle. However, some nonlinear systems fulfil the temporal superposition principle (Kundzewicz, 1982, 1984, 1985) whereas others do not. The authors do not know of any nonlinear system that fulfils the amplitude superposition principle.

There is a class of nonlinear models described by nonlinear partial differential equations that originate from hydrodynamics. To this class belongs the complete St Venant model of open channel flow and its simplifications (e.g. the nonlinear diffusion analogy developed by Price (1973)). The set of partial differential equations of hyperbolic type, introduced by St Venant (1871), reads:

$$\frac{\partial v}{\partial t} + v\frac{\partial y}{\partial x} + \frac{A}{B}\frac{\partial v}{\partial x} = q/B \tag{11}$$

$$\frac{1}{g}\frac{\partial v}{\partial t} + \frac{v}{g}\frac{\partial v}{\partial x} + \frac{\partial y}{\partial x} + S_f - S_o - \frac{q}{gA}(v_q - v) = 0 \tag{12}$$

where equation (11) expresses the conservation of mass and equation (12) expresses the conservation of momentum. The notation used is standard and will not be explained here.

The nonlinearities of equations (11) and (12) are due to the terms:

$$v\frac{\partial v}{\partial x},\ \frac{A}{B}\frac{\partial v}{\partial x},\ \frac{q}{B},\ \frac{v}{g}\frac{\partial v}{\partial x},\ S_f,\ \text{and}\ \frac{q}{gA}(v_o - v)$$

By using another choice of dependent variables (A and Q) one can make the continuity equation linear at the cost of increasing the number of nonlinear terms in the momentum equation (Kundzewicz, 1985).

It is clear that linear hydrodynamic models originate from nonlinear hydrodynamic models. This parentage may not hold in other classes of models. Alternative approaches to nonlinear modelling originate from different ways of introduction of nonlinearity elements to the formulations of linear models developed earlier.

In the following three sections, alternative means of introducing nonlinearity to linear models of dynamic hydrology will be studied. Accounting for nonlinear effects by means of convenient linear mathematics is reviewed first; nonlinear ordinary differential models are then studied; and finally nonlinear integral operators are investigated. No further analysis of nonlinear partial differential equations will be given, as it can be readily found in numerous references on hydrodynamics.

NONLINEAR EFFECTS VIA LINEAR MATHEMATICS

Although hydrological systems and processes are nonlinear, there have been many attempts to model them with the help of linear mathematics. If the data for several events are available, a linear model can be found that best fits these data according to a specified criterion. In this approach, one linear model is supposed to cover optimally (in the average sense) all nonlinear behaviours that occur for the different conditions observed.

A step in the right direction is the use of a model that, although it remains linear for one event, may change from event to event. Thus the event characteristics tune the model parameters whose event-constant values are used in modelling the event.

A very simple approach of this kind is to allow for the dependence of the kernel function on some aggregated characteristics of the input signal (effective rainfall in rainfall-runoff modelling, or inflow to the reach in flood routing). Extending the concepts of Minshall (1960) and of Amorocho (1961), who found a family of rainfall-dependent IUHs, one could proceed as follows:

(a) identify the sets of conceptual model parameters corresponding to a number of different events; and

(b) correlate the conceptual model parameters with event indices. A family of relationships $K = G_1\ (A,V,T)$ and $N = G_2\ (A,V,T)$ where A,V,T are aggregated event indices (amplitude, volume and duration time, respectively) and K,N are conceptual parameters of cascade-type models, can be considered as a simple representation of nonlinearity via linear means. Within one event the system behaves linearly, that is both aspects of the superposition principle are fulfilled.

The idea outlined above can be extended to the general structure of the linear integral operator:

$$y(t) = \int h(\underline{p}_e,\tau)x(t - \tau)d\tau \tag{13}$$

where $\underline{p}_e$ is a vector of event characteristics. This vector remains constant for one event but may vary from event to event. It is not quite clear, however, how to construct the vector $\underline{p}_e$ optimally. Some kind of schematization of input signals may be necessary.

A more advanced application of linear mathematics in modelling nonlinear systems will be considered in the section below on multilinear models.

NONLINEAR DIFFERENTIAL MODELS

There are two distinct ways of introducing nonlinearity to conceptual hydrological models. Firstly, the nonlinear extension of the conceptual storage equation (5) replacing the hydrodynamic law of conservation of momentum can be assumed. Thus in general:

$$f(S,x,\dot{x},\ldots,x^{(m)},y,y,\ldots,y^{(n)}) = 0 \tag{14}$$

where f is a nonlinear function.

Secondly, the summation of equation (5) can be extended in that the coefficients a_m and b_n could depend on x and y, that is:

$$S = \Sigma_{m=1}^{M}\ a_m(x,y)x^{(m)} + \Sigma_{n=1}^{N}\ b_n(x,y)y^{(n)} \tag{15}$$

This latter was the nonlinear hydrological system model formulated by Chow & Kulandaiswamy (1971).

Nonlinear extension of the storage equation

The introduction of nonlinearity to a conceptual model will first be examined by replacing equation (5) by the more general equation (14).

Consider the concept of the linear reservoir consisting of the continuity equation (4) and the special case of the storage equation (6):

$$y = K\ S \tag{16}$$

There have been several attempts to use a more complex relation between the storage and the outflow, one that is likely to better represent the natural hydrological system in question. This gave rise to the idea of a nonlinear reservoir whose outflow law attains a more general form:

$$y = f(S) \tag{17}$$

An example of this sort of structure is a nonlinear reservoir with the power outflow law:

$$y = K\ S^{C} \tag{18}$$

or

$$S = K_1\ y^{C_1} \tag{19}$$

Such a model was used successfully by Laurenson (1964) for rainfall-runoff modelling.

Another concept corresponding to a typical static characteristic from dynamic system theory is a nonlinear reservoir with hysteresis (Fig.2) whose storage law is the storage equation (19) extended by the derivative of outflow:

$$S = K_1\ y^{C_1} + K_2\ \frac{dy}{dt} \tag{20}$$

This equation was used by Prasad (1967) in rainfall-runoff modelling. In his analysis of the generalized Muskingum method, Ding (1974) used a similar extension of equation (18):

$$y = K\ S^{C} + K_3\ \frac{dS}{dt} \tag{21}$$

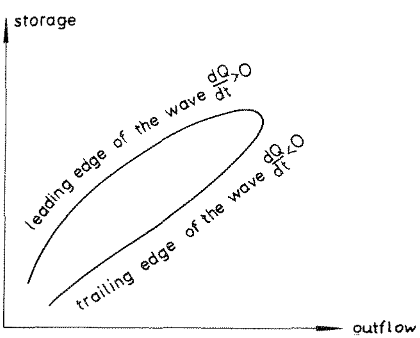

Fig. 2 Reservoir with hysteresis effect.

Yet another type of nonlinear reservoir accounts for a finite storage capacity. One of the possible outflow laws was devised by Kaczmarek (1975):

$$y(t) = a\ [1 - b\ \frac{S_{max}}{S(t)}] + x(t)\ \frac{c}{1 - S(t)/S_{max}} \tag{22}$$

Equation (14) can also serve as the general background to the nonlinear storage law for a generalized nonlinear Muskingum method that was derived by Strupczewski & Kundzewicz (1980) from hydrodynamic premises and took the form:

$$S = f(x^{0.6},\ y^{0.6}) \tag{23}$$

Exemplification of the function f and the details of derivation are given in Strupczewski & Kundzewicz (1980)

Nonlinearity via variable coefficients

The approach resulting from equation (15) will now be investigated. A simple example is again a nonlinear reservoir formed by making the storage coefficient, K, in equation (16) dependent on S, that is

$$y = K^*(S).S \tag{24}$$

Another type of nonlinear reservoir can be formed by assigning constant values of K in equation (16) for particular ranges of S. This produces a threshold effect visible in nonlinear reservoir models with piecewise constant parameters. A simple example of this kind reads:

$$K = \begin{cases} K_1 & \text{for} \quad S \leqslant S_* \\ K_2 & \text{for} \quad S > S_* \end{cases} \tag{25}$$

or

$$y = \begin{cases} K_1\ S & \text{for} \quad S \leqslant S_* \\ K_2\ S & \text{for} \quad S > S_* \end{cases} \tag{26}$$

This corresponds to a nonlinear reservoir with a bottom outlet and with a side outlet that operates if some threshold storage is reached as was used in different combinations by Sugawara *et al.* (1975) in rainfall-runoff modelling.

A special case of a nonlinear reservoir of the threshold type with piecewise constant parameters (i.e. piecewise linear outflow law) is a reservoir with a dead zone as shown in Fig.3. The nonlinear reservoir given by equation (18) can also be put into the framework of equation (24) as:

$$K^*(S) = K\ S^{C-1} \tag{27}$$

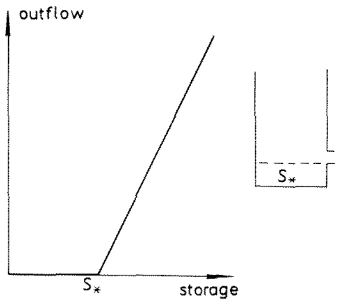

Fig. 3 Reservoir with a dead zone.

Alternatively, other extensions of the equation for a linear reservoir (16) can be used, such as:

$$S = K_1(y)\ y \tag{28}$$

or

$$S = K_2(x)\ y \tag{29}$$

Much attention by hydrologists has been given in the last decade to the Muskingum method with variable coefficients. This is the direct consequence of the analysis of Cunge (1969). Since the finite difference formulation of the Muskingum model is analogous to a certain finite difference scheme of the kinematic wave model (or even the diffusion analogy), the conceptual Muskingum parameters can be determined via the physically sound parameters of the latter hydrodynamic models. If the dependent variable of the hydrodynamic model (e.g. flow rate) varies with the modules of the finite difference scheme, the conceptual parameters of the variable Muskingum model follow these variations. Thus the conceptual parameters of the variable Muskingum equation can follow the relationships obtained by Cunge (1969):

$$K(c) = L/c \tag{30}$$

$$\alpha(Q,c) = 0.5[1 - Q/BcS_oL)] \tag{31}$$

where c is the wave celerity, Q is the flow rate, L is the reach length, S_o is the bottom slope and B is the surface width.

Nonlinear state model

The natural generalization of the nonlinear reservoir model (equations (4) and (17)) is a combination of n such conceptual elements. Each nonlinear reservoir is then responsible for part of the attenuation of the system response. This lumped dynamic model can be represented by a set of ordinary differential equations:

$$\dot{S}_1(t) = -f[S_1(t)] + X(t)$$

$$\dot{S}_2(t) = -f[S_2(t)] + f[S_1(t)]$$

$$\cdots\cdots\cdots\cdots\cdots \qquad (32a)$$

$$\dot{S}_n(t) = -f[S_n(t)] + f[S_{n-1}(t)]$$

$$Y(t) = f[S_n(t)] \qquad (32b)$$

under the initial condition $\underline{S}(O)$, where X is the input signal, S_i is the storage in the i-th reservoir, f( ) represents the outflow-storage relation and Y is the output signal.

It is assumed here that the function, f, is not prescribed but is differentiable for $S_i \geqslant 0$ as many times as required. The vector differential equation (32) can be considered as the definition of a nonlinear operator mapping a space of inflows into a space of corresponding outflows. Hence the change of the trajectories $S_i(t)$ and Y(t) from the steady state $y_o = f(S_{i,o}) = x_o$, due to an input increment, x(t), can be determined by means of a Taylor series expansion (Napiórkowski, 1978). Accordingly, we may divide the trajectories' increments into linear, quadratic, and cubic parts, and a residual error, i.e.

$$\Delta S_i(t) = S_i(t) - S_{i,o} = \delta S_i(t) + \delta^2 S_i(t) + \delta^3 S_i(t) + e(S_i) \qquad (33)$$

$$y(t) = Y(t) - y_o = \delta y(t) + \delta^2 y(t) + \delta^3 y(t) + e(y) \qquad (34)$$

In order to compute the linear (δ), quadratic (δ^2) and cubic (δ^3) components of y(t) and $\Delta S_i(t)$ we make use of the Taylor expansion of the outflow-storage relations about the steady state, $S_{i,o}$:

$$f[S_i(t)] - y_o = a\,\Delta S_i(t) + b[\Delta S_i(t)]^2 + c[\Delta S_i(t)]^3 + e(f) \qquad (35)$$

where

$$a = \frac{df}{dS_i}, \quad b = \frac{1}{2}\frac{d^2 f}{dS_i^2}, \quad c = \frac{1}{6}\frac{d^3 f}{dS_i^3} \qquad (36)$$

Substituting equations (33), (34), (35) and (36) into equation (32) and neglecting the second and higher order terms gives the set of equations for the linear approximation as:

$$\delta S(t) = a\phi\delta S(t) + [1,0\ldots,0]^T x(t) \qquad (37a)$$

$$\delta y(t) = a\delta S_n(t) \qquad (37b)$$

where

$$\phi = \begin{vmatrix} -1, & 0, & \ldots, & 0 \\ 1, & -1, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \ldots, & 0 \end{vmatrix} \tag{38}$$

When the second-order increments are taken into account we get the additional equations:

$$\delta^2\dot{S}(t) = a\phi\delta^2 S(t) + b\phi[\delta S(t)]^2 \tag{39a}$$

$$\delta^2 y(t) = a\delta^2 S_n(t) + b[\delta S_n(t)]^2 \tag{39b}$$

which are linear in $\delta^2 S(t)$ and $\delta^2 y(t)$.

When the third-order increments are also taken into account we have the further equations:

$$\delta^3 S(t) = a\phi\delta^3 S(t) + 2b\phi\delta S(t)\delta^2 S(t) + c\phi[\delta S(t)]^3 \tag{40a}$$

$$\delta^3 y(t) = a\delta^3 S_n(t) + 2b\delta S_n(t)\delta^2 S_n(t) + c[\delta S_n(t)]^3 \tag{40b}$$

which are linear in $\delta^3 S(t)$ and $\delta^3 y(t)$.

Having determined the functions $\delta S(t)$, $\delta^2 S(t)$, $\delta^3 S(t)$ the fourth order increment of the output trajectory can also be obtained in a similar way by expansion of the set (32) up to fourth-order increments.

It should be noted that the argument of the forcing function for equation (39) is the solution of equation (37), and that the arguments of the forcing function for equation (40) are the solutions of equations (37) and (39). It should also be noted that the input, $x(t)$, occurs only in equation (37). Consequently the addition of the components $\delta^2 y(t)$ and $\delta^3 y(t)$ affects only the distribution of the output, and the total volume of these components is zero.

A more detailed description of this approach can be found in Napiórkowski (1978), Napiórkowski & Strupczewski (1979, 1981), and Napiórkowski & O'Kane (1984).

The first-, second- and third-order components described by equations (37), (39) and (40) form the Third Order State Model (TOSM) which was used to represent a catchment response by Napiórkowski (1985). The objective was to solve the problem of identifying the four parameters n, a, b and c of the model for a watershed previously described by Diskin & Boneh (1973). The catchment is that of the Cache River at Forman in southern Illinois. The data of effective rainfall, represented as rectangular pulses with a time interval of 1 day, and surface runoff for eight storms were observed between 1935 and 1951.

The optimal values of the parameters (n, a, b, c) of the TOSM were found to be:

$n = 3$
$a = 0.67$
$b = 5.58 \times 10^{-3}$ $(\mathrm{day}^{-1}\mathrm{mm}^{-1})$
$c = 83.6 \times 10^{-6}$ $(\mathrm{day}^{-1}\mathrm{mm}^{-2})$

An example of the degree of fit to the observed runoff by the TOSM is shown in Fig.4 for one of eight storms (storm no.1). The

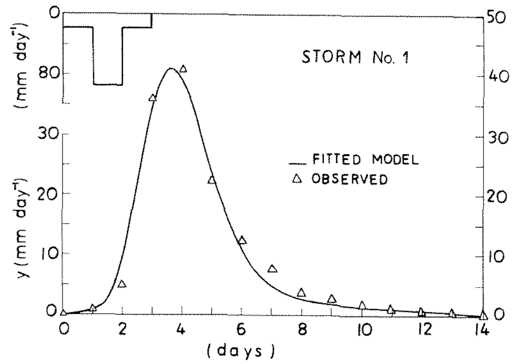

Fig. 4 Comparison of observed runoff and that predicted by the TOSM for storm no. 1.

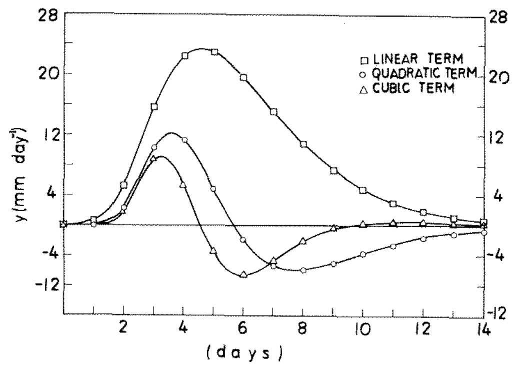

Fig. 5 The linear, quadratic and cubic terms predicted by the TOSM for storm no. 1.

Table 1 Optimal parameters and values of the objective function for models based on a nonlinear cascade

	Linear	Quadratic	Cubic
n	4	3	3
a	1.32	0.75	0.67
$b \times 10^{-3}$	0	6.84	5.58
$c \times 10^{-6}$	0	0	83.6
J	445	233	154

separate linear, quadratic and cubic components for this particular storm are plotted in Fig.5. In Table 1 a comparison is made between the optimal linear ($b = c = 0$), optimal quadratic ($c = 0$) and the cubic model based on a cascade of nonlinear reservoirs.

The satisfactory results of such simulations indicate that the second- and third-order increments produce a marked improvement in the predictive power of the conceptual model when compared with its linearized version, through a steepening of the rising limb and a flattening of the recession.

In the next section it will be shown that the Third Order State Model is equivalent to the three first terms of a Volterra series.

NONLINEAR INTEGRAL MODELS

The linear integral operator given by equations (7) to (10) is a very convenient tool for modelling hydrological processes, which is why there have been several attempts to accommodate nonlinearity within this simple structure.

Variable unit hydrograph

One of the ways of introducing nonlinearity to the linear integral operator modelling of hydrological processes was devised by Amorocho (1967) who suggested a variable unit hydrograph i.e. a family of input-dependent kernels of the operator in equations (7) to (10). This concept was used in practice by Ding (1974) who assumed that the kernel function depended on the dummy time variable, τ, and also on the instantaneous value of the input signal at the time instant $(t - \tau)$, where t is the time instant for which the system response is calculated:

$$y(t) = \int h[\tau, x(t - \tau)]x(t - \tau)dr \qquad (41)$$

To the authors' knowledge this methodology proposed by Ding (1974) was never developed in a rigorous way. Elements of the idea, however, are used in the multilinear models presented below.

Volterra series model

A powerful tool of nonlinear modelling, well established in hydrology, is the Volterra series of integral operators introduced to mathematics by Volterra (1930) and to hydrology by Amorocho & Orlob (1961). Following the notation used by Volterra the model can be written as:

$$y(t) = \Sigma_{i=1}^{\infty} |\int|_i \; h_i(\tau_1, \ldots, \tau_i)\Pi_{k=1}^{i}[x(t - \tau_k)d\tau_k] \qquad (42)$$

The description of dynamic systems by a Volterra series is a generalization of the concept of the transfer function which is of great importance in the analysis, design and control of linear systems. The Volterra series represents an explicit input-output relation for nonlinear systems and consists of an infinite series composed of terms of convolution integrals. The first term is the convolution integral of the first order kernel with the input function, while the n-th order term is an n-fold convolution integral containing the n-th order kernel multiplied by an n-th order product of the input function. The modelling of hydrological processes (such as flood routing, rainfall-runoff) by means of a

Volterra series has been developed independently of other methods of describing dynamic systems, in particular by a state equation formulation. The problem of series identification has been solved by numerical methods applied to an input record and its corresponding output by means of kernel expansion (under an arbitrary assumption as to kernel structure) in orthonormal polynomials (see Amorocho & Brandstetter, 1971; Kuchment, 1972; Papazafiriou, 1976). As was pointed out by Napiórkowski (1978), the fact that the structure of the kernels is not known may lead to a search for the solution within the wrong class of functions. Thus the possibility cannot be excluded that the solution derived hardly reflects reality. Accordingly, the analytical derivation of the kernels of the Volterra series is not another methodological approach, but it helps in the correct formulation of the identification problem.

Napiórkowski (1978) and Napiórkowski & Strupczewski (1979, 1981) have developed the description by state equations and the description by integral series for the case of the nonlinear cascade given by equation (32).

The solution of equation (37a) describing the linear part of the storage trajectory is:

$$\delta S(t) = \int_0^t \exp(a\phi\tau)\ [1,0,\ldots,0]^T\ x(t-\tau)d\tau \tag{43}$$

where $\exp(a\phi t)$ is the transition matrix for equation (37a). One can see that the linear component of the state of the cascade of nonlinear reservoirs can be described as the first term of the Volterra series:

$$\delta S(t) = \int_0^t K_1(\tau)\ x(t-\tau)d\tau \tag{44a}$$

where the vector of linear response kernels is given by:

$$K_1(t) = \exp(a\phi t)\ [1,0,\ldots,0]^T \tag{45a}$$

From equation (37b) one can see that the linear part of the outflow trajectory is:

$$\delta y(t) = \int_0^t h_1(\tau)\ x(t-\tau)d\tau \qquad (44b$$

with

$$h_1(t) = a\ K_{1,n}(t) \tag{45b}$$

where $K_{1,n}$ is the linear state kernel for the n-th reservoir in the cascade.

The solution of equation (39a) describing the quadratic part of the storage trajectory is:

$$\delta^2 S(t) = \int_o^t \exp(a\phi r)\ b\phi[\delta S(t - r)]^2 dr \tag{46}$$

and the transition matrix for equation (39a) is the same as for equation (37a). Having the solution for $\delta S(t)$ from equation (44a) we can insert $[\delta S(t - r)]^2$ in equation (46). The double change of the order of integration between r and τ_1,τ_2 results in the second term of the Volterra series:

$$\delta^2 S(t) = \int_o^t\int_o^t K_2(\tau_1,\tau_2)\ x(t - \tau_1)x(t - \tau_2)d\tau_1 d\tau_2 \tag{47a}$$

where

$$K_2(\tau_1,\tau_2) = \int_o^{\max(\tau_1,\tau_2)} b \exp(a\phi r)\ K_1(\tau_1 - r)K_1(\tau_2 - r)dr \tag{48a}$$

is the vector of the second-order state kernels.

From equation (39b) one can see that the quadratic part of the outflow trajectory is:

$$\delta^2 y(t) = \int_o^t\int_o^t h_2(\tau_1,\tau_2)\ x(t - \tau_1)x(t - \tau_2)d\tau_1 d\tau_2 \tag{47b}$$

with

$$h_2(\tau_1,\tau_2) = a\ K_{2,n}(\tau_1,\tau_2) + b\ K_{1,n}(\tau_1)\ K_{1,n}(\tau_2) \tag{48b}$$

where $K_{2,n}$ is the second-order state kernel for the n-th reservoir and $K_{1,n}$ is the first-order kernel already found in the linear approximation.

Finally, the solution of equation (40a) describing the cubic part of the storage trajectory is:

$$\delta^3 s(t) = \int_o^t \exp(a\phi r)\phi\{2b\delta S(t - r)\delta^2 S(t - r) + c[\delta S(t - r)]^3\}dr \tag{49}$$

where the transition matrix for equation (40a) is the same as for equations (37a) and (39a). Having the solution for $\delta S(t)$ from equation (44a) and for $\delta^2 S(t)$ from equation (47a), we can insert $\delta S(t - r)\delta^2 S(t - r)$ and $[\delta S(t - r)]^3$ in equation (49). The triple change of the order of integration between r and τ_1,τ_2,τ_3 results in the third term of the Volterra series:

$$\delta^3 S(t) = \int_o^t\int_o^t\int_o^t K_3(\tau_1,\tau_2,\tau_3)\ x(t - \tau_1)x(t - \tau_2)x(t - \tau_3)d\tau_1 d\tau_2 d\tau_3 \tag{50a}$$

where

$$K_3(\tau_1,\tau_2,\tau_3) = \int_o^{\max(\tau_1,\tau_2,\tau_3)} \exp(a\phi r)\phi[2bK_1\ (\tau_1 - r)K_2\ (\tau_2 - r,\tau_3 - r) + c\ K_1(\tau_1 - r)K_1\ (\tau_2 - r)K_1\ (\tau_3 - r)]dr \tag{51a}$$

From equation (40b) one can see that the cubic part of the outflow trajectory is:

$$\delta^3 y(t) = \int_o^t \int_o^t \int_o^t h_3(\tau_1,\tau_2,\tau_3) x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) d\tau_1 d\tau_2 d\tau_3 \tag{50b}$$

with

$$h_3(\tau_1,\tau_2,\tau_3) = a\ K_{3,n}(\tau_1,\tau_2,\tau_3) + 2b\ K_{1,n}(\tau_1)\ K_{2,n}(\tau_2,\tau_3)$$
$$+ c\ K_{1,n}(\tau_1)\ K_{1,n}(\tau_2)\ K_{1,n}(\tau_3) \tag{51b}$$

where $K_{3,n}$ is the third-order kernel for the n-th reservoir.

This proves that the Third Order State Model as represented by equations (37), (39) and (40) is equivalent to the sum of the three first terms of the Volterra series given by equations (44), (47) and (50). The structure of the first two Volterra kernels obtained after considerable manipulation was shown by Napiórkowski (1978) to be:

$$h_1(\tau) = a\ H_n(\tau) \tag{52}$$

$$h_2(\tau_1,\tau_2) = b\{H_n(\tau_1)\ \Sigma_{i=1}^n\ H_i(\tau_2) + H_n(\tau_2)\ \Sigma_{i=1}^n\ H_i(\tau_1)$$
$$- H_n[\max(\tau_1,\tau_2)]\} \tag{53}$$

where

$$H_n(t) = (at)^{n-1} \exp(-at)/(n - 1)! \tag{54}$$

The third-order kernel has not been obtained. It is computationally more efficient to calculate the higher order approximations from the state space representation rather than by using the n-fold integrals in the Volterra series.

Multilinear models

It is interesting that several convenient means of modelling nonlinear hydrological systems that seem to occur independently of one another can be embraced by one clearly defined category. This category, named multilinear models (Kundzewicz, 1982, 1984, 1985) contains the concepts called multiple-input linear models, multiple linearization models and nonlinear threshold models used earlier by Becker *et al.* (1981). Also the ideas of Amorocho (1961, 1967) and Ding (1974) are embraced within this category. The unifying term "multilinearity" has been used in mathematics (nonlinear functional analysis) to describe a different idea. It is believed, however, that no ambiguity is caused due to the redefining of the term.

A multilinear model is composed of parallel linear submodels, whose input signals are parts of the exterior input signals, distributed by a definite algorithm (Fig.6).

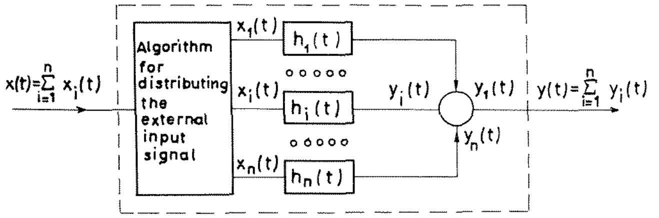

Fig. 6 Principle of multilinear models.

The general mathematical description of a multilinear model composed of n submodels reads:

$$y(t) = \Sigma_{i=1}^{n} \int h_i(\tau) x_i(t - \tau) d\tau \tag{55}$$

under the condition:

$$\Sigma_{i=1}^{n} x_i(t) = x(t) \tag{56}$$

The limits of the integration interval in equation (55) should be chosen in accordance with the discussion of linear models at the start of this paper.

A specific example of an early hydrological application of the multilinear structure of Fig.6 is the rainfall-runoff model introduced by Diskin (1964). The model (Fig.7) consisted of two cascades of linear reservoirs in parallel, with a simple algorithm for the distribution of the input signals by means of a constant coefficient, α.

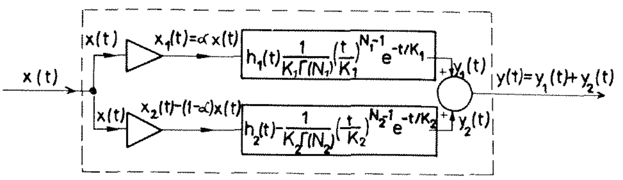

Fig. 7 The Diskin model, an early example of a multilinear model.

The parallel submodels represented two components of the runoff process, namely surface flow and subsurface flow. The assumption of a constant value of the distribution factor, α, assured the linearity of the total model. In fact, from the physical point of view, the value of this parameter could depend on rainfall parameters (e.g. mean intensity, duration time) and on the catchment conditions (e.g. API). However, already the simple linear structure shown in Fig.7 was shown to produce a considerably better simulation of the natural processes than a single cascade of linear reservoirs in series.

The fundamental problem that arises in multilinear modelling is how to distribute the external input signals into inputs to linear submodels.

There are two established answers to this question as illustrated in Figs 8 and 9. The mathematical description of the distribution reads for the amplitude distribution scheme (Fig.8):

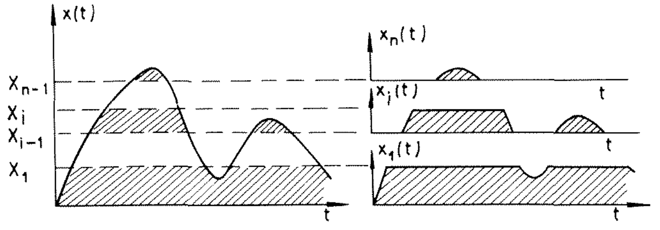

Fig. 8 Amplitude distribution of input signal.

$$x_1(t) = \min[x(t), X_1]$$

....

$$x_i(t) = \max\{0, \min[x(t) - X_{i-1}, X_i - X_{i-1}]\} \tag{57}$$

....

$$x_n(t) = \max[x(t) - X_{n-1}, 0]$$

and for the time distribution scheme (Fig.9):

$$x_1(t) = \{1 - \delta[\min \langle x(t), X_1\rangle - X_1]\}x(t)$$

....

$$\begin{aligned} x_i(t) = 0.5 \ &\{1 - \delta[\min \langle x(t), X_i\rangle - X_i]\} + \\ &+ \{1 - \delta[\max \langle x(t), X_{i-1}\rangle - X_{i-1}]\}\ x(t) \end{aligned} \tag{58}$$

....

$$x_n(t) = \{1 - \delta[\max\langle x(t), X_{n-1}\rangle - X_{n-1}]\}\cdot x(t) \tag{59}$$

where $\delta(x)$ is the Kronecker delta function defined as:

$$\delta(x) = \begin{cases} 0 & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases} \tag{60}$$

and X_k, $k = 1,\dots,n$, are the upper limits of the k-th zone of the external input signal.

The multilinear models corresponding to both methods of distribution of the input signal are used as follows (cf. Fig.6). The input signal is distributed into inputs to submodels, $x_1,\dots,x_n$. Each of the sub-inputs is convoluted by a

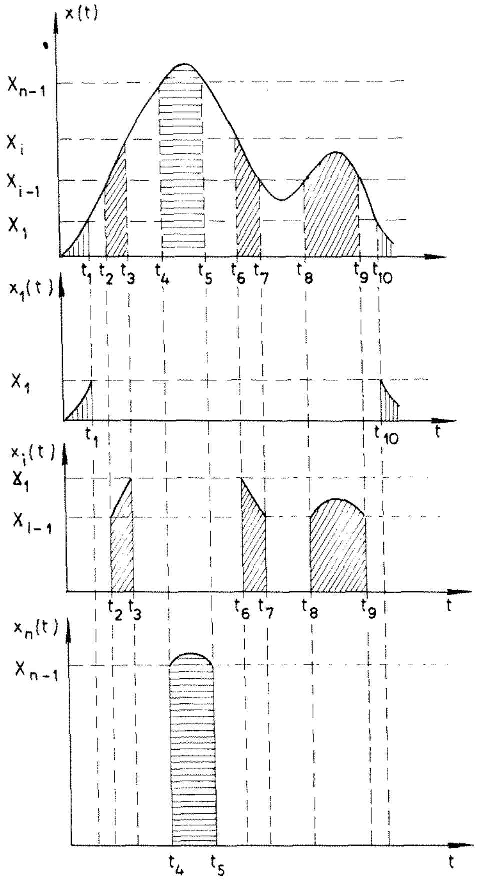

Fig. 9 Time distribution of input signal.

linear integral operator corresponding to the subinput, that is, with the help of the kernels $h_{x1}(\tau),\ldots,h_{xn}(\tau)$. The responses of particular linear submodels are recombined to yield the total response.

It can be demonstrated that the multilinear models fulfil the time superposition principle but do not fulfil the amplitude superposition principle. An extensive discussion of the properties of multilinear models can be found elsewhere (Kundzewicz, 1982, 1984).

Multilinear models have been used widely for operational applications as is convincingly shown in the works of Becker and his collaborators (Becker, 1976; Becker *et al.*, 1981). They used multilinear models in the amplitude distribution scheme (nonlinear threshold model) to forecast flows at several rivers in the German Democratic Republic and in other countries of central Europe.

CONCLUDING REMARKS

Assuming that the accuracy obtained with the help of a linear model is not sufficient, a nonlinear technique should be used. If, within the given class of problems, a rigorous nonlinear hydrodynamic model is available, it is likely to outperform most of the nonlinear models of other types (conceptual or black box system models). If there is no superior hydrodynamic model or if the process is too complicated to be modelled by fluid mechanics laws alone, it may be worthwhile to use a nonlinear model of the conceptual or black box type. In this latter category it is advantageous to use the Volterra integral series that has been proved to perform well for both rainfall-runoff modelling and flood routing. Moreover, the linkage between the Volterra series and arbitrary models (e.g. conceptual) formulated in the state space framework has been established, and in effect the Volterra series is no longer a black box system method.

It is desirable to continue studies on very simple nonlinear models which could be of good value if the large difference between the simplicity of the linear integral operator and the apparent complexity of either the Volterra series or the nonlinear hydrodynamic model discourages the user from applying the more sophisticated method. In such a situation, the use of a multilinear model or of a conceptual nonlinear model should be seriously considered.

In any case, an attempt should be made to strive towards a set of system impulse responses (e.g. IUHs) for different input signals (i.e. effective rainfall intensities) rather than to feel entirely satisfied with a single "mean" impulse response.

REFERENCES

Amorocho, J. (1961) Predicting storm runoff on small experimental watershed - Discussion. *J. Hydraul. Div. ASCE* 87, 185-191.

Amorocho, J. (1967) The nonlinear prediction problem in the study of the runoff cycle. *Wat. Resour. Res.* 3, 861-880.

Amorocho, J. & Brandstetter, A. (1971) Determination of nonlinear response function in rainfall-runoff relations. *Wat. Resour. Res.* 7(5), 1081-1101.

Amorocho, J. & Orlob, G.T. (1961) Nonlinear analysis of hydrologic systems. *Water Resources Center Contribution no.40, University of California, Berkeley.*

Becker, A. (1976) Simulation of nonlinear flow systems by combining linear models. In: *Mathematical Models in Geophysics* (Proc. Moscow Symp., August 1971), 135-142. IAHS Publ. no.116.

Becker, A., Polte, B. & Schmahl, W. (1981) Computerized streamflow forecasting system for the Elbe river in GDR. Proc. International Conf. on Numerical Modelling of River, Channel and Overland Flow for Water Resources and Environmental Applications, Bratislava.

Chow, V.T. & Kulandaiswamy, V.C. (1971) General hydrologic system model. *J. Hydraul. Div. ASCE* 97 (HY6), 791-804.

Cunge, J.A. (1969) Au sujet d'une méthode de calcul de propagation de crues (méthode Muskingum). *J. Hydraul. Res.* 7, 205-230.

Ding, J.Y. (1974) Variable unit hydrograph. *J. Hydrol.* 22, 53-69.

Diskin, M. (1964) A basic study of linearity of the rainfall-runoff process in watersheds. PhD thesis, Univ. Illinois, Urbana.

Diskin, M.H. & Boneh, A. (1973) Determination of optimal kernels for second order stationary surface runoff systems. *Wat. Resour. Res.* 9(2), 311-325.
Dooge, J.C.I. (1973) Linear theory of hydrologic systems. *Tech. Bull. no.1468,* Agricultural Research Service, US Dept Agriculture, Washington, DC.
Dooge, J.C.I. & Harley, B.M. (1967) Linear theory of open channel flow, Part I - Complete linear solution of routing problem. Dept Civil Engng, Univ. College Cork (unpublished memorandum).
Horton, R.E. (1938) The interpretation and application of runoff plot experiments with reference to soil erosion problems. *Soil Sci. Soc. Am. Proc.* 3, 340-349.
Izzard, C.F. (1944) The surface-profile of overland flow. *AGU Trans.* 25, 959-968.
Kaczmarek, Z. (1975) Storage systems dependent on multivariable stochastic processes. *IIASA Research Rep., Laxenburg.*
Keefer, T.N. (1976) Comparison of linear systems and finite difference flow-routing techniques. *Wat. Resour. Res.* 12, 997-1006.
Keefer, T.N. & McQuivey, R.S. (1974) Multiple linearization flow routing model. *J. Hydraul Div. ASCE* 100, 1031-1046.
Kuchment, L.S. (1972) *Matematiceskoje Modelirovanije Recnogo Stoka.* Gidrometeoizdat, Leningrad.
Kundzewicz, Z.W. (1982) Parameteruntersuchungen zur Berechnung des Hochwasserablaufs in Gerinnen. *Mitt. Inst. Wasserbau III, Univ. Karlsruhe* Heft 23.
Kundzewicz, Z.W. (1984) Multilinear flood routing. *Acta Geophys. Polonica,* Special Issue on Systems Hydrology, 32, 419-445.
Kundzewicz, Z.W. (1985) Modele hydrologiczne ruchu fal powodziowych. *Wydawnictwa Geologiczne, Warsaw.*
Laurenson, E.M. (1964) A catchment storage model for runoff routing. *J. Hydrol.* 2, 141-163.
Minshall, N.E. (1960) Predicting storm runoff on small experimental watershed. *J. Hydraul Div. ASCE* 86, 17-38.
Napiórkowski, J.J. (1978) Identyfikacja koncepcyjnego modelu zbiornikowego opisanego szeregiem Volterry. DSc Thesis, Institute of Geophysics, Polish Academy of Sciences, Warsaw.
Napiórkowski, J.J. (1985) The optimization of a third-order surface runoff model. In: *Scientific Procedures Applied to the Planning, Design and Management of Water Resources Systems* (Proc. Hamburg Symp., August 1983), 161-172. IAHS Publ. no.147.
Napiórkowski, J.J. & O'Kane, P. (1984) A new nonlinear conceptual model of flood waves. *J. Hydrol.* 69, 49-58.
Napiórkowski, J.J. & Strupczewski, W.G. (1979) The analytical determination of the kernels of the Volterra series describing the cascade of nonlinear reservoirs. *J. Hydrol. Sci.* 6, 121-142.
Napiórkowski, J.J. & Strupczewski, W.G. (1981) The properties of the kernels of the Volterra series describing the flow deviations from a steady state in an open channel. *J. Hydrol.* 52, 185-198.
Nash, J.E. (1959) Systematic determination of unit hydrograph parameters. *J. Geophys. Res.* 64, 111-115.
Papazafiriou, A.C. (1976) Linear and nonlinear approaches for short-term runoff estimations in time-invariant open hydrologic systems.

J. Hydrol. 30, 65-80.
Prasad, R. (1967) A nonlinear hydrologic system response model. *J. Hydraul. Div. ASCE* 93, 201-221.
Price, R.K. (1973) Variable parameter diffusion method for flood routing. *Report IT 115, Hydraulics Research Station, Wallingford, Oxfordshire, UK.*
Strupczewski, W.G. & Kundzewicz, Z.W. (1980) Muskingum method revisited. *J. Hydrol.* 48, 327-342.
Sugawara, M., Ozaki, E., Katsuyama, Y. & Watanabe, I. (1975) Tank model. Proc. IAHS Workshop on Mathematical Models (Bratislava, September 1975).
St Venant, B. (1871) Théorie du mouvement nonpermanent des eaux avec application aux crues des rivières et à l'introduction des marées dans leur lit. *C.R. Acad. Sci., Paris* 73, 148-154, 237-240.
Volterra, V. (1930) *Theory of Functionals and of Integral and Integro-Differential Equations.* Blackie & Sons, London.

Received 10 January 1985; accepted 18 December 1985.