On stochastic properties of nonlinear hydrological and water resources systems

О стохастических свойствах некоторых нелинейных гидрологических и водохозяйственных систем

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Introduction

One of the most fundamental problems in the field of water resources control is the problem of a choice of a suitable transformation model and its identification. The main widely accepted assumption concerning a structure of a model is its linearity. This assumption is usually introduced due to computational simplicity involved and due to development of linear dynamical system theory. Unfortunately, in many cases the physical structure of a system does not allow acceptance of linearity hypothesis - a linear model cannot describe a phenomenon accurately enough. Thus one is forced to apply methods of nonlinear systems analysis.

In hydrology, as in many other branches of science, one can observe different methods of nonlinear systems analysis, developed independently of one another. In many cases the complexity of a real system does not allow establishing adequate equations of process dynamics. In such cases it seems more efficient to look for a mathematical model of a "black box" type, that ensures the best description of a process in the sense of criterion used.

The two approaches of this type already applied to nonlinear hydrological modelling are based on some generalization of the well known convolution integral linear model /1/

$$y(t)=\int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau + y_0 \qquad /1/$$

where $x/t/$ - input to a system
$y/t/$ - output from a system
$h/t/$ - impulse response /instantaneous hydrograph/

The first possibility of introduction of nonlinearity is considering that the kernel h/t/ depends on the input to the system i.e. /Ding 1975/

$$y(t)=\int_{-\infty}^{+\infty} h[\tau, x(t-\tau)]\, x(t-\tau)\, d\tau + y_0 \qquad /2/$$

The most attractive, in the authors' opinion, structure of nonlinear "Black box" model is that of a Volterra series type /Volterra, 1913/ when taking into account only some first terms of the series

$$y(t)=\sum_{i=1}^{\infty} |\int|_i\, h_i(\tau_1,\dots,\tau_i) \prod_{k=1}^{i} x(t-\tau_k)\, d\tau_k + y_0 \qquad /3/$$

where the meaning of the Volterra symbol $|\int|_p$ is $\underbrace{\int\int\dots\int}_{p \text{ integrals}}$

Models of this structure were already subject to considerations of hydrologists /Amorocho and Orlob1961, Kuchment 1962, Diskin and Boneh 1972, Quimpo 1975 and others/.

The description of a nonlinear system by means of integral operators seems to be superior to a nonlinear differential equation model. The virtues of integral approach are: its brevity, independence of a degree of differential equations and in the stochastic approach avoiding complicated mathematical formalism of Ito stochastic differentials /Firla, 1971/.

One can show the equivalence of description of a system by means of nonlinear differential equations and by means of integral operator /3/. There is anyhow no physical interpretation found so far of higher order operator kernels, although there must exist some conceptual interpretation according to a penultimate statement. The structure of a second order kernel suggested by Diskin and Boneh /1972/cannot be considered a "conceptual model".

Quite a popular type of nonlinear hydrological models is routing in nonlinear reservoirs /nonlinearity introduced by dynamic equations of reservoirs or some threshold structures/. This method does not seem to lead to the development of a simple for practical purposes transformation of flow in a cascade of nonlinear reservoirs. A nonlinear structure corresponding to Nash's cascade of linear reservoirs does not have any practical value.

In this paper the analysis of some stochastic properties of systems/3/ will be performed. Stochastic identification of kernels of operators and the transformation of chosen stochastic processes in these systems will be dealt with.

Analysis of Volterra series models for a Gaussian white noise input

Let us assume the input process be a stationary Gaussian process featuring zero mean and a know autocorrelation function. One can search for such an input process a p-th order nonlinear Volterra type model being optimal in a mean square sense. It is clear that the resulting model is optimal only for this class of input processes only for which it has been calculated.

First consider a Gaussian white noise fed to the input of the system. Its characteristics are following:

$$E\left\{x(t)\right\} = 0 \qquad /4/$$

$$R(\tau) = E\left\{x(t)\,x(t+\tau)\right\} = K\cdot\delta(\tau) \qquad /5/$$

The assumption of white noise input seems reasonable when there is no information concerning a time structure of an input process. Due to some "filtering" properties of a Dirac delta function found in a white noise autocorrelation formula one gains relative simplicity of calculations.

The structure of a system taken for the following developments is equivalent to the series /3/ and consists of G-functionals suggested by Wiener /1958/. The series reads

$$y(t)=\sum_{m=0}^{\infty} G_m[k_m; x(t)] \qquad /6/$$

$$G_0[k_0; x(t)]=k_0 \qquad /7a/$$

$$G_1[k_1; x(t)]=\int_{-\infty}^{+\infty} k_1(\tau)\, x(t-\tau)\, d\tau \qquad /7b/$$

$$G_2[k_2; x(t)]=\iint_{-\infty}^{+\infty} k_2(\tau_1,\tau_2)\, x(t-\tau_1)\times x(t-\tau_2)\, d\tau_1 d\tau_2 - K\int_{-\infty}^{+\infty} k_2(\tau,\tau)\, d\tau \qquad /7c/$$

where $k_n/t/$ is a kernel of the n-th integral transformation.

Each functional $G_m[k_m; x(t)]$ consists of integral Volterra operator of the m-th degree and of some lower order operators determined directly by a basis kernel function. These lower order operators provide orthogonality of G_m-operator to all other Wiener's G_n-operators $n \neq m$ i.e.

$$E\{G_n[k_n; x(t)]\times G_m[k_m; x(t)]\}=0 \quad \text{for } n \neq m \qquad /8/$$

Schetzen /1974/ showed that kernels of the n-th order model being optimal in a mean square sense i.e.

$$E\{e_n^2(t)\}=E\{[y(t)-y_n(t)]^2\}=min \qquad /9/$$

for a Gaussian white noise input read

$$k_n(\tau_1,\dots,\tau_n)=\frac{1}{n!\,K^n}E\Big\{\Big[y(t)-\sum_{m=0}^{n-1}G_m(k_m;x(t))\Big]\times \prod_{i=1}^{n}x(t-\tau_i)\Big\} \quad \tau_i\geq 0 \quad i=1,2,\dots,n \qquad /10/$$

or zero otherwise.

We can thus augment the model to a higher degree due to the application of the so far determined optimal kernels of orders less than and equal to n.

Another Schetzen's formula

$$k_m(\sigma_1,\dots,\sigma_m)=\begin{cases}\frac{1}{m!\,K^m}E\{y(t)\,G_m[d_m;x(t)]\}; & \sigma_i\geq 0 \\ & i=1,\dots,m \\ 0 \text{ for any } \sigma_i<0 & m=0,\dots,m\end{cases} \qquad /11/$$

$$\text{where} \quad d_m(\tau_1,\dots,\tau_m)=\prod_{i=1}^{m}\delta(\tau_i-\sigma_i)$$

/invalid for the case when two or more σ's are equal/ has been used by Quimpo /1975/ to the analysis of hydrological systems described by Volterra equations.

Let us deal with the transformation of a Gaussian white noise in systems described by equations /3/ and /6,7 abc/. We shall focus our attention on the operator consisting of the first three elements of the Volterra series. It seems to be reasonable sinee the first three elements of this series proved to yield quite an accurate approximation. Taking into account a greater number of elements one encounters far greater computational effort and data quality requirements.

It can be shown /Papoulis 1965/ that the transformation of a stationary stochastic process in any deterministic system yields a stationary output process.

By making use of the properties of zero mean Gaussian processes /Wiener 1958, Sage and Melsa 1971/

$$E\left\{x(t-\tau_1)\dots x(t-\tau_i)\right\} = \begin{cases} 0 & i \text{ - odd number} \\ \text{sum of expected values of all the possible pairs combination} & i \text{ - even number} \end{cases} \qquad /12/$$

the following formulae for the crosscorelation function can be obtained:

$$\begin{aligned} R_{yx}(\tau) &= E\left\{x(t)\, y(t+\tau)\right\} = E\left\{x(t)\times\right. \\ &\times\Big[\iint\limits_{-\infty}^{+\infty} k_2(\tau_1,\tau_2)\, x(t+\tau-\tau_1)\, x(t+\tau-\tau_2)\, d\tau_1 d\tau_2 + \\ &+ \int\limits_{-\infty}^{+\infty} k_1(\tau_1)\, x(t+\tau-\tau_1)\, d\tau_1 + \Big(k_0 - K\int\limits_{-\infty}^{+\infty} k_2(\tau_1,\tau_1)\, d\tau_1\Big)\Big]\Big\} = \qquad /13/ \\ &= \int\limits_{-\infty}^{+\infty} k_1(\tau_1)\, E\left\{x(t)\, x(t+\tau-\tau_1)\right\} d\tau_1 = \\ &= K\int\limits_{-\infty}^{+\infty} k_1(\tau_1)\,\delta(\tau-\tau_1)\, d\tau_1 = K\cdot k_1(\tau) \end{aligned}$$

and similarly for autocorrelation of the output process

$$\begin{aligned} R_{yy}(\tau) &= E\left\{y(t)\, y(t+\tau)\right\} = \\ &= k_0^2 + K\int\limits_{-\infty}^{+\infty} k_1(\tau_1)\, k_1(\tau_1+\tau)\, d\tau_1 + \qquad /14/ \\ &+ 2K^2 \iint\limits_{-\infty}^{+\infty} k_2(\tau_1,\tau_2)\, k_2(\tau_1+\tau,\tau_2+\tau)\, d\tau_1 d\tau_2 \end{aligned}$$

If operators of the /3/ type were considered apart from Wiener's G-structure providing orthogonality the transformation of a zero mean Gaussian white noise yields the formulae

$$\begin{aligned} R_{yy}(\tau) &= k_0^2 + K\int\limits_{-\infty}^{+\infty} k_1(\tau_1)\, k_1(\tau_1+\tau)\, d\tau_1 + \Big[K\int\limits_{-\infty}^{+\infty} k_2(\tau_1,\tau_1)\, d\tau_1\Big]^2 + \\ &+ 2Kk_0\int\limits_{-\infty}^{+\infty} k_2(\tau_1,\tau_1)\, d\tau_1 + \qquad /15/ \\ &+ 2K^2\iint\limits_{-\infty}^{+\infty} k_2(\tau_1,\tau_2)\, k_2(\tau_1+\tau,\tau_2+\tau)\, d\tau_1 d\tau_2 \end{aligned}$$

and

$$R_{yx}(\tau) = K \cdot k_1(\tau) \qquad /16/$$

It was shown by Diskin and Boneh /1972/ that for the case of physical systems directly subject to continuity principle /e.g. surface runoff process/ one obtains:

$$\int_{-\infty}^{+\infty} k_2(\tau, \tau)\, d\tau = 0 \qquad /17/$$

Thus for n = 2 Wiener's G-operator is equivalent to a homogeneous Volterra operator

$$\int\!\!\int_{-\infty}^{+\infty} k_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2$$

Analysis of Volterra series integral models for a Gaussian Markovian input process

Schetzen /1974/ suggested the method for obtaining a nonlinear optimal model for a Gaussian input in the form of a non-white noise. We shall apply a similar procedure for a nonlinear system fed by a Markovian process with known autocorrelation function:

$$R(\tau) = \sigma^2 e^{-\alpha|\tau|} \qquad /18/$$

and corresponding power density function

$$S(\omega) = \frac{2\sigma^2 \alpha}{\alpha^2 + \omega^2} \qquad /19/$$

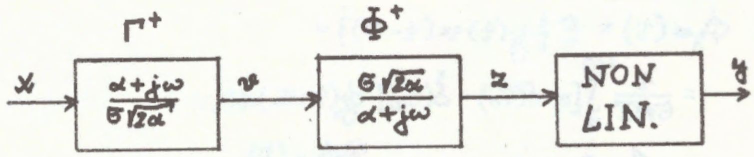
The main point of this method is application of the lemma, which states that a mean square error of a linear system depends only on a signal power spectrum /Deutsch, 1965/. The complete model consists thus of three parts. The first part transforms linearly the Markovian input into a white noise. The second part, also linear transforms the resulting white noise into a process spectrally equivalent to an input Markovian process.

Since the power density function meets the conditions of Paley-Wiener's criterion /Deutsch, 1965/, we can factorize it /Lee, 1960/ so as to obtain

$$S(j\omega) = S^{+}(j\omega)\, S^{-}(j\omega) = \frac{\sigma\sqrt{2\alpha}}{\alpha + j\omega} \cdot \frac{\sigma\sqrt{2\alpha}}{\alpha - j\omega} \qquad /20/$$

After setting a complex variable $s = \sigma + j\omega$ in the place of imaginary $j\omega$ it is easy to see that $S^{+}(s)$ has a pole in the left half plane only and thus can be considered a transfer function of a physically realisable stable linear system.

Due to this approach we can deal with a statistically simple white noise process instead of the original Markovian process. Thus the theory presented in the former section may be applied.



$$S_{xx}(\omega) = \frac{2\alpha\sigma^2}{\alpha^2 + \omega^2} \qquad S_{vv}(\omega) = 1 \qquad S_{zz}(\omega) = \frac{2\alpha\sigma^2}{\alpha^2 + \omega^2}$$

Taking inverse Fourier transforms the following formulae are obtained:

$$\gamma^{+}(t) = \frac{1}{\sigma\sqrt{2\alpha}} \left[\alpha\, \delta(t) + \dot{\delta}(t) \right] \qquad /21/$$

$$\Phi^{+}(t) = \sigma\sqrt{2\alpha}\; e^{-\alpha t} \qquad /22/$$

Now we can make use of the fact that the optimal nonlinear model for the white noise input has kernels described by the relations /10, 11/. According to the structure of fig.1 the conceptual white noise inprocess is not available for measurements. Since the white noise results from a linear transformation of the original Markovian process, it reads

$$v(t) = \int_{0}^{+\infty} \gamma^{+}(\tau)\, x(t-\tau)\, d\tau \qquad /23/$$

Thus we obtain:

$$\Phi_{yv^n}(\tau_1,\dots,\tau_n) = E\{y(t)v(t-\tau_1)\dots v(t-\tau_n)] =$$

$$= \int_{0}^{+\infty}\!\!\int_n \frac{1}{\sigma\sqrt{2\alpha}}\left[\alpha\delta(\sigma_1)-\dot{\delta}(\sigma_1)\right]\dots\frac{1}{\sigma\sqrt{2\alpha}}\left[\alpha\delta(\sigma_m)-\dot{\delta}(\sigma_m)\right]\times \qquad /24/$$

$$\times E\{y(t)x[t-(\tau_1+\sigma_1)]\dots x[t-(\tau_n+\sigma_m)]\,d\sigma_1\dots d\sigma_m$$

For n = 1

$$\Phi_{yv}(\tau) = E\{y(t)v(t-\tau)\} =$$

$$= \frac{1}{\sigma\sqrt{2\alpha}}\int_{0}^{+\infty}\left[\alpha\,\delta(\sigma_1)-\dot{\delta}(\sigma_1)\right]\phi_{yx}(\tau+\sigma_1)\,d\sigma_1 = \qquad /25/$$

$$= \frac{1}{\sigma\sqrt{2\alpha}}\left[\alpha\cdot\phi_{yx}(\tau) - \frac{\partial\phi_{yx}(\tau)}{\partial\tau}\right]$$

For n = 2

$$\Phi_{yv^2}(\tau_1,\tau_2) = \frac{1}{2\sigma\alpha}\left[\alpha^2\phi_{yx^2}(\tau_1,\tau_2) - \alpha\frac{\partial\phi_{yx^2}(\tau_1,\tau_2)}{\partial\tau_1}\right.$$

$$\left. - \alpha\frac{\partial\phi_{yx^2}(\tau_1,\tau_2)}{\partial\tau_2} + \frac{\partial^2\phi_{yx^2}(\tau_1,\tau_2)}{\partial\tau_1\partial\tau_2}\right] \qquad /26/$$

The last formulae enable the determination of kernels of optimal second order model. These kernels form a nonlinear model of two last blocks represented in fig.1. To find the optimal model of the type /6/ it is necessary to modify the kernels obtained according to the formulae /10-11, 24-26/ by a multiple convolution with an impulse response of a block according to the procedure given by Schetzen /1974/.

When transforming a Gaussian Markovian noise in the second order nonlinear model featuring orthogonality properties the following formula for autocorrelation of the output function are obtained:

$$E\{y(t)\,y(t+\tau)\} = F_2(\tau) + 2\sigma^2 \iint_{I_2} h_2(\tau_1,\tau_2)\, e^{-\alpha(\tau_1-\tau_2)}\, d\tau_1 d\tau_2 + \\ -(2h_0+1)\int h_2(\tau_1,\tau_1)\, d\tau_1 \qquad /27/$$

where

$$F_2(\tau) = h_0^2 + 2\sigma^2 \Big\{ \iint_{I_2} h_0 h_2(\tau_1,\tau_2)\, e^{-\alpha(\tau_1+\tau_2)}\, d\tau_1 d\tau_2 + \\ + \iint_{I_4} h_2(\tau_1,\tau_2)\, h_2(\tau_3,\tau_4)\, e^{\alpha(\tau_4-\tau_3+\tau_2-\tau_1)} \prod_{i=1}^{4} d\tau_i + \\ + \Big[\iint_{I_2} h_1(\tau_1)\, h_1(\tau_2)\, e^{\alpha(\tau_1+\tau_2)}\, d\tau_1 d\tau_2\Big]\, e^{-\alpha t} + \\ + \Big[2 \iint_{I_4} h_2(\tau_1,\tau_2)\, h_2(\tau_3,\tau_4)\, e^{\alpha(\tau_4+\tau_3-\tau_2-\tau_1)} \prod_{i=1}^{4} d\tau_i\Big]\, e^{-2\alpha\tau} \Big\} \qquad /28/$$

For the general second order model without the properties of orthogonality /formula 3/ is obtained

$$E\{y(t)\,y(t+\tau)\} = F_2(\tau) \qquad /29/$$

The crosscovariance formula is valid for both approaches

$$E\{x(t)\,y(t+\tau)\} = \sigma^2 \int h_1(\tau_1)\, e^{-\alpha(\tau-\tau_1)}\, d\tau_1 \qquad /30/$$

The above results have academic meaning. In case of development of conceptual nonlinear model of Volterra type /e.g. interpretation of a second order kernel analogical to Nash's structure/ the application of the foregoing formulae may ease, after analytical integration of suggested kernels, the identification of conceptual model parameters.

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Summary

The nonlinear hydrological system is examined. The nonlinearity of the system is modelled by means of a Volterra functional series. This approach is getting recently some interest of scientists modelling the surface runoff systems and the transformation of flow in open channels. The models presented may be applied in water resources systems technique for prediction of inflows to reservoirs, for flood control etc.

The optimal models of this type for white noise input and for Gaussian Markovian input are discussed. Since for practical applications a few first elements of Volterra series give usually a satisfactory approximation, the authors focus their attention on a two elements Volterra model.

The transformation of white noise and of Gaussian Markovian process in the optimal models is examined and the formulae for output autocorrelation and input-output crosscorelation are presented. The method might be useful for identification of conceptual kernels of nonlinear hydrological models.

Резюме

В статье изучается нелинейная гидрологическая система. Нелинейность системы моделируется при помощи ряда вольтерровых функционалов. В последние годы этот подход часто используется при моделировании поверхностного стока и трансформации волны в открытом русле. Представлены модели применимы в системах водного хозяйства - для предсказания притока в резервуар, управления наводнением итд.

Обсуждаются оптимальные модели в условиях неопределенности типа белого шума и марковского нормального процесса. Из-за удовлетворительных результатов вольтерровый ряд приближается двумя первыми слагаемыми. Для оптимальных и общих моделей исследуется трансформация белого шума и марковского процесса. Приводятся функции автокорреляции для выхода и взаимокорреляции - вход-выход. Предлагаемый метод применим в идентификации концептуальных ядер в нелинейных гидрологических моделях.