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THE LINEAR DOWNSTREAM RESPONSE OF A GENERALIZED UNIFORM CHANNEL

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Abstract

The linearised solution for the downstream movement of flood wave in a semi-infinite channel is derived for the general case of any shape of channel and any friction law. The resulting impulse response is seen to differ from the classical case of a wide rectangular channel with Chezy friction only in the values of two parameters. These values can be determined for the general case by differentiation of the equation for the friction slope with respect to discharge and to area of flow.

1. LINEARISATION OF THE ST. VENANT EQUATIONS

1.1. Basic equations of flood routing. The movement of flood waves in rivers is studied on the basis of a one-dimensional analysis so that the independent variables are the elapsed time t and the single space dimension x in the direction of flow. The most important problem in flood routing is the downstream problem i.e. the prediction of the flood characteristics at a downstream section on the basis of a knowledge of the flow characteristics at an upstream section and the hydraulic characteristics of the channel between the two sections. Other problems of importance are:

(a) the upstream problem which involves predicting the effect on the upstream channel reach of the changes in the flow conditions at a given section;

(b) the tributary problem which deals with the effect of tributary inflow on conditions in the main channel both upstream and downstream of the point of entry;

(c) the lateral inflow problem in which there is a distributed inflow to the channel reach.

The above classification and description applies only to a tranquil or subcritical flow in which the Froude number is less than one. For rapid or supercritical flow there is no upstream effect. The present paper concentrates on the case of tranquil flow and on the downstream problem.

When only one space dimension is taken into account, the equation of continuity in the absence of lateral inflow is given by:

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0, \tag{1}$$

where $Q(x, t)$ is the discharge, $A(x, t)$ is the cross-sectional area, x is the distance from the upstream boundary and t is the elapsed time.

An equation for the conservation of linear momentum is usually derived on the assumption that:

a) the slope of the channel bed is small and uniform;

b) vertical acceleration can be neglected so that hydrostatic pressure prevails at every point of the cross-section;

c) the frictional resistance is the same as for steady uniform flow;

d) the velocity is uniformly distributed over each cross-section.

For these assumptions we have the equation originally written by Saint-Venant (1871) as:

$$\frac{\partial z}{\partial x}+\frac{v}{g}\frac{\partial v}{\partial x}+\frac{1}{g}\frac{\partial v}{\partial t}+\frac{\tau_0}{\gamma R}=0, \tag{2a}$$

where $z(x, t)$ is the elevation of the water surface above a fixed horizontal datum, $v(x, t)$ is the average velocity in the cross-section, $\tau_0(x, t)$ is the average shear stress along the perimeter of the cross-section, γ is the weight density of the water, and $R(x, t)$ is the hydraulic radius (i.e. the ratio of area to wetted perimeter) of the cross section. This momentum equation is more usually written as (Cunge et al., 1980; Henderson, 1966):

$$\frac{\partial y}{\partial x}+\frac{v}{g}\frac{\partial v}{\partial x}+\frac{1}{g}\frac{\partial v}{\partial t}=S_0-S_f, \tag{2b}$$

where $y(x, t)$ is the depth of flow, $S_0(x)$ is the bottom slope and $S_f(x, t)$ is the friction slope defined by

$$\tau_0=\gamma R S_f, \tag{3}$$

which is the equilibrium condition for steady uniform flow.

The friction slope depends on the type of friction law assumed, the shape and roughness of the cross-section, the flow at the section and the depth of flow. For our purpose it is more convenient to replace the depth of the flow by the area of flow which is a function of it. Accordingly the friction slope can be written in the very general form:

$$S_f=f(A, Q, \text{shape}, \text{roughness}). \tag{4a}$$

For any given shape and roughness of the cross-section and any given friction law—whether laminar, smooth turbulent or rough turbulent (Chezy, Manning or logarithmic) — the friction slope can be expressed as a function of flow (Q) and the area of flow (A). Thus for Chezy friction we have in general that

$$S_f = \frac{Q^2}{C^2 A^2 R(A)}, \tag{4b}$$

in which $R(x, t)$ can be expressed as a function of $A(x, t)$. For the particular case of a wide rectangular channel the friction slope for the case of Chezy friction is given by:

$$S_f = \frac{BQ^2}{C^2 A^3}, \tag{4c}$$

where B is the constant width of the channel and C is the Chezy friction parameter. For the Manning formula and wide rectangular channel the corresponding expression in metric units is

$$S_f = n^2 \frac{B^{4/3} Q^2}{A^{10/3}}, \tag{4d}$$

where n is the Manning friction parameter. The results in the present paper apply to any shape of section and to any friction law.

1.2. Linearisation in discharge and area. The momentum equation given by equation (2b) above is obviously non-linear in character and thus no closed form solution of this non-linear flood routing problem is available. One approach towards the finding of a satisfactory approximate solution is through the linearisation of the non--linear equation. The first attempt at a linearisation of the complete equation seems to be due to Deymié (1935). Further work has been done by Massé (1939), Supino (1950), Lighthill and Whitham (1955), Dooge and Harley (1967), Brutsaert (1973). All of these studies were confined to Chezy friction law and a wide rectangular channel.

Any solution of a linearised problem must of necessity be only an approximation to the solution of the original non-linear equation. The question of how good that approximation is can only be properly evaluated if the linearised solution is compared to the complete non-linear solution for that given problem. Nevertheless, a linearised solution may give insight into the nature of the solution of the complete non-linear problem and may do so to a greater extent than the non-linear solution of a simplified version of the complete equation. However, in the case of unsteady flow in open channels there are some phenomena which cannot be reproduced by a linearised solution. For example, a shock will occur in open channel flow whenever two like characteristics intersect one another. In the case of a linearised equation, the like characteristics are all parallel to one another and therefore can never intersect in order to indicate shock formation. In contrast the non-linear kinematic wave solution will indicate the formation of shock waves which would not occur if the complete equation were used.

Since the continuity equation given by equation (1) is already linear in $Q(x, t)$ and $A(x, t)$, it seems appropriate to adopt discharge and area as the dependent variables and to express the non-linear momentum equation in terms of the same variable. This may be done through the use of the diagnostic equation:

$$Q = vA \tag{5}$$

which by definition connects the discharge $Q(x, t)$, the mean velocity $v(x, t)$ and the area of flow $A(x, t)$. When equation (5) is used to eliminate velocity from equation (2b) we obtain

$$(1-F^2)g\bar{y}\frac{\partial A}{\partial x}+\frac{2Q}{A}\frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial t}=gA(S_0-S_f), \tag{6}$$

$\frac{\partial T}{\partial t} = \phi$

where $\bar{y}(x, t)$ is the hydraulic mean depth defined by

$$\bar{y}(x,t)=\frac{A(x,t)}{T(x,t)} \tag{7a}$$

and $T(x, t)$ represents the width of the channel at the water surface and is defined by

$$T(x,t)=\frac{dA}{dy}. \tag{7b}$$

and $F(x, t)$ is the Froude Number defined by

$$F^2(x,t)=\frac{Q^2T}{gA^3} \tag{8}$$

which is an important parameter of the flow conditions.

Equation (6) can conveniently be linearised by treating the unsteady flow as a perturbation or deviation from a steady flow condition and neglecting second and higher order terms. The form of the linearised equation is particularly simple if the perturbation is taken about an initial condition of steady uniform flow. For the case of initial steady uniform flow we would write the total flow (Q) and the cross-sectional area (A) as

$$Q(x,t)=Q_0+Q'(x,t)+e_Q(x,t), \tag{9a}$$

$$A(x,t)=A_0+A'(x,t)+e_A(x,t), \tag{9b}$$

where Q_0 is the reference condition of steady uniform flow, A_0 is the cross-sectional area corresponding to this flow, Q' and A' are the first-order increments and e_Q and e_A represent the higher order terms (i.e. error of the linear approximation).

When equations (9) are substituted in equation (1) and the higher order terms neglected, we obtain as the continuity equation for the perturbations Q' and A':

$$\frac{\partial Q'}{\partial x}+\frac{\partial A'}{\partial t}=0 \tag{10}$$

which is identical in form to equation (1). When the non-linear terms in equation (6) are expanded in Taylor series around the uniform steady state (Q_0, A_0) for the increments defined by equations (9) and the higher order terms than linear are neglected, we obtain the linearised momentum equation given by:

$$(1-F_0^2)g\bar{y}_0\frac{\partial A'}{\partial x}+2v_0\frac{\partial Q'}{\partial x}+\frac{\partial Q'}{\partial t}=gA_0\left(-\frac{\partial S_f}{\partial Q}Q'-\frac{\partial S_f}{\partial A}A'\right), \tag{11}$$

where the derivatives of the friction slope $S_f(x, t)$ with respect to discharge Q and area A on the right hand side of the equation are evaluated at the reference conditions. The left hand side of equation (11) has the same form as in equation (6) from which it is derived except that the coefficients are frozen at their reference values. Since for the reference condition of uniform steady flow the total derivative in the friction slope must be zero, we can write

$$c_k=-\frac{\dfrac{\partial S_f}{\partial A}}{\dfrac{\partial S_f}{\partial Q}}=\frac{dQ}{dA}, \tag{12}$$

where c_k is the kinematic wave speed (Kleitz, 1877; Lighthill and Whitham, 1955; Seddon, 1900). We may for convenience define m as the ratio of the kinematic wave speed given by equation (12) to the average velocity of flow at the reference condition

$$m=\frac{c_k}{\dfrac{Q_0}{A_0}}. \tag{13}$$

The parameter m is a function of the shape of channel and of area of flow (A). For wide rectangular channels with Chezy friction m is always equal to 3/2 and with Manning friction always equal to 5/3. For shapes of channel other than wide rectangular m will take on different values. This can be illustrated for the case of a channel with a triangular cross-section. For such a triangular flume with Chezy friction, m is equal to 5/4 and with Manning friction to 4/3.

The variation of the friction slope with discharge at the reference condition for all frictional formulas for rough turbulent flow can be taken as

$$\frac{\partial S_f}{\partial Q}=\frac{2S_0}{Q_0}. \tag{14}$$

Using this value the right hand side of equation (11) can be written as

$$\text{R.H.S.}=2gA_0S_0\left(m\frac{A'}{A_0}-\frac{Q'}{Q_0}\right). \tag{15}$$

If we wish to carry out flood routing in terms of the flow, it is convenient to transform the two first order linear equations given by equations (10), (11) in the dependent variables $Q'(x, t)$ and $A'(x, t)$ into a single second order partial differential equation in the single dependent variable $Q'(x, t)$. This may be done by:

a) differentiating equation (10) with respect to x,

b) differentiating equation (11) with respect to t,

c) making the necessary substitutions in order to eliminate the variable $A'(x, t)$.

The resulting equation

$$(1-F_0^2)g\bar{y}_0\frac{\partial^2 Q'}{\partial x^2}-2v_0\frac{\partial^2 Q'}{\partial x\,\partial t}-\frac{\partial^2 Q'}{\partial t^2}=gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial Q'}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial Q'}{\partial t}\right) \tag{16}$$

is a second order differential equation for the perturbation $Q'(x, t)$ from the steady uniform reference flow Q_0. Equation (16) is the generalised form of the equation derived by Deymié (1935) and Massé (1939) for the special case of a wide rectangular channel with Chezy friction.

A single second order differential equation can also be obtained in terms of the perturbation $A'(x, t)$ from the reference area A_0. In this case it is necessary to eliminate $Q'(x, t)$ from equations (10), (11). This is done by differentiating the continuity equation with respect to time and the momentum equation with respect to distance. When this is done the resulting linearised equation is given by

$$(1-F_0^2)g\bar{y}_0\frac{\partial^2 A'}{\partial x^2}-2v_0\frac{\partial^2 A'}{\partial x\,\partial t}-\frac{\partial^2 A'}{\partial t^2}=gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial A'}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial A'}{\partial t}\right), \tag{17}$$

which is seen to be identical in form to equation (16) except for the dependent variable. This indicates that the mathematical problem involved in the solution of the linearised equation is the same in each case but of course the boundary conditions will vary with the choice of dependent variable. This invariance of the form of the basic linearised equation for the two variables was noted by Dooge and Harley (1967) for the special case of a wide rectangular channel with Chezy friction.

1.3. Linear equation for other dependent variables. The result obtained at the end of the last section can be generalised to a large number of choices for the dependent variable. This is most conveniently shown by working in terms of a perturbation potential $U'(x, t)$ which was introduced by Deymié (1935) and developed by Supino (1950). This perturbation potential can be defined as a function whose partial derivative with respect to x gives the perturbation from the reference area:

$$\frac{\partial U'}{\partial x}=A'(x, t) \tag{18a}$$

and whose partial derivative with respect to time gives minus the perturbation from the reference discharge:

$$\frac{\partial U'}{\partial t}=-Q'(x, t). \tag{18b}$$

The definitions given by equations (18) above ensures that the equation of continuity for the perturbations in flow and area given by equations (10) is automatically satisfied. Substitution from equations (18) into equation (11) gives us a second order partial differential equation for the perturbation potential

$$(1-F_0^2)g\bar{y}_0\frac{\partial^2 U'}{\partial x^2}-2v_0\frac{\partial^2 U'}{\partial x\,\partial t}-\frac{\partial^2 U'}{\partial t^2}=gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial U'}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial U'}{\partial t}\right), \tag{19}$$

which again is identical in mathematical form to equation (16). The linearised equation for a number of other choices of dependent variable can be obtained from equation (19) above. Since equation (19) is a linear equation, then any derivative of the perturbation potential $U'(x, t)$ is also a solution of the equation. Hence by equation (18a) the perturbation from the area of flow $A'(x, t)$ is a solution and by equation (18b) the perturbation of the flow $Q'(x, t)$ is also a solution. Similarly, any linear combination of solutions is also a solution of the basic linear equation. Thus the perturbation of the velocity from its reference value is defined by

$$v(x, t)=v_0+v'(x, t)+e_v(x, t), \tag{20a}$$

where $v'(x, t)$ is the first order perturbation and e_v the higher order terms. From equation (5) the first order perturbation can be written as

$$v'(x, t)=\frac{Q(x, t)}{A(x, t)}-\frac{Q_0}{A_0}-e_v(x, t). \tag{20b}$$

Substitution from equations (9) and neglect of higher order terms gives us for $v'(x, t)$, the first order perturbation in the velocity, the relationship

$$\frac{v'}{v_0}=\frac{Q'}{Q_0}-\frac{A'}{A_0} \tag{21}$$

which indicates that the first order perturbation in the velocity $v'(x, t)$ is a linear combination of the perturbation in the flow and the perturbation in the area and not the ratio of these two perturbations. Since both of the latter are solutions of equation (19), then the perturbation in the velocity defined by equation (21) will also be a solution.

Similarly, the perturbation in the surface width of the channel can be shown to be a solution. Since for any given shape, the area and the surface width are both unique functions of depth of flow we can write the surface width as a function of the area of flow

$$T(x, t)=T[A(x, t)], \tag{22}$$

where the function $T[\ \ ]$ depends on the shape of the channel. The first order perturbation in $T(x, t)$ will be given by

$$T'(x, t)=\left[\frac{dT}{dA}\right]_0 A'(x, t) \tag{23}$$

and hence will be a solution of any linear equation of which $A'(x, t)$ is a solution.

In the characteristic form of the equation of unsteady flow in open channels, the celerity defined by

$$c(x, t)=\sqrt{\frac{gA}{T}} \tag{24}$$

is used as a dependent variable. The total perturbation in the celerity is defined by

$$c(x,t)=c_0+c'(x,t)+e_c(x,t). \tag{25}$$

To a first order of approximation the perturbation $c'(x,t)$ is given by

$$\frac{c'}{c_0}=0.5\left(\frac{A'}{A_0}-\frac{T'}{T_0}\right) \tag{26}$$

which indicates that the first order perturbation in the celerity $c'(x,t)$ is a linear combination of the perturbation of the area $A'(x,t)$ and the perturbation in the surface width $T'(x,t)$. Since the latter two variables are solutions of the general equation, then the perturbed celerity $c'(x,t)$ will also be a solution.

As a final example, we could take as the dependent variable the Froude number defined by (8):

$$F^2=\frac{Q^2T}{gA^3}\,.$$

Proceeding in the same way, we could define the deviation from the reference Froude number F by:

$$F(x,t)=F_0+F'(x,t)+e_F(x,t). \tag{27}$$

To a first order of approximation we would have on the basis of equation (8)

$$\frac{F'}{F_0}=\frac{Q'}{Q_0}-1.5\frac{A'}{A_0}+0.5\frac{T'}{T_0}, \tag{28}$$

so the first order perturbation of the Froude number $F'(x,t)$ will also be a solution of the general linearised equation since it is a linear combination of three variables which are themselves all solution of it.

The list of candidate dependent variables is endless and we could include first order approximations to changes in the specific energy or total momentum or similar combinations of the basic variables. In general it can be said that the linearised form of the St. Venant equations is given by

$$(1-F_0^2)\,g\bar{y}_0\frac{\partial^2 f}{\partial x^2}-2v_0\frac{\partial^2 f}{\partial x\,\partial t}-\frac{\partial^2 f}{\partial t^2}=gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial f}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial f}{\partial t}\right), \tag{29}$$

where $f(x,t)$ is any linear function of the perturbation potential defined by equation (18). The choice of the dependent variable is a matter of convenience and it is obvious that the most convenient dependent variable will be the one in which the boundary conditions are directly given. Accordingly, if the upstream hydrograph in a flood routing problem is given in terms of discharge, then discharge is obviously the appropriate dependent variable for use in a linearised solution. If, on the other hand, the upstream hydrograph is given in terms of water level, it would be more convenient to solve the problem in terms of area of flow or surface width or some other variable which is a direct and unique function of water level.

The solution of the general linear equation for unsteady downstream flow in an uniform open channel subject to given initial conditions and appropriate boundary conditions will be discussed in the next section. The complete linear flood routing model is a hyperbolic partial differential equation (i.e. it has two real characteristics) and so two boundary conditions are required. Because the number of conditions specified at the upstream end and the downstream end must be exactly equal to the number of characteristics originating at that boundary, it is essential to examine the directions of these characteristics when it comes to the correct formulation of the flood routing problem.

The directions of the characteristics in the (x, t) plane are given

$$c_{1,2}=\frac{dx}{dt}=\frac{Q_0}{A_0}\pm\sqrt{\frac{gA_0}{T_0}}=v_0\frac{(F_0\pm 1)}{F_0}, \tag{30}$$

which gives the celerity of both the primary and secondary waves in the system. In the case of rapid flow, i.e. Froude number greater than 1, the celerity of both waves is positive and as a result two boundary conditions are required at the upstream end of the reach. In the case of the movement of flood waves, which is the major concern in practice, the flow is tranquil, i.e. the Froude number is less than 1, the celerity of the secondary wave is negative and one boundary condition must be prescribed at each end of the channel reach.

2. SOLUTION OF GENERAL LINEAR EQUATION

2.1. Solution in transform domain. The general linear equation for unsteady flow in open channels can conveniently be solved by the use of the Laplace transform technique. The Laplace transform $L(x, s)$ of the dependent variable $f(x, t)$ is defined as:

$$L(x,s)=\int_0^\infty \exp(-st)f(x,t)\,dt. \tag{31}$$

Since equation (29) represents perturbations from an initial steady condition, the initial value of the dependent variable $f(x, t)$ and its derivatives will all be zero. For this case of zero initial conditions, equation (29) when transformed to the Laplace transform domain becomes

$$(1-F_0^2)g\bar{y}_0\frac{d^2L}{dx^2}-2v_0 s\frac{dL}{dx}-s^2L=gA_0\left(-\frac{\partial S_f}{\partial A}\frac{dL}{dx}+sL\frac{\partial S_f}{\partial Q}\right). \tag{32}$$

The above equation is a second-order homogeneous ordinary differential equation for the Laplace transform $L(x, s)$ as a function of x. The solution can be written in the general form:

$$L(x,s)=A_1(s)\exp[\lambda_1(s)]+A_2(s)\exp[\lambda_2(s)], \tag{33}$$

where λ_1 and λ_2 are the roots of the characteristic equation for equation (32):

$$(1-F_0^2)g\bar{y}_0\lambda^2+\left(-2v_0 s+gA_0\frac{\partial S_f}{\partial A}\right)\lambda-\left(s^2+gA_0\frac{\partial S_f}{\partial Q}s\right)=0 \tag{34}$$

and $A_1(s)$, $A_2(s)$ are functions of s to be determined so the appropriate boundary conditions are satisfied. Since equation (34) is a quadratic equation, the solution can be written directly. Using the standard expression for the solution of the quadratic equation and gathering terms in powers of the transform variable s, we obtain the expression

$$\lambda_{1,2} = es + f \pm \sqrt{as^2 + bs + c}, \tag{35}$$

where the parameters are given in terms of the hydraulic variables by the following relationship:

$$a = \frac{1}{g\bar{y}_0(1-F_0^2)^2}, \tag{36a}$$

$$b = T_0 \frac{Q_0(1-F_0^2)\dfrac{\partial S_f}{\partial Q} - \dfrac{A_0 F_0^2 \partial S_f}{\partial A}}{Q_0(1-F_0^2)^2} = \frac{2S_0}{v_0\bar{y}_0} \frac{1+(m-1)F_0^2}{(1-F_0^2)^2}, \tag{36b}$$

$$c = \frac{T_0^2\left(\dfrac{\partial S_f}{\partial A}\right)^2}{4(1-F_0^2)^2} = m^2\left(\frac{S_0}{\bar{y}_0}\right)^2 \frac{1}{(1-F_0^2)^2}, \tag{36c}$$

$$d = \frac{b^2}{4} - ac, \tag{36d}$$

$$e = \left(\frac{1}{g\bar{y}_0}\right)^{0.5} \frac{F_0}{(1-F_0^2)}, \tag{36e}$$

$$f = -\frac{\dfrac{T_0 \partial S_f}{\partial A}}{2(1-F_0^2)} = m\frac{S_0}{\bar{y}_0(1-F_0^2)}. \tag{36f}$$

Since e and f are positive for $F<1$ and since the Laplace transform $L(x, s)$ must vanish for $s \to \infty$, the positive root in equation (35) must represent waves travelling upstream and the negative root waves travelling downstream.

The functions $A_1(s)$ and $A_2(s)$ can be found from the equations for the upstream $f_u(t)$ and the downstream $f_d(t)$ boundary conditions. At the upstream boundary $(x=0)$ we have

$$f_u(s) = A_1(s) + A_2(s) \tag{37}$$

and at the downstream boundary $(x=L)$ we have

$$f_d(s) = A_1(s)\exp[\lambda_1(s)L] + A_2(s)\exp[\lambda_2(s)L], \tag{38}$$

where $f_u(s)$ and $f_d(s)$ are the Laplace transforms of $f_u(t)$ and $f_d(t)$, respectively. Solving equations (37) and (38) for the unknown functions and substituting these values in equation (33) we get

$$L(x, s) = h_u(x, s) f_u(s) + h_d(x, s) f_d(s), \tag{39}$$

where

$$h_u(x, s) = \exp[(es+f)x] \frac{\mathrm{sh}\sqrt{as^2+bs+c}(L-x)}{\mathrm{sh}\sqrt{as^2+bs+c}\,L} \tag{40}$$

is the system function (i.e. the Laplace transform of the impulse response) for an upstream input and

$$h_d(x,s)=\exp[-(es+i)(L-x)]\frac{\operatorname{sh}\sqrt{as^2+bs+c}\,x}{\operatorname{sh}\sqrt{as^2+bs+c}\,L} \tag{41}$$

is the system function for a downstream input.

In the present paper we will concentrate on the downstream wave which is of primary importance in flood routing. To filter out the upstream wave we can set $L\to\infty$ in equation (39) and deal with the limiting case of downstream flow for semi-infinite reach (i.e. the case where the downstream control is so distant from the section of interest that the downstream boundary condition has no influence). In such a case the upstream transfer function is given by:

$$h_u(x,s)=\exp(exs+fx-x\sqrt{as^2+bs+c}) \tag{42}$$

and the downstream transfer function

$$h_d(x,s)=0. \tag{43}$$

2.2. Solution in the time domain. While, as we shall see later, the Laplace transform of the solution gives a good deal of information about the properties of the general solution for the downstream channel response, it is desirable to obtain an explicit formulation of the solution in the time domain. This may be done by using the standard transform pair given by Doetsch (1961) which gives for the expression in the transform domain

$$\exp(-x\sqrt{as^2+bs+c})-\exp\left(\frac{-bx}{2\sqrt{a}}-\sqrt{a}\,xs\right) \tag{44a}$$

the corresponding function in the time domain

$$\sqrt{\frac{d}{a}}\,x\exp\left(\frac{-bt}{2a}\right)\frac{I_1\left[\frac{\sqrt{d}}{a}\sqrt{t^2-ax^2}\right]}{\sqrt{t^2-ax^2}}\,1(t-\sqrt{a}\,x), \tag{44b}$$

where $I_1[\ \ ]$ is a modified Bessel function of the first kind and $1(\ \ )$ is a unit step function.

By adopting this standard transform pair it is possible to invert equation (42) to the time domain where the solution is found to have two distinct parts so that we can write

$$h_u(x,t)=h_u^1(x,t)+h_u^2(x,t). \tag{45}$$

The first part of the solution which may be termed the head of the wave is given by

$$h_u^1(x,t)=\delta[t-(\sqrt{a}-e)x]\exp\left[-\left(\frac{b}{2\sqrt{a}}-f\right)x\right] \tag{46}$$

and the second part of the solution which may be termed the body of the wave given by:

$$h_u^2(x,t)=\exp(fx)\sqrt{\frac{d}{a}}\,x\exp\left(\frac{b}{2\sqrt{a}}(t+ex)\right)\frac{I_1\left[\frac{\sqrt{d}}{a}\sqrt{(t+ex)^2-ax^2}\right]}{\sqrt{(t+ex)^2-ax^2}}\,1[t-(\sqrt{a}-e)x]. \tag{47}$$

It must be stressed that above solution will be the same for any choice of dependent variable. As indicated in the Station 1.3, the dependent variable $f(x, t)$ may be either the perturbation potential, or any linear function of it.

The head of the wave as given by equation (46) clearly consists of a delta function which travels downstream and whose volume declines exponentially. Equation (46) can be more conveniently written as

$$h_u^1(x,t)=\delta\left(t-\frac{x}{c_1}\right)\exp(-px), \tag{48}$$

where the celerity of the head of the wave (c_1) is given by:

$$c_1=\frac{1}{\sqrt{a}-e}=v_0+\sqrt{g\bar{y}_0} \tag{49}$$

and is clearly seen to be the dynamic speed. The parameter p characterising the rate of the attenuation of the wave is given by:

$$p=\frac{b}{2\sqrt{a}}-f=\frac{S_0}{\bar{y}_0}\,\frac{1-(m-1)F_0}{(1+F_0)F_0}. \tag{50}$$

Equations (49), (50) are the generalised form of the equations for a wide rectangular channel with Chezy friction given by Dooge and Harley (1967).

Since the solution applies only to tranquil flow i.e. for Froude Numbers less than one, the value of the parameter p will always be positive. Accordingly the head of the wave as represented by equation (48) will consist of a delta function which travels downstream at the dynamic wave speed and whose volume is exponentially decreasing. When this part of the linear channel response is convoluted with a given input it produces a contribution to the downstream outflow which will take form of a dynamic wave moving with a celerity given by equation (49), of the same shape as the input but with its volume (and all of the ordinates) being reduced exponentially.

The remaining and major part of the solution to the linear channel response is the body of the wave as given by equation (47). This equation for the body of the wave can be written as

$$h_u^2(x,t)=\exp(-rt+ox)\,h\left(\frac{x}{c_1}-\frac{x}{c_2}\right)\frac{I_1\left[2h\sqrt{\left(t-\frac{x}{c_1}\right)\left(t-\frac{x}{c_2}\right)}\right]}{\sqrt{\left(t-\frac{x}{c_1}\right)\left(t-\frac{x}{c_2}\right)}}\,1\left(t-\frac{x}{c_1}\right), \tag{51}$$

where the parameter c_2 is the upstream characteristic and is given by

$$c_2=-\frac{1}{(\sqrt{a}+e)}=v_0-\sqrt{g\bar{y}_0}. \tag{52}$$

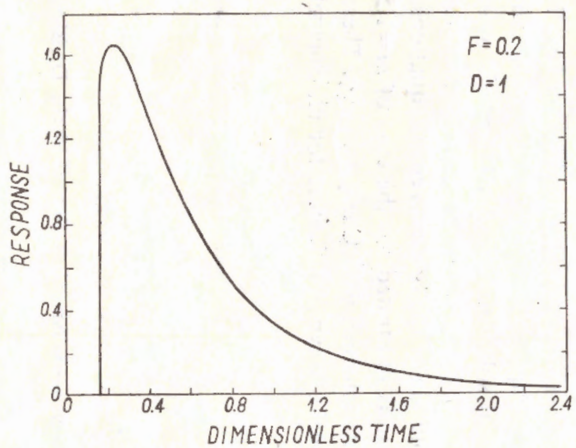

Fig. 1. Body of the wave for short channel $F=0.2$

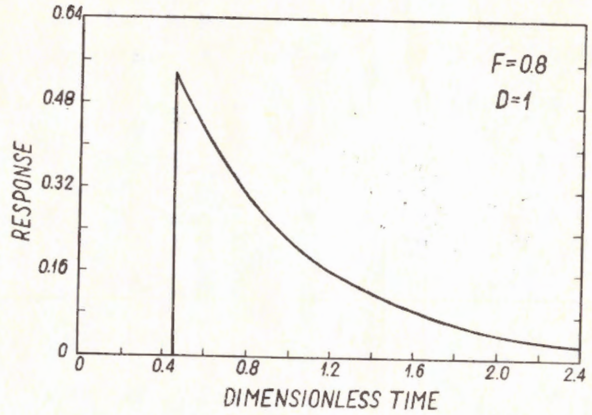

Fig. 2. Body of the wave for short channel $F=0.8$

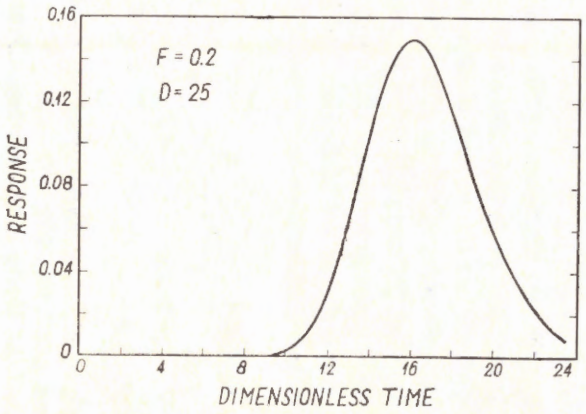

Fig. 3. Body of the wave for long channel $F=0.2$

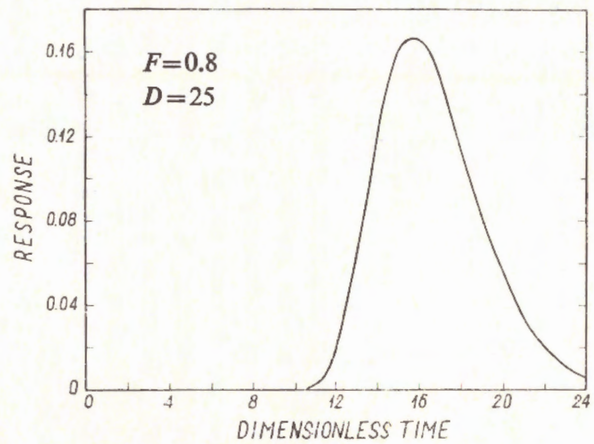

Fig. 4. Body of the wave for long channel $F=0.8$

The remaining parameters (r, o, h) are given by:

$$r=\frac{b}{2a}=\frac{S_0 v_0}{\bar{y}_0}\frac{1+(m-1)F_0^2}{F_0^2}, \tag{53a}$$

$$o=f-\frac{be}{2a}=(m-1)\frac{S_0}{\bar{y}_0}, \tag{53b}$$

$$h=\frac{\sqrt{d}}{2a}=\frac{S_0 v_0}{\bar{y}_0}\frac{\sqrt{(1-F_0^2)[1-(m-1)^2F_0^2]}}{2F_0^2}. \tag{53c}$$

The above expression are generalised forms of those obtained by Dooge and Harley (1967) for a wide rectangular channel with Chezy friction.

The shape of the body of the wave for any given shape of cross-section and any given friction law depends on the dimensionless length of the channel ($D=S_0 L/\bar{y}_0$) and on the Froude number of the reference flow (F_0). Figs 1, 2, 3 and 4 show the variation in shape for the case of a wide rectangular channel with Chezy friction. Figs 1 and 2 show the dimensionless shape of a relatively short channel ($S_0 L/\bar{y}_0=1$) for $F=0.2$ and 0.8, respectively. Figs 3 and 4 show the corresponding shapes for a relatively long channel ($S_0 L/\bar{y}_0=25$).

3. CONCLUSIONS

A linearised solution for the downstream movement of a flood wave in a semi-infinite channel is derived for the general case of any shape of channel and any friction law. It gives insight into the nature of the solution of the complete non-linear problem and will do so more effectively than the non-linear analytical solution based on kinematic wave theory which has the serious disadvantage of predicting the formation of shock waves which do not occur in nature or in a numerical solution of the full St. Venant equations. The shape of the complete linear channel response is illustrated for the case of $m=1.5$ (which corresponds to the case of a wide rectangular channel with Chezy friction). These hydrograph shapes for both short and long channel are typical of these for all channel shapes and friction laws encountered in practice.

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ODPOWIEDŹ IMPULSOWA DLA KANAŁU PRYZMATYCZNEGO PRZY GÓRNYM WARUNKU BRZEGOWYM

Streszczenie

Wyznaczono rozwiązanie liniowe ruchu fali powodziowej dla półnieskończonego kanału pryzmatycznego o dowolnym kształcie i dowolnego prawa tarcia. Otrzymana odpowiedź impulsowa różni się od odpowiedzi impulsowej dla przypadku klasycznego (szerokie koryto prostokątne i prawo tarcia Chezy'ego) jedynie wartościami dwóch parametrów. Wartości tych parametrów można otrzymać różniczkując równanie opisujące prawo tarcia względem przepływu i powierzchni przepływu.