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A C T A G E O P H Y S I C A P O L O N I C A
Vol. XXXV, no. 4 1987

PROPERTIES OF THE GENERALIZED DOWNSTREAM CHANNEL RESPONSE

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Abstract

The properties of the impulse response for a linearised channel of any shape and any friction law are studied using the cumulants and shape factors of the general response and the amplitude and phase spectra. It is confirmed that even for this very general case the average downstream movement is given exactly by the kinematic approximation. It is shown that for very long waves the attenuation approaches zero whereas for very short waves the amplitude decreases exponentially with distance.

1. THE GENERAL LINEAR RESPONSE

The classical solution for the unsteady movement of a flood wave in a river channel is based on the Saint-Venant (1871) equations which represent a one-dimensional analysis. The equation of continuity is given by

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0, \tag{1}$$

where $A(x, t)$ is the area of flow and $Q(x, t)$ the flow rate at a given cross section. The equation for the conservation of linear momentum is given by

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{\partial y}{\partial x}=g(S_0-S_f), \tag{2}$$

where $v(x, t)$ is the mean velocity of flow, $y(x, t)$ is the depth of flow, S_0 is the bottom slope and S_f is the friction slope which is a function of the channel shape and roughness and of the area of flow. An insight into the nature of the solutions of the full non-linear

St. Venant equations can be obtained by studying analytically the properties of the linearised forms of the equations.

While the earlier studies of the linear channel response were on the basis of a wide rectangular channel with Chezy friction, the authors have recently extended the analysis to cover any shape of channel and any friction law (Dooge, 1980; Dooge et al., 1987). If the linearisation is based on a reference condition of flow Q_0 and area of flow A_0 and neglected of higher order terms then it can be shown (Dooge et al., 1987) that the governing linear equation is

$$(1-F_0^2)\,g\bar{y}_0\frac{\partial^2\psi}{\partial x^2}-2v_0\frac{\partial^2\psi}{\partial x\,\partial t}-\frac{\partial^2\psi}{\partial t^2}=gA_0\left(\frac{\partial S_f}{\partial Q}\frac{\partial\psi}{\partial t}-\frac{\partial S_f}{\partial A}\frac{\partial\psi}{\partial x}\right), \tag{3}$$

where $\psi(x, t)$ is the perturbation in the flow $Q'(x, t)$ or in the area $A'(x, t)$ or in any other variable of interest. F_0, $\bar{y}_0$, and v_0 denote the Froude Number, the hydraulic mean depth and the mean velocity of flow for reference condition, respectively.

For the case of a semiinfinite uniform channel with an impulsive input at the upstream end the Laplace transform of the linear channel response is given by

$$h(x, s)=\exp(exs+\sqrt{c}x-x\sqrt{as^2+bs+c}), \tag{4}$$

where the coefficients are related to the parameters of the channel as follows:

$$a=\frac{1}{g\bar{y}_0(1-F_0^2)^2}, \tag{5a}$$

$$b=\frac{2S_0}{v_0\bar{y}_0}\,\frac{1+(m-1)F_0^2}{(1-F_0^2)^2}, \tag{5b}$$

$$c=m^2\left(\frac{S_0}{\bar{y}_0}\right)^2\frac{1}{(1-F_0^2)^2}, \tag{5c}$$

$$d=\frac{b^2}{4}-ac, \tag{5d}$$

$$e=\left(\frac{1}{g\bar{y}_0}\right)^{0.5}\frac{F_0}{1-F_0^2} \tag{5e}$$

as can be readily verified. In the above equation m denotes the ratio of the kinematic wave speed to the average velocity of flow at the reference condition

$$m=\frac{\left(\dfrac{dQ}{dA}\right)}{\left(\dfrac{Q_0}{A_0}\right)}. \tag{6}$$

The discussion of the properties of the linear channel response in later sections of this paper are based directly on the Laplace transform of this solution as given by equation (4). Inversion of equation (4) to the time domain indicates that the general linear downstream channel response consists of two parts: (a) the discontinuous head of the wave and (b) the more slow-moving body of the wave. The head of this response to a delta function input at the upstream end of channel reach is a delta function of decreasing volume given by:

$$h_1(x,t)=\delta\left(t-\frac{x}{c_1}\right)\exp(-px), \tag{7}$$

where c_1 is the downstream characteristic velocity:

$$c_1=v_0+\sqrt{g\bar{y}_0} \tag{8}$$

and the parameter p characterises the exponential decline in volume and is given by

$$p=\frac{b}{2\sqrt{a}}-\sqrt{c} \tag{9a}$$

which in terms of the hydraulic parameters is

$$p=\frac{S_0}{\bar{y}_0}\frac{1-(m-1)F_0}{(1+F_0)F_0} \tag{9b}$$

and is positive for all cases of stable open channel flow.

The body of the wave is given by

$$h_2(x,t)=\exp(-rt+ox)h\left(\frac{x}{c_1}-\frac{x}{c_2}\right)\frac{I_1\left[2h\sqrt{\left(t-\frac{x}{c_1}\right)\left(t-\frac{x}{c_2}\right)}\right]}{\sqrt{\left(t-\frac{x}{c_1}\right)\left(t-\frac{x}{c_2}\right)}}1\left(t-\frac{x}{c_1}\right), \tag{10}$$

where c_2 is the upstream characteristic velocity

$$c_2=-\frac{1}{\sqrt{a}+e}=v_0-\sqrt{g\bar{y}_0} \tag{11}$$

and the other parameters are given by

$$r=\frac{S_0 v_0}{\bar{y}_0}\frac{1+(m-1)F_0^2}{F_0^2}, \tag{12}$$

$$o=(m-1)\frac{S_0}{\bar{y}_0}, \tag{13}$$

$$h=\frac{S_0 v_0}{\bar{y}_0}\frac{\sqrt{(1-F_0^2)[1-(m-1)^2F_0^2]}}{2F_0^2} \tag{14}$$

while $I_1[\ ]$ is a modified Bessel function of the first kind and $1(\ )$ is a unit step function.

The results in equations (7) to (14) are generalizations (Dooge, 1980; Dooge et al., 1987) of the results previously obtained by Dooge and Harley (1967) for the special case of a wide rectangular channel with Chezy friction.

2. MOMENTS AND CUMULANTS OF GENERAL RESPONSE

The use of moments to characterise a distribution so widely used in statistics was introduced into hydrology by Nash (1959). Since then it has been widely used both in unit hydrograph analysis and in flood routing to study the properties of linear responses and to compare the various models proposed for use in representing the linear channel response or the unit hydrograph. Since the linear channel response is a function of both x and t we can describe it in terms of its moments with respect to x or its moments with respect to t. The moments with respect to time are more convenient for the case of the downstream wave discussed in this paper. In any case where the form of a function is known, the moments about the time origin can be determined as follows

$$U'[h(x,t)] = \int_0^\infty h(x,t)\,t^R dt = (-1)^R \frac{d^R}{ds^R}[h(x,s)]_{s=0}, \tag{15}$$

where U' is the R-th moment of the function $h(x, t)$ about the time origin and $h(x, s)$ is the corresponding Laplace transform.

In establishing theoretical relationships, it is more convenient to replace the moments by the cumulants which are related to them (Kendall and Steward, 1963). While the moments are generated by the Laplace transform as indicated by equation (15) the cumulants are generated by the logarithm of the Laplace transform. Accordingly they are given by

$$k_R[h(x,t)] = (-1)^R \frac{d^R}{ds^R}\{\ln[h(x,s)]\}_{s=0}. \tag{16}$$

The first cumulant is identical with the first moment about the origin; the second cumulant is identical to the second moment about the centre; the third cumulant is identical to the third moment about the centre; but in the case of higher moments and cumulants this identity does not exist.

For the fourth cumulant we have

$$k_4 = U_4 - 3(U_2)^2 \tag{17}$$

which is termed excess kurtosis in statistics. For the fifth cumulant we also have a relatively simple relationship

$$k_5 = U_5 - 10U_3U_2. \tag{18}$$

It is clear from the above that in the case of the linear channel response it is possible to derive the values of the moments or of the cumulants from the solution in the Laplace transform domain given by equation (4) even if the explicit solution in the time domain given by equations (7) and (10) is not available. Substituting from equation (4) in equation

(15) we obtain for the moments

$$U'[h(x,t)]=(-1)^R \frac{d^R}{ds^R}[\exp(exs+\sqrt{c}x-x\sqrt{as^2+bs+c})]_{s=0}. \tag{19}$$

For the case of the cumulants the exponential form of equation (4) gives us a simplified generating function since substituting from equation (4) into equation (16) gives

$$k_R[h(x,t)]=(-1)^R \frac{d^R}{ds^R}[(-\sqrt{as^2+bs+c}+es+\sqrt{c})x]_{s=0}, \tag{20}$$

where the parameters a, b, c, and e have the values given by equations (5).

Recently a general but complicated expression for any order of cumulant of the linear channel response has been derived by Romanowicz et al. (1986). However, only few first cumulants are used in practice. In fact, they are relatively easy to obtain from implicit equation (20).

The first cumulant, which is equal to the first moment about the origin or the lag of the linear channel response, is a most important characteristic for hydrological purposes. Its value may be determine by taking the first derivative in equation (20) and then putting $s=0$. The first cumulant is given by:

$$k_1=\left(\frac{b}{2\sqrt{c}}-e\right)x. \tag{21a}$$

Substituting from equations (5b), (5c), and (5e) gives us

$$k_1=\frac{x}{mv_0}. \tag{21b}$$

The first cumulant (and therefore the first moment) for any length of channel is given by the time taken for a kinematic wave to traverse the length of channel. This simple and exact result indicates that the empirical expression for the movement of the peak of flood wave given by Kleitz (1877) and Seddon (1900)

$$c_k=\frac{dQ}{dA} \tag{22}$$

holds exactly in the linearised case for the average movement of the wave.

In order to obtain the second cumulant it is necessary to evaluate equation (20) for the second derivative. We get

$$k_2=\left(\frac{b^2}{4c}-a\right)\frac{x}{\sqrt{c}}. \tag{23a}$$

Substituting from equations (5a), (5b), and (5c) we obtain

$$k_2=\frac{1}{m}[1-(m-1)^2F_0^2]\left(\frac{y_0}{S_0x}\right)\left(\frac{x}{mv_0}\right)^2 \tag{23b}$$

for the second cumulant (i. e. the second moment about the centre) in terms of the hydraulic characteristics of the channel at the reference conditions.

The third and next cumulants can be obtained in a similar fashion. Evaluating the third, fourth and fifth derivatives at $s=0$ we get the following expressions for respective cumulants:

$$k_3=\frac{3}{m^2}[1-(m-1)^2F_0^2][1+(m-1)F_0^2]\left(\frac{\bar{y}_0}{S_0 x}\right)^2\left(\frac{x}{mv_0}\right)^3, \tag{24a}$$

$$k_4=\frac{15}{m^3}[1-(m-1)^2F_0^2]\left[1-\frac{(m^2-10m+10)F_0^2}{5}+(m-1)^2F_0^4\right]\left(\frac{\bar{y}_0}{S_0 x}\right)^3\left(\frac{x}{mv_0}\right)^4, \tag{25a}$$

$$k_5=\frac{105}{m^4}[1-(m-1)^2F_0^2][1+(m-1)F_0^2]\times$$
$$\times\left[1-\frac{(3m^2-14m+14)F_0^2}{7}+(m-1)^2F_0^4\right]\left(\frac{\bar{y}_0}{S_0 x}\right)^4\left(\frac{x}{mv_0}\right)^5. \tag{26a}$$

Substituting lower order cumulants we get convenient for computation forms:

$$k_3=\frac{3}{m}[1+(m-1)F_0^2]\left(\frac{\bar{y}_0}{S_0 x}\right)k_1 k_2, \tag{24b}$$

$$k_4=\frac{3(1-F_0^2)}{m}\left(\frac{\bar{y}_0}{S_0 x}\right)k_2^2+\frac{4}{3}\frac{k_3^2}{k_2}, \tag{25b}$$

$$k_5=5k_3\left[\frac{7}{9}\left(\frac{k_3}{k_2}\right)^2-3F_0^2\left(\frac{\bar{y}_0}{S_0 x}\right)^2k_1^2\right]. \tag{26b}$$

For any given shape of channel and friction law the cumulants are functions of the time of passage of a kinematic wave through the channel (x/mv_0), the dimensionless length of the channel ($S_0 x/\bar{y}_0$), and the Froude Number for the reference flow condition (F_0).

3. COMPARISON OF SHAPE FACTORS

The lower moments or cumulants have been used in hydrology to characterise the response of catchment components or the total response of the catchment. The first moment or cumulant gives the lag of the hydrologic system and is of great importance in characterising the response. Nash (1959) introduced the idea of using dimensionless moments to describe the shape of the unit hydrograph. He defined the R-th dimensionless moment as

$$m_R=\frac{U_R U_0^{R-1}}{(U_1')^R}, \tag{27}$$

where U_R is the R-th moment about the centre of area, U_0 is the area of the distribution (usually normalised to unity) and U_1' is the first moment about the origin.

Dooge and Harley (1967) adapted this approach by replacing the R-th moment (U_R) by the R-th cumulant (k_R) to form shape factors for normalised linear channel responses defined by:

$$S_R = \frac{k_R}{k_1^R} \tag{28}$$

and applied these shape factors to the linear channel response of a wide rectangular channel with Chezy friction. For the more general case dealt with in the present paper these shape factors can readily be derived from the expressions for the cumulants derived in Section 2. Since the area under the linear channel response is unity, the shape factor S_2 is found by combining equations (23b) and (21b) to obtain:

$$S_2 = \frac{1}{m}[1-(m-1)^2F_0^2]\left(\frac{\bar{y}_0}{S_0 x}\right) \tag{29}$$

and the shape factor S_3 is got by combining equations (21b) and (24a) to obtain:

$$S_3 = \frac{3}{m^2}[1-(m-1)^2F_0^2][1+(m-1)F_0^2]\left(\frac{\bar{y}_0}{S_0 x}\right)^2. \tag{30}$$

According if the general solution for the linear channel response is plotted on a shape factor diagram of S_3 versus S_2 it will be represented by the line:

$$S_3 = 3\,\frac{1+(m-1)F_0^2}{1-(m-1)^2F_0^2}(S_2)^2 \tag{31}$$

for any given value of the parameter m and the Froude Number F_0. Such a shape factor diagram can be used to compare conveniently the general linear channel response for the linearised St. Venant equations with conceptual models proposed for use in flood routing and so evaluate the applicability of the latter models.

The relationship for the higher cumulants can also be readily derived. Thus we have:

$$S_4 = 15\,\frac{1-(m^2-10m+10)\dfrac{F^2}{5}+(m-1)^2F_0^4}{[1-(m-1)^2F_0^2]^2}(S_2)^3 \tag{32}$$

for the fourth order shape factor and

$$S_5 = 105\,\frac{[1+(m-1)F_0^2]\left[1-(3m^2-14m+14)\dfrac{F_0^2}{7}+(m-1)^2F_0^4\right]}{[1-(m-1)^2F_0^2]^3}(S_2)^4 \tag{33}$$

or alternatively:

$$S_5 = \frac{35}{3}\,\frac{1-(3m^2-14m+14)\dfrac{F_0^2}{7}+(m-1)^2F_0^4}{[1-(m-1)^2F_0^2][1+(m-1)F_0^2]}(S_3)^2 \tag{34}$$

for the fifth order shape factor.

An alternative approach to removing the effect of area from moments or cumulants is to use the second moment about the centre (which is the same as the second cumulant) as the basis of dimensionless moments or dimensionless cumulants. This approach was applied to statistical distributions by Pearson (1948) and to flood routing by Strupczewski and Kundzewicz (1980). In this method the R-th order shape can be defined as:

$$f_R = \frac{k_R}{(k_2)^{R/2}} \tag{35}$$

for the case where the area under the function is unity. For the case of the general linear channel response we have for the third order shape factor:

$$f_3 = \frac{3}{\sqrt{m}} \frac{1+(m-1)F_0^2}{[1-(m-1)^2F_0^2]^{0.5}} \left(\frac{\bar{y}_0}{S_0 x}\right)^{0.5} \tag{36}$$

and for the fourth order shape factor:

$$f_4 = \frac{15}{m} \frac{1-(m^2-10m+10)\frac{F_0^2}{5}+(m-1)^2F_0^4}{1-(m-1)^2F_0^2} \left(\frac{\bar{y}_0}{S_0 x}\right). \tag{37}$$

The relationship between the above factors can be written:

$$f_4 = \frac{5}{3} \frac{1-(m^2-10m+10)\frac{F_0^2}{5}+(m-1)^2F_0^4}{[1+(m-1)F_0^2]^2} (f_3)^2 \tag{38}$$

which corresponds to equation (32) in the approach where the first cumulant is used to remove the effect of scale.

The higher cumulants can be similarly dealt with. Thus equations (23b) and (26a) can be combined to give:

$$f_5 = \frac{105}{m^{1.5}} \frac{\left[1-(3m^2-14m+14)\frac{F_0^2}{7}+(m-1)^2F_0^4\right][1+(m-1)F_0^2]}{[1-(m-1)^2F_0^2]^{1.5}} \left(\frac{\bar{y}_0}{S_0 x}\right)^{1.5} \tag{39}$$

for the fifth order shape factor. This can be related to the third order factor by:

$$f_5 = \frac{35}{9} \frac{1-(3m^2-14m+14)\frac{F_0^2}{7}+(m-1)^2F_0^4}{[1+(m-1)F_0^2]^2} (f_3)^3. \tag{40}$$

The above relationship can all be readily evaluated for particular values of the parameter m and the Froude Number.

4. AMPLITUDE AND PHASE SPECTRA

In previous sections the problem of the general linearised solution has been discussed in terms of the impulse response of the linearised St. Venant equations. As an alternative the solution of the latter equation can be described in terms of the frequency response

(e. g. Osiowski, 1972). In the present section the frequency approach is used and expressions derived for the amplitude spectrum and frequency spectrum of the complete linearised St. Venant equations.

In Section 1 the Laplace transform of the impulse response was defined as

$$h(x, s)=\exp(exs+\sqrt{c}x-x\sqrt{as^2+bs+c}),$$

where a, b, c, and e, are parameters which depend on the hydraulic parameters of the channel and the reference flow as indicated in equations (5).

The impulse response (4) describes all transfer properties of the linearised St. Venant equations for any input function and zero initial conditions. Sometimes it is convenient to employ only a part of the function $h(x, s)$ on the imaginary axis $s=i\omega$ i.e. to replace the Laplace transform by the Fourier transform. The function

$$h(x, i\omega)=h(x, s)|_{s=i\omega} \tag{41}$$

is called an amplitude-phase characteristic of the system or a frequency transfer function. The quantities

$$A(x, \omega)=|h(x, i\omega)|, \tag{42a}$$

$$\varphi(x, \omega)=\arg[h(x, i\omega)] \tag{42b}$$

which fulfil the relations

$$A(x, -\omega)=A(x, \omega), \tag{43a}$$

$$\varphi(x, -\omega)=-\varphi(x, \omega) \tag{43b}$$

are called the amplitude characteristic and the phase characteristic, respectively. From equations (42) one can see that

$$h(x, i\omega)=A(x, \omega)\exp[i\varphi(x, \omega)]. \tag{44}$$

So, the amplitude and phase characteristics determine changes in amplitude and phase caused by the system for sinusoidal input function with frequency ω.

The frequency transfer function for the linearised St. Venant equations can be obtained from the Laplace transform (4) by taking only the imaginary part of the complex variable $s=i\omega$ is given by:

$$h(x, i\omega)=\exp(x\sqrt{c}+ie\omega x-x\sqrt{-a\omega^2+c+ib\omega}). \tag{45}$$

In order to determine the amplitude and phase characteristics we must separate the real and imaginary parts of the square root in equation (45). The remaining two terms separate automatically. The calculation are carried out for $\omega \geqslant 0$ only. The frequency characteristics for $\omega<0$ are determined from equations (43).

The complex quantity whose square root is required is

$$(-a\omega^2+c)+ib\omega \tag{46a}$$

which can be expressed in polar form as

$$A+iB=C(\cos\theta+i\sin\theta), \tag{46b}$$

where

$$A=-a\omega^2+C, \tag{47a}$$

$$B=b\omega, \tag{47b}$$

$$C=(A^2+B^2)^{0.5}, \tag{47c}$$

$$\theta=\arctan\frac{B}{A}. \tag{47d}$$

The square root of the expression (46b) is given by:

$$(A+iB)^{0.5}=C^{0.5}\left[\cos\left(k\pi+\frac{\theta}{2}\right)+i\sin\left(k\pi+\frac{\theta}{2}\right)\right] \quad \text{for} \quad k=1,2 \tag{48}$$

and by use of the half-angle relations:

$$(A+iB)^{0.5}=\pm\left(\frac{C}{2}\right)^{0.5}[(1+\cos\theta)^{0.5}+i(1-\cos\theta)^{0.5}]. \tag{49}$$

It is important to note that as ω varies from 0 to $+\infty$ the angle θ defined by equation (47d) varies from 0 to π, $\theta/2$ falls in the first quadrant.

Since

$$\cos\theta=\frac{A}{C} \tag{50}$$

it follows that

$$(A+iB)^{0.5}=\pm\frac{(C+A)^{0.5}+i(C-A)^{0.5}}{\sqrt{2}}, \tag{51}$$

where the positive root represents waves travelling upstream, while the negative root represents waves travelling downstream.

The amplitude can now be written by means of equations (42a), (45), (47), and (51) as

$$A(x,\omega)=\exp\left(\sqrt{c}x-\frac{\{[b^2\omega^2+(-a\omega^2+c)^2]^{0.5}-a\omega^2+c\}^{0.5}x}{\sqrt{2}}\right), \tag{52a}$$

where ω is the frequency and a, b, and c are the parameters defined by equations (5). Similarly the phase can be written by means of equations (42b), (45), (47), and (51) as

$$\varphi(x,\omega)=ex\omega-\frac{x}{\sqrt{2}}\{[b^2\omega^2+(-a\omega^2+c)^2]^{0.5}+a\omega^2-c\}^{0.5}, \tag{53a}$$

where the parameters are the same as in equation (52). Substituting of equations (5) into

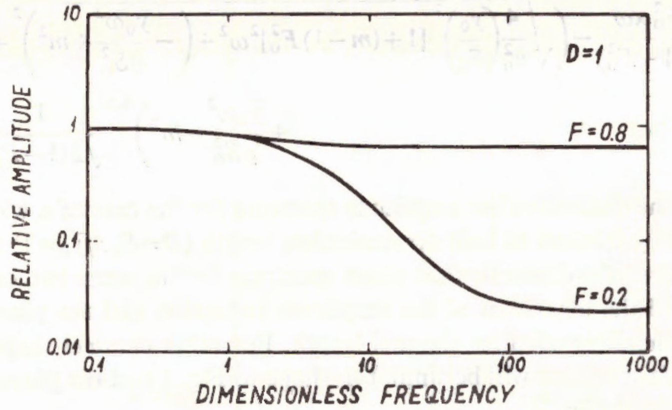

Fig. 1. Amplitude spectrum for unit length

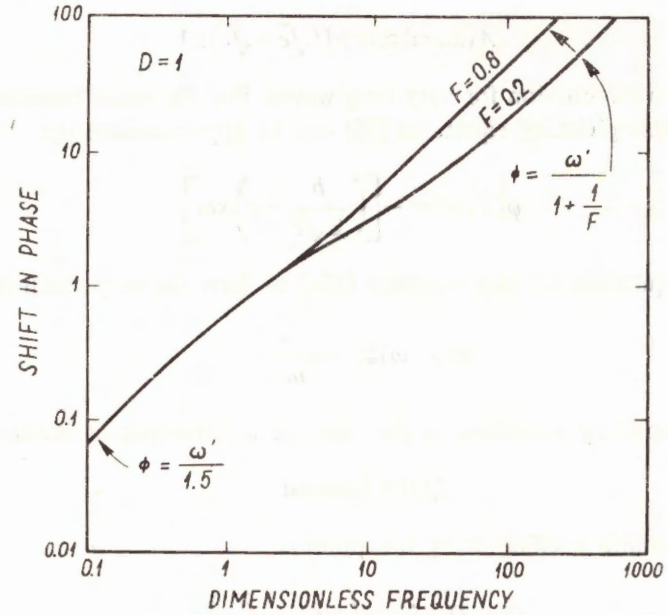

Fig. 2. Phase spectrum for unit length

equations (52a) and (53a) we have

$$A(x,\omega)=\exp\left[\frac{m}{1-F_0^2}\frac{S_0x}{\bar{y}_0}-\left(\sqrt{\frac{4}{v_0^2}\left(\frac{\bar{y}_0}{S_0}\right)^2[1+(m-1)F_0^2]^2\omega^2+\left(-\frac{\bar{y}_0\omega^2}{gS_0^2}+m^2\right)^2}+\right.\right.$$

$$\left.\left.-\frac{\bar{y}_0\omega^2}{gS_0^2}+m^2\right)^{0.5}\frac{1}{\sqrt{2}(1-F_0^2)}\frac{S_0x}{\bar{y}_0}\right] \qquad (52.b)$$

$$\varphi(x,\omega)=\frac{F_0^2 x\omega}{v_0(1-F_0^2)}-\left(\sqrt{\frac{4}{v_0^2}\left(\frac{\bar{y}_0}{S_0}\right)^2[1+(m-1)F_0^2]^2\omega^2+\left(-\frac{\bar{y}_0\omega^2}{gS_0^2}+m^2\right)^2}+\right.$$
$$\left.+\frac{\bar{y}_0\omega^2}{gS_0^2}-m^2\right)^{0.5}\frac{1}{\sqrt{2}(1-F_0^2)}\frac{S_0 x}{\bar{y}_0}. \qquad (53.b)$$

Fig. 1 shows the dimensionless amplitude spectrum for the case of a wide rectangular channel with Chezy friction of unit dimensionless length ($D=S_0x/\bar{y}_0=1$) for $F=0.2$ and $F=0.8$. Fig. 2 shows the dimensionless phase spectrum for the same two cases. For other lengths of channel the logarithm of the amplitude reduction and the phase shift will be proportional to the dimensionless channel length. For other channel shapes and friction law, the amplitude spectrum will be similar in shape to Fig. 1 and the phase spectrum will be similar in shape to Fig. 2.

It is instructive to examine the form of the amplitude and phase spectra for the limiting values of the frequency ω. For very low frequencies i. e. very long waves, the amplitude given by equations (52) can be approximated by

$$A(x,\omega)\cong\exp[(\sqrt{c}-\sqrt{c})x] \qquad (54)$$

so that there is no attenuation for very long waves. For the same condition of very low frequency the phase given by equations (53) can be approximated by:

$$\varphi(x,\omega)=-\left[\left(\frac{b}{2\sqrt{c}}-e\right)x\omega\right]. \qquad (55a)$$

Substituting of equations (5) into equation (55a) we have for very small frequencies

$$\varphi(x,\omega)\cong-\omega\frac{x}{mv_0}. \qquad (55b)$$

The upstream boundary condition in the form of a harmonic oscillation

$$f_u(t)=f_0\cos\omega t \qquad (56a)$$

results in a harmonic oscillation at the point x

$$f(x,t)=f_0\cos\left(\omega t-\omega\frac{x}{mv_0}\right). \qquad (56b)$$

The phase velocity of the above wave

$$c_k=mv_0 \qquad (57)$$

corresponds to the kinematic wave speed.

At the other extreme of very high frequencies, i.e. very short waves, the amplitude approaches the value given by

$$A(x,\omega)=\exp\left[-\left(\frac{b}{2\sqrt{a}}-\sqrt{c}\right)x\right]. \qquad (58a)$$

From equation (9b) we have for very high frequencies

$$A(x,\omega)=\exp\left[-\frac{1-(m-1)F_0}{F_0(1+F_0)}S_0\frac{x}{\bar{y}_0}\right]=\exp(-px). \tag{58b}$$

It will be noted that the amplitude for infinite frequency does not decay to zero thus indicating infinite power. However, the amplitude will decrease to zero if the length of the channel becomes infinite.

The phase for very short waves is found from equations (53)

$$\varphi(x,\omega)\cong-(\sqrt{a}-e)x. \tag{59a}$$

From equation (8) we get that

$$\varphi(x,\omega)\cong-\frac{\omega x}{v_0+\sqrt{g\bar{y}_0}}=-\frac{\omega x}{c_1}. \tag{59b}$$

For the case of a very short wave as the upstream boundary condition

$$f_u(t)=f_0\cos\omega t \tag{60a}$$

the resulting harmonic oscillation at point x takes form

$$f(x,t)=\exp(-px)\cos\left(\omega t-\frac{\omega x}{c_1}\right) \tag{60b}$$

which corresponds to head of the wave (see equation (7)) travelling with the phase velocity $c_1=v_0+\sqrt{g\bar{y}_0}$ and attenuation $\exp(-px)$.

Problem CPBP-03.09 Manuscript received 3 November 1986

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WŁASNOŚCI UOGÓLNIONEJ ODPOWIEDZI IMPULSOWEJ DLA ZLINEARYZOWANYCH RÓWNAŃ RUCHU PRZY GÓRNYM WARUNKU BRZEGOWYM

Streszczenie

Rozważania dotyczą kanału pryzmatycznego o dowolnym kształcie i dowolnego prawa tarcia. W analizie odpowiedzi impulsowej wykorzystano jej kumulanty, współczynniki kształtu i charakterystyki widmowe. Dla dyskutowanego ogólnego przypadku potwierdzono, że średni ruch w dół cieku jest opisany dokładnie za pomocą równania fali kinematycznej. Wykazano, że dla bardzo długich fal tłumienie dąży do zera, a dla fal krótkich zmniejsza się wykładniczo z odległością.