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# On backwater effects in linear diffusion flood routing

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ABSTRACT The effect of the downstream boundary condition on the flow in a channel reach as predicted by the linear diffusion analogy model is analysed by both Laplace transformation and by modal analysis. In each case the solution takes the form of an infinite series. It is shown that the series based on Laplace transformation is highly convergent for small times and the series based on modal analysis is highly convergent for medium and longer times.

*Sur les effets de remous dans le calcul de propagation des crues par la diffusion linéaire*
RESUME L'effet des conditions aux limites aval sur le débit d'un bief de rivière, tel q'il est prévu par le modèle analogique de diffusion linéaire, est analysé ici par la transformation de Laplace et aussi par l'analyse modale. Dans ces deux cas, la solution prend la forme d'une série infinie. On montre aussi que la série basée sur la transformation de Laplace converge rapidement pour des temps courts et que la série basée sur l'analyse modale converge rapidement pour des temps moyens et longs.

## INTRODUCTION

The linear parabolic model of unsteady river flow (diffusion analogy) has the advantage of being a simple means of flood routing. The analysis of the order of magnitude of elements of the St Venant equations of open channel flow (Henderson, 1966; Kuchment, 1972) indicates that the inertia terms may in many cases be neglected. Different derivations of the linear diffusion analogy models appear in hydrological literature. Some of them are presented as hydrodynamic models obtained by simplification of the complete model (neglect or approximation of several terms), with in some cases linearization, while others are presented as conceptual models. It is common to both types of approaches that only an upstream boundary condition is used, i.e. the assumption of a semi-infinite reach is made.

This assumption of a semi-infinite reach allows a simple form of solution to be obtained, but a disadvantage of the approach is the limitation to the case when the backwater effects are negligible. There is actually no theoretical reason why the backwater effect should not be included in the linear diffusion analogy model, so that both a downstream and an upstream boundary condition could be included together with a finite river reach rather than a semi-infinite reach. At the cost of introducing more complicated mathematics than in the semi-infinite case, the solution of the finite reach problem can significantly increase the range of validity of the linear diffusion analogy model. One can use the linear framework for the simulation of the system reactions to sudden changes occurring at the lower boundary of the reach (e.g. dam break, obstruction of flow, change of discharge policy at the regulation structure or flood from a tributary). The semi-infinite model obviously fails in all these situations.

## MATHEMATICAL FORMULATION OF THE PROBLEM

Let us assume that the channel reach is of finite length L and that the activating inputs are physically located at both boundaries to allow for the possibility that the backwater effect cannot be neglected. Assume that the system be described by the linear convective-diffusion equation

$$\frac{\partial Q}{\partial t} + c\,\frac{\partial Q}{\partial x} = D\,\frac{\partial^2 Q}{\partial x^2} \tag{1}$$

where $Q(x, t)$ is the flow rate, c is the advective velocity, and D is the coefficient of hydraulic diffusivity. These parameters can be related to the hydraulic properties and the reference flow condition for a uniform channel. Several approaches are possible, all based on both linearization and simplification of the hydrodynamic equation of the open channel flow (Forcheimer, 1930; Hayami, 1951; Lighthill & Whitham, 1955; Dooge & Harley, 1967; Dooge & Napiórkowski, 1982).

The initial condition is given by

$$Q(x, 0) = Q_o(x) \qquad 0 < x < L \tag{2}$$

and the boundary conditions by

$$Q(0, t) = Q_u(t) \tag{3}$$

$$Q(L, t) = Q_d(t) \tag{4}$$

If the flow is initially steady and uniform along the whole reach, the above formulation is valid not only for flow rate perturbations from a reference value but also for instantaneous flow rate values.

The essential problem involved in the solution of equation (1) subject to equations (2), (3), (4) is to find the flow rates in an intermediate cross section. This can be accomplished analytically in at least two ways: (a) using the Laplace transform method; (b) using modal analysis (separation of variables).

The theoretical principles of these two methods can be found in the extensive mathematical literature (the Laplace transform method in Doetsch, 1961; or Osiowski, 1972; and the modal analysis method in Carrier & Pearson, 1976).

## USE OF DIMENSIONLESS VARIABLES

It is convenient to analyse the problem and to evaluate the result in terms of dimensionless independent variables. The flow at any point and any time can be written as

$$Q(x, t) = f(x, t, L, c, D, Q_o, Q_u, Q_d) \tag{5}$$

If we define a dimensionless flow rate as

$$\underline{Q}(x, t) = Q(x, t)/Q_o' \tag{6}$$

where $Q_o'$ is either the steady uniform reference discharge or some convenient constant reference discharge and the boundary conditions as

$$\underline{Q}_u(t) = Q_u(t)/Q_o' \tag{7}$$

and

$$\underline{Q}_d(t) = Q_d(t)/Q_o' \tag{8}$$

then the basic equation can be written as

$$\underline{Q}(x, t) = f(x, y, L, c, D, \underline{Q}_o, \underline{Q}_u, \underline{Q}_d) \tag{9}$$

Since the five remaining dimensional variables in equation (9) are all kinematic (i.e. have dimensions only of length and time) they can be reduced to three dimensionless variables.

If we wish to study in particular the effect of the distance (L) between the upstream and downstream boundaries on the flow at an intermediate fixed point, x, then it is convenient to include L in only one of three dimensionless variables so that its effect can be isolated. When this is done we get

$$\underline{Q}(x, t) = f(cx/D, c^2t/D, cL/D, \underline{Q}_o, \underline{Q}_u, Q_d) \tag{10}$$

In effect, we have a dimensionless distance given by

$$\underline{x} = cx/D \tag{11}$$

and a dimensionless time given by

$$\underline{t} = c^2t/D \tag{12}$$

and a dimensionless length of channel given by

$$\underline{L} = cL/D \tag{13}$$

Equation (1) now becomes

$$\frac{\partial Q(\underline{x},\ \underline{t})}{\partial \underline{t}} + \frac{\partial Q(\underline{x},\ \underline{t})}{\partial \underline{x}} = \frac{\partial^2 Q(\underline{x},\ \underline{t})}{\partial \underline{x}^2} \tag{14}$$

which is completely dimensionless.

This equation can be reduced to a pure diffusion form, that is

$$\frac{\partial^2 Q'}{\partial \underline{x}^2} = \frac{\partial Q'}{\partial \underline{t}} \tag{15}$$

by using the auxiliary transformation

$$Q'(\underline{x},\ \underline{t}) = \exp(-\underline{x}/2 + \underline{t}/4)\ \underline{Q}(\underline{x},\ t) \tag{16}$$

The transformation used changes the initial condition to

$$Q'(\underline{x},\ 0) = \exp(-\underline{x}/2)\underline{Q}_o(x) = q_o(\underline{x}) \tag{17}$$

and the boundary condition to

$$Q'(0,\ \underline{t}) = \exp(\underline{t}/4)\ \underline{Q}_u(\underline{t}) = Q'_u(\underline{t}) \tag{18}$$

$$Q'(\underline{L},\ \underline{t}) = \exp(-\underline{L}/2 + \underline{t}/4)\ \underline{Q}_d(\underline{t}) = Q'_d(\underline{t}) \tag{19}$$

The equation to be solved is now in canonical form as equation (15).

## LAPLACE TRANSFORM METHOD

The formulation in the Laplace transform domain of the problem given by equation (15), the initial condition (17) and the boundary conditions (18, 19) reads

$$\frac{d^2 q}{d\underline{x}^2} - sq = q_o(\underline{x}) \tag{20}$$

where $q(\underline{x},\ s)$ is the Laplace transform of the function $Q'(\underline{x},\ \underline{t})$.

The general solution for equation (20) is the sum of the general solution of the corresponding homogeneous equation and a particular solution of the inhomogeneous equation. The general solution of the homogeneous equation is given by

$$q_h(\underline{x},\ s) = A(s)\ \exp(-\sqrt{s}\ \underline{x}) + B(s)\ \exp(\sqrt{s}\ \underline{x}) \tag{21}$$

The solution of equation (20) for the case of $q_o(\underline{x}) = \delta(\underline{x})$ is given by the Green's function (Osiowski, 1972)

$$K(x) = \mathrm{sh}(\sqrt{s}\ \underline{x})/\sqrt{s} \tag{22}$$

where sh(·) is the function hyperbolic sine. The general solution of the inhomogeneous equation is

$$\begin{aligned} q(\underline{x},\ s) &= A(s)\exp(-\sqrt{s}\ \underline{x}) + B(s)\exp(\sqrt{s}\ \underline{x}) \\ &\quad + \int_0^{\underline{x}} \mathrm{sh}[\sqrt{s}\ (\underline{x} - a)]\ q_o(a)\,da/\sqrt{s} \end{aligned} \tag{23}$$

The functions A(s) and B(s) can be found from the equations for the upstream and downstream boundary conditions.

At the upstream boundary ($\underline{x} = 0$) we have

$$q_u(s) = A(s) + B(s) \tag{24}$$

and at the downstream boundary ($\underline{x} = \underline{L}$) we have

$$q_d(s) = A(s)\exp(-\sqrt{s}\,\underline{L}) + B(s)\exp(\sqrt{s}\,\underline{L}) + \int_0^{\underline{L}} \mathrm{sh}[\sqrt{s}\,(\underline{L} - a)]q_o(a)\,da/\sqrt{s} \tag{25}$$

where $q_u(s)$ and $q_d(s)$ are the Laplace transforms of $Q_u'(\underline{t})$ and $Q_d'(\underline{t})$ respectively. Solving equations (24, 25) for the unknown functions in equation (23) we obtain

$$A(s) = \frac{q_u(s)\exp(\sqrt{s}\,\underline{L}) - q_d(s) + \int_0^{\underline{L}} \mathrm{sh}[\sqrt{s}\,(\underline{L} - a)]q_o(a)\,da/\sqrt{s}}{\exp(\sqrt{s}\,\underline{L}) - \exp(-\sqrt{s}\,\underline{L})} \tag{26}$$

$$B(s) = \frac{q_d(s) - q_u(s)\exp(-\sqrt{s}\,\underline{L}) - \int_0^{\underline{L}} \mathrm{sh}[\sqrt{s}\,(\underline{L} - a)]q_o(a)\,da/\sqrt{s}}{\exp(\sqrt{s}\,\underline{L}) - \exp(-\sqrt{s}\,\underline{L})} \tag{27}$$

and substituting these values in equation (23) we get

$$q(\underline{x}, s) = q_u(s)\frac{\mathrm{sh}[\sqrt{s}\,(\underline{L} - \underline{x})]}{\mathrm{sh}\sqrt{s}\,\underline{L}} + q_d(s)\frac{\mathrm{sh}\sqrt{s}\,\underline{x}}{\mathrm{sh}\sqrt{s}\,\underline{L}} - \frac{\mathrm{sh}\sqrt{s}\,\underline{x}}{\sqrt{s}\,\mathrm{sh}\sqrt{s}\,\underline{L}}\int_0^{\underline{L}} \mathrm{sh}[\sqrt{s}\,(\underline{L} - a)]q_o(a)\,da + \int_0^{\underline{x}} \mathrm{sh}[\sqrt{s}\,(\underline{x} - a)]q_o(a)\,da/\sqrt{s} \tag{28}$$

which gives in the Laplace transform domain the relationship between the flow at an intermediate point and the given flows at the upstream and downstream boundaries. Assume, that zero initial condition holds along the whole reach considered, that is $q_o(\underline{x}) = 0$ for $0 < \underline{x} < \underline{L}$. Then the system response to excitation at both terminating cross sections reads in the Laplace transform domain

$$q(\underline{x}, s) = h_u'(\underline{x}, s).q_u(s) + h_d'(\underline{x}, s).q_d(s) \tag{29}$$

where

$$h_u'(\underline{x}, s) = \mathrm{sh}[\sqrt{s}\,(\underline{L} - \underline{x})]/\mathrm{sh}(\sqrt{s}\,\underline{L}) \tag{30}$$

is the system function (i.e. the Laplace transform of the impulse response) for an upstream input and

$$h_d'(\underline{x}, s) = \mathrm{sh}(\sqrt{s}\,\underline{x})/\mathrm{sh}(\sqrt{s}\,\underline{L}) \tag{31}$$

is the system function for a downstream input.

The inverse Laplace transform of these complicated equations can be obtained in series form (Doetsch, 1961). We can write equation (30) as

$$h'_u(\underline{x},\ s) = \{\exp(-\sqrt{s}\ x) - \exp[-\sqrt{s}(2\underline{L} - \underline{x})]\}[1 - \exp(-\sqrt{s}2\underline{L})]^{-1} \tag{32a}$$

and expand the denominator to give

$$h'_u(\underline{x},\ s) = [\exp(-\sqrt{s}\underline{x}) - \exp[-\sqrt{s}(2\underline{L} - \underline{x})]\Sigma_{n=0}^{\infty}\exp(-\sqrt{s}\ 2n\underline{L}) \tag{32b}$$

Separating the two parts of denominator gives

$$h'_u(\underline{x},\ s) = \Sigma_{n=0}^{\infty}\exp[-\sqrt{s}(2n\underline{L} + \underline{x})] - \Sigma_{n=0}^{\infty}\exp[-\sqrt{s}(2n\underline{L} + 2\underline{L} - \underline{x})] \tag{32c}$$

and simplifying the second term we can write

$$h'_u(\underline{x},\ s) = \Sigma_{n=0}^{\infty}\exp[-\sqrt{s}(2n\underline{L} + \underline{x})] - \Sigma_{n=1}^{\infty}\exp[-\sqrt{s}(2n\underline{L} - \underline{x})] \tag{32d}$$

which can be combined into a single series

$$h'_u(\underline{x},\ s) = \Sigma_{-\infty}^{+\infty}\exp[-\sqrt{s}(2n\underline{L} + \underline{x})] \tag{32e}$$

For $0 < \underline{x} < \underline{L}$, the following transform pair is applicable

$$F(s) = \exp(-a\sqrt{s}); \quad f(t) = a\exp(-a^2/4t)/(2\sqrt{\pi}t^{3/2}); \quad a > 0$$

because both $2n\underline{L} + \underline{x} > 0$ and $2n\underline{L} - \underline{x} > 0$. Therefore we have after interchanging the sum and the inverse Laplace operator the inverse transform of function (30) in the form

$$h'_u(\underline{x},\ \underline{t}) = L^{-1}\{h'_u(\underline{x},\ s)\} = \Sigma_{-\infty}^{+\infty}\frac{2n\underline{L} + \underline{x}}{2\sqrt{\pi}\underline{t}^{3/2}}\exp[\frac{-(2n\underline{L} + \underline{x})^2}{4\underline{t}}] \tag{33}$$

A similar development in series of equation (31) leads to

$$h'_d(\underline{x},\ \underline{t}) = L^{-1}\{h'_d(\underline{x},\ s)\} = \Sigma_{-\infty}^{+\infty}\frac{2n\underline{L} + \underline{L} - \underline{x}}{2\sqrt{\pi}\underline{t}^{3/2}}\exp[\frac{-(2n\underline{L} + \underline{L} - \underline{x})}{4\underline{t}}] \tag{34}$$

The original function $Q'(\underline{x},\ \underline{t})$ in the time domain is determined from the corresponding boundary conditions through the relationship

$$Q'(\underline{x},\ \underline{t}) = Q'_u(\underline{t}) * h'_u(\underline{x},\ \underline{t}) + Q'_d(\underline{t}) * h'_d(\underline{x},\ \underline{t}) \tag{35}$$

Since the multiplication of the functions in equation (29) becomes convolution on inversion to the time domain, returning to dimensionless variables of the convective diffusion analogy of equation (14) one obtains

$$\begin{aligned} \underline{Q}(\underline{x},\ \underline{t}) &= \exp(\underline{x}/2 - \underline{t}/4)\ \{[\exp(\underline{t}/4)\ \underline{Q}_u(\underline{t})] * h'_u(\underline{x},\ \underline{t}) \\ &\quad + [\exp(-\underline{L}/2 + \underline{t}/4)\ \underline{Q}_d(\underline{t})] * h'_d(\underline{x},\ \underline{t})\} \\ &= \underline{Q}_u(\underline{t}) * h_u(\underline{x},\ \underline{t}) + \underline{Q}_d(\underline{t}) * h_d(\underline{x},\ \underline{t}) \end{aligned} \tag{36}$$

where

$$h_u(\underline{x},\ t) = \exp(\underline{x}/2 - \underline{t}/4)\ h'_u(\underline{x},\ t) \tag{37}$$

and

$$h_d(\underline{x},\ \underline{t}) = \exp[-(\underline{L} - \underline{x})/2 - \underline{t}/4]\ h'_d(\underline{x},\ \underline{t}) \tag{38}$$

are, respectively, the impulse responses in the time domain for the upstream and downstream boundary conditions.

## MODAL ANALYSIS

Another powerful method for the solution of the linear diffusion analogy model (15) for a finite reach is modal analysis, which enables the partial differential equation to be transformed to a set of ordinary differential equations by means of the method of eigenfunction expansions (Carrier & Pearson, 1976).

The solution of the problem can be found by separating the variables and writing

$$Q'(\underline{x},\ \underline{t}) = \alpha(\underline{t})\ \beta(\underline{x}) \tag{39}$$

Substitution from equation (39) into the differential equation (15) gives us the double relationship

$$\frac{1}{\alpha(\underline{t})}\ \frac{d\alpha}{d\underline{t}} = \frac{1}{\beta(\underline{x})}\ \frac{d\beta}{d\underline{x}} = -\lambda \tag{40}$$

The two members of equation (40) each depend on a single independent variable $\underline{x}$ or $\underline{t}$ and are linked through the constant $-\lambda$.

In effect the separation of the variable leads to the method of eigenfunction expansion and the solution is sought in the form of the following series

$$Q'(\underline{x},\ \underline{t}) = \Sigma_n \alpha_n(\underline{t}) \beta_n(\underline{x}) \tag{41}$$

where the set of functions $\beta_n(\underline{x})$ is the set of eigenfunctions associated with the related homogeneous problem (i.e. with both boundary conditions equal to zero), and the set of functions $\alpha_n(\underline{t})$ are Fourier coefficients of $Q'(\underline{x},\ \underline{t})$ relative to the system $\{\beta_n(\underline{x})\}$.

One can see from equation (40) that the method of eigenfunction expansion leads to the eigenvalue problem

$$\begin{aligned} &\frac{d^2}{d\underline{x}^2}\ \beta(\underline{x}) + \lambda\beta(\underline{x}) = 0 \\ &\beta(0) = 0, \qquad \beta(\underline{L}) = 0 \end{aligned} \qquad 0 < \underline{x} < \underline{L} \tag{42}$$

for which the eigenvalues are

$$\lambda_n = (n\pi/\underline{L})^2 \tag{43}$$

and the corresponding eigenfunctions are

$$\beta_n(\underline{x}) = \sin(n\pi\underline{x}/\underline{L}) \qquad n = 1, 2, \ldots \tag{44}$$

For these eigenfunctions the normalizing constants are given by

$$\int_o^{\underline{L}} \beta_n^2(\underline{x})\ d\underline{x} = \underline{L}/2 \tag{45}$$

Because of the orthogonality of the eigenfunctions, that is

$$\int_o^{\underline{L}} \beta_i(\underline{x})\ \beta_j(\underline{x})\ d\underline{x} = \begin{cases} \underline{L}/2 & i = j \\ 0 & i \neq j \end{cases} \tag{46}$$

the Fourier coefficients are related to the solution by the formula

$$\begin{aligned} \alpha_n(\underline{t}) &= \int_o^{\underline{L}} Q'(\underline{x},\ \underline{t})\ \beta_n(\underline{x})\ d\underline{x}/(\underline{L}/2) \\ &= \int_o^{\underline{L}} Q'(\underline{x},\ \underline{t})\ \sin(n\pi\underline{x}/\underline{L})\ d\underline{x}/(\underline{L}/2) \end{aligned} \tag{47}$$

By integrating the righthand side of equation (47) twice by parts and making use of the original equation (15) one gets

$$\begin{aligned} \alpha_n(t) &= 2[Q_u'(\underline{t}) - (-1)^n\ Q_d'(\underline{t})]/(n\pi) \\ &\quad - \frac{2\underline{L}}{n^2\pi^2} \int_o^{\underline{L}} \frac{\partial Q'(\underline{x},\ \underline{t})}{\partial \underline{t}} \sin\ (\frac{n\pi\underline{x}}{\underline{L}})\ d\underline{x} \end{aligned} \tag{48}$$

The time derivative of equation (47) reads

$$\frac{d\alpha_n(\underline{t})}{d\underline{t}} = \frac{2}{\underline{L}} \int_o^{\underline{L}} \frac{\partial Q'(\underline{x},\ \underline{t})}{\partial \underline{t}} \sin\ (\frac{n\pi\underline{x}}{\underline{L}})\ d\underline{x} \tag{49}$$

After simple manipulation (addition of equation (49) to the multiplied equation (48)) to eliminate the term containing the integral, one can obtain an infinite set of ordinary differential equations

$$\frac{d\alpha_n(\underline{t})}{dt} + (\frac{n\pi}{\underline{L}})^2\ \alpha_n(\underline{t}) = \frac{2n\pi}{\underline{L}}\ [Q_u'(\underline{t}) - (-1)^n Q_d'(\underline{t})] \tag{50}$$

which is equivalent to the original differential equation (15). The initial conditions corresponding to the respective terms of the Fourier development of the initial condition are

$$\alpha_n(0) = \frac{2}{\underline{L}} \int_o^{\underline{L}} q_o(\underline{x})\ \sin(n\pi\underline{x}/\underline{L}) d\underline{x} \tag{51}$$

For convenience we can introduce an additional variable

$$G(t) = Q_u'(\underline{t}) - (-1)^n Q_d'(t) \tag{52}$$

which reflects the impact of both boundary conditions.

The Laplace transform solution of equation (50) with the condition (51) in the complex domain reads

$$\alpha_n(s) = [2n\pi/\underline{L}\ g(s) + \alpha_n(0)]/(s + n^2\pi^2/\underline{L}^2) \tag{53}$$

where g(s) is the Laplace transform of G($\underline{t}$).

If zero initial conditions hold along the whole reach ($q_o(\underline{x}) = 0$ that is $\alpha_n(0) = 0$ for all n), the Laplace transform solution is simplified to the product form:

$$\alpha_n(s) = \frac{2n\pi/\underline{L}}{s + n^2\pi^2/\underline{L}^2} \cdot g(s) \tag{54}$$

The inverse Laplace transform yields the following convolution type response

$$\alpha_n(\underline{t}) = 2n\frac{\pi}{\underline{L}^2} \exp(-n^2\pi^2\underline{t}/\underline{L}^2) * G(\underline{t}) \tag{55}$$

The general solution given by equation (41) can now be written in terms of the upstream and downstream boundary conditions as

$$Q'(\underline{x}, \underline{t}) = h_u''(\underline{x}, \underline{t}) * Q_u'(\underline{t}) + h_d''(\underline{x}, \underline{t}) * Q_d'(\underline{t}) \tag{56}$$

and by comparison of equations (44, 52, 55) the impulse response for an upstream disturbance can be written as

$$h_u''(\underline{x}, \underline{t}) = 2\pi/\underline{L}^2 \ \Sigma_{n=1}^{\infty} n \cdot \sin(n\pi\underline{x}/\underline{L}) \exp(-n^2\pi^2\underline{t}/\underline{L}^2) \tag{57}$$

and the impulse response for a disturbance at the downstream boundary as

$$h_d''(\underline{x}, \underline{t}) = -2\pi/\underline{L}^2 \ \Sigma_{n=1}^{\infty} (-1)^m \ n \cdot \sin(n\pi\underline{x}/\underline{L}) \exp(-n^2\pi^2\underline{t}/\underline{L}^2) \tag{58}$$

Returning to the original dimensionless variable of equation (15) we get

$$Q(\underline{x}, \underline{t}) = h_u(\underline{x}, \underline{t}) * \underline{Q}_u(\underline{t}) + h_d(\underline{x}, \underline{t}) * \underline{Q}_d(\underline{t}) \tag{59}$$

where

$$h_u(\underline{x}, \underline{t}) = \exp(\underline{x}/2 - \underline{t}/4) \ h_u''(\underline{x}, \underline{t}) \tag{60}$$

$$h_d(\underline{x}, \underline{t}) = \exp[-(\underline{L} - \underline{x})/2 - \underline{t}/4 \ h_d''(\underline{x}, \underline{t}) \tag{61}$$

Equations (60) and (61) give the impulse responses for disturbances at the upstream and downstream boundary respectively.

It remains to confirm that solution given by equations (57) and (58) satisfy boundary conditions as well as the differential equation. The eigenfunction expansion postulated by equation (41) relates to the corresponding homogeneous problem and hence applies to the open interval defined by

$$0 < \underline{x} < \underline{L} \tag{62}$$

and the validity of the solution at $x = 0$ and $x = L$ must be separately verified. For the impulse response function given by equation (57) it is necessary to show that

$$h''_u(x, t) \to \delta(t) \text{ as } x \to 0 \tag{63}$$

and that

$$h''_u(x, t) \to 0 \text{ as } x \to L \tag{64}$$

The crucial test therefore is to verify that the integral of the impulse response function

$$I_u = \int_o^\infty h''_u(x, t)dt \tag{65}$$

has the value of unity for $x = 0$ and zero for $x = L$.

Since the impulse response function for an upstream disturbance is given by equation (57) the integral required is given by

$$I_u = 2 \sum_{n=1}^\infty \sin(n\pi x/L)/(n\pi) \tag{66}$$

so that the problem reduces to an evaluation of the sum of an infinite series. This can be readily obtained by taking the Fourier expansion of

$$f(x) = x \text{ for } 0 < x < 2L \tag{67}$$

by writing

$$f(x) = a_o/2 + \sum_{n=1}^\infty a_n \cos(n\pi x/L) + \sum_{=1}^\infty b_n \sin(n\pi x/L) \tag{68}$$

The Fourier coefficients in equation (68) are easily determined since

$$a_o = 1/L \int_o^{2L} x \, dx = 2L \tag{69a}$$

$$a_n = 1/L \int_o^{2L} x \cos(n\pi x/L)dx = 0 \tag{69b}$$

$$b_n = 1/L \int_o^{2L} x \sin(n\pi x/L)dx = 2L/(n\pi) \tag{69c}$$

Substitution of the coefficients from equations (69) into equation (68) gives

$$x = L - 2L \sum_{n=1}^\infty \sin(n\pi x/L)/(n\pi) \tag{70}$$

so that the sum of the series in equation (66) and hence the value of the integral in equation (65) is given by

$$I_u = (L - x)/L \tag{71}$$

It follows at once that the integral has the value unity for $x = 0$ and the value zero for $x = L$.

A similar line of reasoning based on the Fourier expansion of x in the interval

$$-\underline{L} < \underline{x} < \underline{L} \tag{72}$$

leads to the result that

$$I_d = \int_0^\infty h''_d(\underline{x}, \underline{t})d\underline{t} \tag{73}$$

is given by the series

$$I_d = -2 \sum_{n=1}^{\infty} (-1)^n \sin(n\pi\underline{x}/\underline{L})/(n\pi) \tag{74}$$

and that the sum of the series is given by

$$I_d = \underline{x}/\underline{L} \tag{75}$$

which has the required values of zero at $\underline{x} = 0$ and unity at $\underline{x} = \underline{L}$.

## NUMBER OF TERMS REQUIRED FOR COMPUTATION

The two alternative approaches, by means of the Laplace and the modal analysis techniques, to flood routing in a finite reach result in two equivalent pairs of expressions for the upstream and downstream transfer functions: equations (37) and (38) for the Laplace transform approach, and equations (60) and (61) for the modal analysis approach. It is instructive to ask the question: how many terms in the series are required in each case for practical calculations?

For small values of $\underline{t}$ or large values of $\underline{L}$ the Laplace transform method is superior because then the factors $\exp[-(2n\underline{L} + \underline{x})^2/4\underline{t}]$ are all small and only a few terms of the series in equation (37) and (38) are needed. For the limiting case of $\underline{L} \to \infty$, i.e. a semi-infinite channel, all terms are zero except the first. The convergence for a finite reach is illustrated by evaluation of particular terms of series (33) for the case of $\underline{x} = 5$ and $\underline{L} = 10$ given in Table 1. It can be seen from Table 1 that the series based on Laplace transform derivation is highly convergent for values $\underline{t}$ less than 10, is slowly convergent for $\underline{t} = 100$, and is oscillatory for $\underline{t} = 1000$.

For the particular case of $\underline{x} = \underline{L}/2$ the convergence will be the same for the series reflecting the response to the downstream boundary condition since for this value equation (34) is identical to equation (33). The values given by equations (37) and (38) for

**Table 1** Evaluation of the first seven terms of series (33) for values of $\underline{x} = 5$, $\underline{L} = 10$

| n \ $\underline{t}$ | 1 | 10 | 100 | 1000 |
|---|---|---|---|---|
| 0 | $0.272 \times 10^{-2}$ | $0.239 \times 10^{-1}$ | $0.133 \times 10^{-2}$ | $0.443 \times 10^{-4}$ |
| −1 | $-0.158 \times 10^{-23}$ | $-0.483 \times 10^{-3}$ | $-0.241 \times 10^{-2}$ | $-0.126 \times 10^{-3}$ |
| 1 | (0) | $0.365 \times 10^{-7}$ | $0.148 \times 10^{-2}$ | $0.191 \times 10^{-3}$ |
| −2 | (0) | $-0.156 \times 10^{-13}$ | $-0.462 \times 10^{-3}$ | $-0.230 \times 10^{-3}$ |
| 2 | (0) | $0.414 \times 10^{-22}$ | $0.804 \times 10^{-4}$ | $0.242 \times 10^{-3}$ |
| −3 | (0) | $-0.703 \times 10^{-33}$ | $-0.806 \times 10^{-5}$ | $-0.230 \times 10^{-3}$ |
| 3 | (0) | (0) | $0.474 \times 10^{-6}$ | $0.202 \times 10^{-3}$ |

**Table 2** Evaluation of the first 10 terms of series (57) for values of $\underline{x} = 5$, $\underline{L} = 10$

| n \ $\underline{t}$ | 1 | 10 | 100 | 1000 |
|---|---|---|---|---|
| 1 | $0.569 \times 10^{-1}$ | $0.234 \times 10^{-1}$ | $0.325 \times 10^{-5}$ | (0) |
| 2 | $0.595 \times 10^{-8}$ | $0.171 \times 10^{-9}$ | $0.632 \times 10^{-25}$ | (0) |
| 3 | $-0.775 \times 10^{-1}$ | $-0.262 \times 10^{-4}$ | (0) | (0) |
| 4 | $-0.728 \times 10^{-8}$ | $-0.489 \times 10^{-14}$ | (0) | (0) |
| 5 | $0.266 \times 10^{-1}$ | $0.604 \times 10^{-11}$ | (0) | (0) |
| 6 | $0.303 \times 10^{-8}$ | $0.393 \times 10^{-22}$ | (0) | (0) |
| 7 | $-0.349 \times 10^{-2}$ | $-0.437 \times 10^{-21}$ | (0) | (0) |
| 8 | $-0.255 \times 10^{-9}$ | $-0.522 \times 10^{-34}$ | (0) | (0) |
| 9 | $0.191 \times 10^{-3}$ | $0.108 \times 10^{-34}$ | (0) | (0) |
| 10 | $0.122 \times 10^{-10}$ | (0) | (0) | (0) |

the original dimensionless variables will not be equal but since $\underline{x} = \underline{L}/2$ we have

$$h_d(\underline{L}/2,\ \underline{t}) = \exp(-\underline{L}/2)\ h_u(\underline{L}/2,\ \underline{t})$$

and the rate of convergence will remain the same in the two cases.

For large time periods or short lengths of channel, the modal analysis solution is preferable due to the heavily damped nature of the exponential terms. The first few elements of the series are significantly greater than the others. This is illustrated in Table 2 for the same values of $\underline{x} = 5$ and $\underline{L} = 10$. It can be seen from Table 2 that the series based on modal analysis is highly convergent for values of $\underline{t} \geq 10$. For a value of $\underline{t} = 1$, the modal analysis series is only slowly convergent but the sum of the 10 terms shown is equal to single term approximation of the Laplace transform series.

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