(7) Napiórkowski

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Section 1.2

THE USE OF VOLTERRA SERIES IN THE MODELLING OF FLOW IN AN OPEN CHANNEL

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1. Alternative descriptions of open channel flow.

In recent years, an interest has arisen in studying the relationship between the continuum mechanics approach to flow problems based on an internal viewpoint and the systems approach based on an external viewpoint[1]. Comparisons have nearly always been confined to the linearised versions of the two types of model. In the present paper a linkage is obtained between a simplified non-linear version of the St. Venant equations and the linear and quadratic terms of the alternative Volterra series representation.

The Volterra series representation is a generalisation of the transformation model of a linear time-invariant system represented by :

$$y(t) = \int_0^t h(\tau)\, u(t-\tau)\, d\tau \qquad (1)$$

where u (t) is the input to the system, y (t) is the output from the system, and h (t) is the impulse response of the system. For non-linear systems this external description is generalised to:

$$y(t) = \sum_{m=1}^{\infty} \int_0^t \dots \int_0^t h_m(\tau_1, \dots, \tau_m) \prod_{i=1}^{m} u(t-\tau_i)\, d\tau_i \qquad (2)$$

where h_n ($\tau_1, \tau_2, \dots, \tau_m$) is the n dimensional kernel of the n - th term of the Volterra series. The numerical determination of h_n from input and output data is theoretically feasible but in practice becomes quite difficult for kernels of higher than second order. The problem tackled in the present paper is the prediction of the kernels of the Volterra series for a physically based mathematical model.

1. Dooge, James C.I. (1969). "Alternative approaches to flow problems" General Lecture to XV111 Congress of IAHR, Cagliari, September, 1979.

The internal description of one-dimensional unsteady flow in an open channel is based on the principle of conservation of mass and the principle of conservation of linear momentum in the direction of motion. The equation of continuity is given by :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{3}$$

where Q (x,t) is the discharge and A (x,t) is the cross-sectional area of flow. The dynamic St. Venant equation based on momentum for a uniform channel is given by :

$$\frac{\partial y}{\partial x} + \frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{g}\frac{\partial u}{\partial t} = S_o - S_f \tag{4}$$

where y (x,t) is the depth of flow, u (x,t) is the velocity of flow, S_o is the constant bottom slope (conventionally taken as positive when downward) and S_f is the friction slope which is only equal to the bottom slope for steady uniform flow.

The formulation of the problem is completed by specifying the shape of the prismatic channel, the friction law used to estimate S_f, together with the initial and boundary conditions. The shape may be conveniently defined by relating the area of flow, which appears in equation (3), to the depth of the flow, which appears in equation (4), as follows :

$$A(x,t) = f_A\,[y(x,t)] \tag{5}$$

The friction law connects the friction slope to the parameter of flow and may be written in some general form such as :

$$S_f = f_F\,[Q, y, \text{shape}, \text{roughness}] \tag{6a}$$

In practice the most widely used friction law is that of Manning :

$$S_f = n^2 \frac{Q^2}{A^2 R^{4/3}} \tag{6b}$$

where the area A (y) and the hydraulic radius R (y) depend only on the shape of the channel and on the depth of flow y (x,t), and n is a roughness parameter.

2. Simplification of basic equations.

In order to compare the internal description given by equations (3), (4), (5) and (6) and the external description given by equation (2) without becoming drowned in complex algebraic expressions, the internal description is replaced by a simpler but still non-linear model and the Volterra series is restricted to the first two i.e. the linear and quadratic terms. However, the approach used is equally valid for the full St. Venant equations and for any order of Volterra kernel.

The first simplification is to transform equations (3) and (4) from partial to ordinary differential equations i.e. to replace the physically based distributive model by a physically based lumped model of the channel reach being studied. The equation of continuity given by equation (3) above can readily be integrated along the reach to give :

$$Q_1(t) - Q_2(t) = \frac{dS}{dt} \tag{7}$$

where Q_1 (t) is the upstream inflow, Q_2 (t) is the downstream outflow, and S (t) is the total storage in the reach. For the low values of the Froude number usually encountered in lowland rivers and canals ($0.1 < F < 0.3$), the dynamic equation given by equation (4) can be approximated very closely by the convective-diffusive form :

$$\frac{\partial y}{\partial x} = S_o - S_f(x,t) \tag{8}$$

The latter equation when integrated along the reach can be expressed in the lumped form :

$$y_2(t) - y_1(t) = S_o \cdot L - \int_1^2 S_f(x,t)\,dx \qquad (9)$$

where the friction slope S_f will vary in space and time because of its dependence on the parameters of flow through equation (6).

The dynamic equation can, however, be expressed in completely lumped form by the use of a further assumption. If it is assumed that (over the length of reach concerned) the water level varies linearly throughout the reach, then the term on the left hand side of equation (8) will be a function of time only, and the same must be true of the friction slope S_f on the right hand side of the same equation. Accordingly equation (9) now becomes :

$$y_2(t) - y_1(t) = S_o \cdot L - S_f(t) \cdot L \qquad (10)$$

It is convenient to represent this friction slope which varies only in time, as a ratio of the bottom slope of the channel by defining the function :

$$b(t) = \frac{S_f(t)}{S_o} \qquad (11)$$

This function will have a value of 1 for steady uniform flow, a constant value other than 1 for steady non-uniform flow, and will vary with time but not with distance in the case of unsteady flow. The function b (t) defined by equation (11) is obviously related to the looped rating curve which is characteristic of unsteady flow.

It remains to see if the combination of equations (5) to (11) is sufficient to solve the flood routing problem in which either Q_1 (t) or y_1 (t) is given and wither Q_2 (t) or y_2 (t) is to be predicted. The lumped equation of continuity is given by :

$$\frac{dS}{dt} = Q_1(t) - Q_2(t) \qquad (7)$$

and the lumped dynamic equation by :

$$y_2(t) - y_1(t) = S_o \cdot L\,[1 - b(t)] \qquad (12)$$

For the flood routing problem these two equations contain five unknown functions of time and hence three further equations linking these unknown functions are required. Two of these are provided by writing equation (11) for each end of the reach and using the assumed friction law and the known shape of cross section to write :

$$b(t) = f_b\,(Q_1, y_1) \qquad (13)$$

$$b(t) = f_b\,(Q_2, y_2) \qquad (14)$$

and thus providing a set of four equations in five unknowns.

The remaining equation required is provided by relating the storage in the reach which appears in equation (7) to the upstream and downstream depths at the same instant of time. By definition we have :

$$S(t) = \int_1^2 A(x,t)\,dx \tag{15}$$

and we have already assumed that at any instant the depth varies linearly from y_1 at the upstream end to y_2 at the downstream end. We can use the latter assumption to rewrite equation (15) as :

$$S(t) = \frac{L}{y_2 - y_1} \int_{y_1}^{y_2} A(y)\,dy \tag{16}$$

Since the variation of area with depth is known this can be integrated for any length of channel reach. If the variation of area with depth is smooth, then it can be shown that when the water surface is linear the storage is given by :

$$S = L \sum_{r=0}^{\infty} \left(\frac{d^r A}{dy^r}\right)_{y_0} \frac{1}{(r+1)!} \sum_{i=0}^{i=r} (y_2 - y_0)^{r-i} (y_1 - y_0)^{i} \tag{17}$$

which provides the fifth equation required.

The simplified model examined in this paper is thus defined by equations (7), (12), (13), (14) and (17). It remains to show how the kernels in the Volterra series representation of equation (2) can be derived from these five equations for a given shape of channel and for a given friction law. The approach used is to examine the first order, second order, and higher order perturbations in the five dependent variables due to the perturbation of the system from a steady state trajectory.[1,2]

1. Findeisen W., Szymanowski J., and Wierzbicki A. (1977). Theory and computational methods of optimization. (In Polish). WNT Warsaw. 1977.
2. Napiorkowski, Jaroslaw J. (1978). Identification of conceptual reservoir model described by Volterra series. (In Polish). D. Sc. Thesis. Institute of Geophysics, Polish Academy of Sciences.

3. Determination of first order kernel.

If the initial condition is taken as that of a steady uniform flow (Q_0) in the channel reach then the upstream inflow (Q_1) can be written as :

$$Q_1(t) = Q_0 + \Delta Q_1(t) \tag{18}$$

where the perturbation in inflow is a given function of time. Since the kernel of the first term in the Volterra series represents only linear effects we are interested only in the first order variations in the five dependent variables.

$$\Delta Q_2(t) = \delta Q_2(t) + \delta^2 Q_2(t) + \ldots \tag{19}$$

$$\Delta S(t) = \delta S(t) + \delta^2 S(t) + \ldots \tag{20}$$

$$\Delta b(t) = \delta b(t) + \delta^2 b(t) + \ldots \tag{21}$$

$$\Delta y_1(t) = \delta y_1(t) + \delta^2 y_1(t) + \ldots \tag{22}$$

$$\Delta y_2(t) = \delta y_2(t) + \delta^2 y_2(t) + \ldots \tag{23}$$

Substitution from equations (18) to (23) in the five model equations (7), (12), (13), (14), and (17) and neglect of the perturbations higher than the first in equations (19) to (23) gives us a set of five linear equations which constitute the linearised version of the physically based model being studied. Thus the continuity equation represented by equation (7) becomes :

$$\frac{d}{dt}\left[\delta S(t)\right] = \Delta Q_1(t) - \delta Q_2(t) \tag{24}$$

and the dynamic equation represented by equation (12) becomes :

$$\delta y_2(t) - \delta y_1(t) = - S_o \cdot L \cdot \delta b(t) \qquad (25)$$

Since the perturbations are taken around the initial uniform flow Q_0, the equations (13), (14) and (17) can be expanded around this condition and the first order approximations written as :

$$\delta b(t) = \frac{\partial f_b}{\partial Q} \cdot \Delta Q_1(t) + \frac{\partial f_b}{\partial y} \delta y_1(t) \qquad (26)$$

$$\delta b(t) = \frac{\partial f_b}{\partial Q} \cdot \delta Q_2(t) + \frac{\partial f_b}{\partial y} \delta y_2(t) \qquad (27)$$

$$\delta S(t) = \frac{\partial f_s}{\partial y} \delta y_1(t) + \frac{\partial f_s}{\partial y} \delta y_2(t) \qquad (28)$$

with all the partial derivatives evaluated at the initial condition about which the perturbation takes place.

In order to connect the internal linear description given by equations (24) to (28) with the kernel of the first term of the Volterra series, it is necessary to reduce the five equations to a relationship between the first order variation in the output (δQ_2) and the total variation in the input (ΔQ_1). By subtracting equation (27) from equation (26) and substituting the result in equation (25) we obtain :

$$\frac{\partial f_b}{\partial Q} \left[\Delta Q_1(t) - \delta Q_2(t) \right] = - \frac{\partial f_b}{\partial y} \cdot S_o \cdot L \cdot \delta b(t) \qquad (29)$$

which does not contain the variation in the upstream and downstream depths. By substituting from equations (26) and (27) for these variations in equation (28) we obtain :

$$\frac{\partial f_b}{\partial y} \delta S(t) = - \frac{\partial f_s}{\partial y} \frac{\partial f_b}{\partial Q} \left[\Delta Q_1(t) + \delta Q_2(t) \right] + 2 \frac{\partial f_s}{\partial y} \delta b(t) \qquad (30)$$

The next step is to eliminate the slope variation function $\delta b(t)$ from equations (29) and (30). When this is done we obtain an expression for the first order variation in storage as function of the total variation in inflow and the first order variation in outflow in the following form :

$$\delta S(t) = - \left(\frac{\frac{\partial f_s}{\partial y} \cdot \frac{\partial f_b}{\partial Q}}{\frac{\partial f_b}{\partial y}} \right) \left[\left(1 + \frac{2}{\frac{\partial f_b}{\partial y} S_o L} \right) \Delta Q_1 + \left(1 - \frac{2}{\frac{\partial f_b}{\partial y} S_o L} \right) \delta Q_2 \right] \qquad (31)$$

Equation (31) obviously corresponds to the linearised Muskingum assumption. This correspondence is dealt with in more detail elsewhere.[1]

The simplified form of the dynamic equation given by equation (31) can for convenience be written as :

$$\delta S(t) = C_1 \Delta Q_1(t) + C_2 \delta Q_2(t) \qquad (32)$$

1. Dooge James C.I., Strupczewski W.G. and Napiorkowski Jaroslaw J. (1980). Hydrodynamic derivation of storage parameters of the Muskingrem model. Submitted to Journal of Hydrology. November, 1980.

where C_1 and C_2 are constants depending on the hydraulic properties of the channel and defined by :

$$C_1 = - \frac{\frac{\partial f_s}{\partial y} \cdot \frac{\partial f_b}{\partial Q}}{\frac{\partial f_b}{\partial y}} \left(1 + \frac{2}{\frac{\partial f_b}{\partial y} S_o L} \right) \tag{33}$$

$$C_2 = - \frac{\frac{\partial f_s}{\partial y} \frac{\partial f_b}{\partial Q}}{\frac{\partial f_b}{\partial y}} \left(1 - \frac{2}{\frac{\partial f_b}{\partial y} \cdot S_o L} \right) \tag{34}$$

An equation linking the input and output only can now be obtained by eliminating the variation in channel storage between the continuity equation :

$$\frac{d}{dt} \left[\delta S(t) \right] = \Delta Q_1(t) - \delta Q_2(t) \tag{24}$$

and the dynamic equation in the form of equation (32) above. This elimination gives us :

$$\begin{aligned} \delta Q_2(t) + C_2 \cdot \frac{d}{dt} \left[\delta Q_2(t) \right] &= \\ = \Delta Q_1(t) - C_1 \cdot \frac{d}{dt} \left[\Delta Q_1(t) \right] \end{aligned} \tag{35}$$

as the input-output equation we have been seeking. Since equation (35) is linear with constant coefficients the solution is given by :

$$\delta Q_2(t) = h_1(t) * \Delta Q_1(t) \tag{36}$$

where h_1 (t) is the solution of the equation,

$$h_1(t) + C_2 \cdot \frac{dh_1}{dt} = \delta(t) - C_1 \delta'(t) \tag{37}$$

where δ(t) is the Dirac delta function. This solution is easily shown to be :

$$h_1(t) = \frac{1}{C_2} \left(1 + \frac{C_1}{C_2} \right) \exp\left(-\frac{t}{C_2} \right) - \frac{C_1}{C_2} \delta(t) \tag{38}$$

Because of the convolution form of equation (36), the impulse response defined by equation (38) is obviously the kernel of the first term of the Volterra series.

4. Determination of second order kernel.

The kernel of the second term in the Volterra series representation of equation (2) can be similarly derived once the first order kernel is known. The algebra is of necessity more complex and consequently some detail is omitted in the following discussion to avoid undue length.

If the second order variations in storage and outflow are taken into account, the equation of continuity given by equation (7) becomes :

$$\frac{d}{dt} \left[\delta S(t) + \delta^2 S(t) \right] = \Delta Q_1(t) - \delta Q_2(t) - \delta^2 Q_2(t) \tag{39}$$

Combining this with the first order continuity equation given by equation (24), we obtain for the second order continuity equation

$$\frac{d}{dt}\left[\delta^2 S(t)\right] = -\,\delta^2 Q_2(t) \tag{40}$$

Similarly, we can derive the second order dynamic equation as :

$$\delta^2 y_2(t) - \delta^2 y_1(t) = -\,S_o L\,\delta^2 b(t) \tag{41}$$

from equation (12) and equation (25). The relationship between the first and second variations can be determined by taking both first and second order terms in the expansions of b (t) and S (t) around the initial condition. For b (t) at the upstream end of the reach we get :

$$\begin{aligned}\delta^2 b(t) = \frac{1}{2}\frac{\partial^2 f_b}{\partial Q^2}(\Delta Q_1)^2 + \frac{\partial f_b}{\partial Q}\cdot\frac{\partial f_b}{\partial y}(\Delta Q_1)(\delta y_1)\\ + \frac{1}{2}\frac{\partial^2 f_b}{\partial y^2}(\delta y_1)^2 + \frac{\partial f_b}{\partial y}\,\delta^2 y_1(t)\end{aligned} \tag{42}$$

and for b(t) at the downstream end :

$$\begin{aligned}\delta^2 b(t) = \frac{1}{2}\frac{\partial^2 f_b}{\partial Q^2}(\delta Q_2)^2 + \frac{\partial f_b}{\partial Q}\cdot\frac{\partial f_b}{\partial y}(\delta Q_2)(\delta y_2)\\ + \frac{1}{2}\frac{\partial^2 f_b}{\partial y^2}(\delta y_2)^2 + \frac{\partial f_b}{\partial y}\cdot\delta^2 y_2(t) + \frac{\partial f_b}{\partial Q}\,\delta^2 Q_2(t)\end{aligned} \tag{43}$$

Similarly we can write for the second order variation in storage :

$$\begin{aligned}\delta^2 S(t) = \frac{1}{2}\frac{\partial^2 f_1}{\partial y^2}\left[(\delta y_1)^2 + (\delta y_2)^2\right]\\ + \left(\frac{\partial f_1}{\partial y}\right)^2(\delta y_1)(\delta y_2) + \frac{\partial f_1}{\partial y}\left[\delta^2 y_1(t) + \delta^2 y_2(t)\right]\end{aligned} \tag{44}$$

In equations (42), (43), and (44), all the partial derivatives are evaluated at the initial condition. The five equations (39) to (44) define the second order corrections to the first order solution given by use of the linear kernel defined by equation (38).

In order to derive the second order kernel, the same procedure is used as in the derivation of the first order kernel. Equations (41) to (44) are quadratic in the first order variations and linear in the second order variations. Once $\delta Q_2(t)$ has been determined from equation (36), then $\delta b(t)$ can be obtained from equation (29) and $\delta S(t)$ from equation (32). The values of $\Delta Q_1(t)$ and $\delta b(t)$ can be used in equation (26) to obtain $\delta y_1(t)$ and the values of $\delta Q_2(t)$ and $\delta b(t)$ in equation (27) to obtain $\delta y_2(t)$. Thus all the quadratic terms in equations (41) to (44) can be determined and the only unknowns are the second order variations $\delta^2 y_1$, $\delta^2 y_2$, $\delta^2 b$, $\delta^2 S$ and $\delta^2 Q_2$. Since the four equations are linear in the five unknown second variations, they can be readily reduced as before to a single equation involving only the variations in storage and outflow. It can be shown[1] that this reduced equation is of the form :

$$\begin{aligned}\delta^2 S(t) = C_2\,\delta^2 Q_2(t) + A_{11}(\Delta Q_1)^2 + A_{12}(\Delta Q_1)(\delta Q_2)\\ + A_{22}(\delta Q_2)^2\end{aligned} \tag{45}$$

where C_2 has the value given by equation (34) and the other coefficients are also functions of the hydraulic properties of the channel. Equation (45) can be written as :

$$\delta^2 S(t) = C_2\left[\delta^2 Q_2(t) + g(t)\right] \tag{46}$$

1. Napiorkowski, Jaroslaw J. (1978). Identification of conceptual reservoir model described by Volterra series. (In Polish). D. Sc. Thesis. Institute of Geophysics, Polish Academy of Sciences.

where g(t) is known from the first order solution and is of the form indicated by equation (45). The combination of equation (40) and equation (46) allows to eliminate $\delta^2 S(t)$ and write :

$$- \delta^2 Q(t) = C_2 \frac{d}{dt} \left[\delta^2 Q_2(t) + g(t) \right] \tag{47}$$

Equation (47) has the solution :

$$\delta^2 Q(t) = - g'(t) * \exp\left(- \frac{t}{C_2}\right) \tag{48a}$$

which can be written alternatively as :

$$\delta^2 Q(t) = \frac{1}{C_2} g(t) * \exp\left(- \frac{t}{C_2}\right) - g(t) \tag{48b}$$

in which form the taking of derivatives is avoided.

Since the function g(t) is given by :

$$g(t) = \frac{A_{11}}{C_2} (\Delta Q_1)^2 + \frac{A_{12}}{C_2} (\Delta Q_1)(\delta Q_2) + \frac{A_{22}}{C_2} (\delta Q_2)^2 \tag{49}$$

the convolution in equation (48) will involve three separate integrals. It can be shown[1] that g(t) can be expressed as :

$$g(t) = \alpha_1 \left[f(t) \right]^2 + \alpha_2 f(t) \cdot \Delta Q_1(t) + \alpha_3 \left[\Delta Q_1(t) \right]^2 \tag{50}$$

where f(t) is the function :

$$f(t) = \exp\left(- \frac{t}{C_2}\right) * \Delta Q_1(t) \tag{51}$$

The evaluation of each of the integrals gives rise to a part of the quadratic kernel defined by :

$$\delta^2 Q_2(t) = h_2(t_1, t_2) ** (\Delta Q_1)^2 \tag{52}$$

which can consequently be written as :

$$h_2(t_1, t_2) = K_1(t_1, t_2) + K_2(t_1, t_2) + K_3(t_1, t_2) \tag{53}$$

where each term on the right hand side arises from a separate integral. Napiorkowski[1] has shown that the result of the integrations are :

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1. Napiorkowski, Jaroslow J. (1978). Identification of conceptual reservoir model described by Volterra series. (In Polish). D. Sc. Thesis. Institute of Geophysics, Polish Academy of Sciences.

$$K_1(t_1,t_2) = \alpha_2 \exp\left[-\frac{1}{C_2}\max(t_1,t_2)\right] - 2\alpha_1 \exp\left[-\frac{t_1+t_2}{C_2}\right] \tag{54a}$$

$$K_2(t_1,t_2) = \frac{\alpha_2}{2C_2}\exp\left[-\frac{1}{C_2}\max(t_1,t_2)\right] - \frac{\alpha_2}{2}\exp\left(-\frac{t_1}{C_2}\right)\delta(t_2) - \frac{\alpha_2}{2}\exp\left(-\frac{t_2}{C_2}\right)\delta(t_1) \tag{54b}$$

$$K_3(t_1,t_2) = \frac{\alpha_3}{2C_2}\left[\exp\left(-\frac{t_2}{C_2}\right) + \exp\left(-\frac{t_1}{C_2}\right)\right]\delta(t_2-t_1) - \alpha_3\,\delta(t_1)\,\delta(t_2) \tag{54c}$$

The substitution of these values from equation (54) into equation (53) gives the complete quadratic kernel to be used in equation (52) to determine the second order correction to the estimated output.

5. Results for a wide rectangular channel.

The results obtained above will be illustrated for the simple case of a wide rectangular channel with Manning friction. In this case equation (11) will take the form :

$$b(t) = \left[\frac{Q(t)}{Q_0}\right]^2 \left[\frac{y_0}{y(t)}\right]^{\frac{10}{3}} \tag{55}$$

For a rectangular channel equation (17) will take the form :

$$S(t) = W\,\frac{L}{2}\,[y_1(t) + y_2(t)] \tag{56}$$

where W is the width of the channel and L is its length. The simple form of equation (56) is true only for a rectangular channel. For the more general case

of :

$$A = K\cdot y^m \tag{57}$$

it can be readily shown by means of equation (16) that :

$$S(t) = \frac{kL}{m+1}\left(\frac{y_2^{m+1} - y_1^{m+1}}{y_2 - y_1}\right) \tag{58}$$

The relationship between the first order variations can be written as :

$$\delta S(t) = \frac{kL}{2}\,m\,[\delta y_2(t) + \delta y_1(t)] \tag{59}$$

which is equivalent to :

$$\delta S(t) = \frac{L}{2}\,[\delta A_2(t) + \delta A_1(t)] \tag{60}$$

For our special case of a rectangular channel both equation (56) and equation (60) give :

$$\delta S(t) = \frac{WL}{2}\,[\delta y_2(t) + \delta y_1(t)] \tag{61}$$

For other shapes equation (59) or equation (60) should be used.

For a wide rectangular channel with Manning friction the value of C_1 in equation (33) will be given by :

$$C_1 = 0.3\,\frac{W\cdot y_0\cdot L}{Q_0}\left(1 - 0.6\,\frac{y_0}{S_0 L}\right) \tag{62}$$

and the value of C_2 in equation (34) by :

$$C_2 = 0.3 \ \frac{W \cdot y_o \cdot L}{Q_o} \left(1 + 0.6 \ \frac{y_o}{S_o L}\right) \tag{63}$$

Substitution of these values for C_1 and C_2 in equation (38) determines the first order kernel completely.

In the case of the quadratic kernel, the procedure is the same but of necessity more complex. The values of A_{11}, A_{12} and A_{22} in equation (45) are determined from the properties of the channel. The values of α_1, α_2 and α_3 in equation (50) can then be determined from the known values of C_1, C_2, A_{11}, A_{12} and A_{22}. All of the constants required for the computation of the quadratic kernel in accordance with equation (54) are then available.

A numerical experiment was run to illustrate the effect of taking the second order effect into account. The channel is taken as rectangular, 100 m in width, with a bottom slope of $S_o = 0.000248$ and Manning roughness of $n = 0.025$. The length of reach was taken as 50 km. For the computation shown, the initial steady flow was taken as 200 m^3/S and the upstream inflow, relative to this, as :

$$\Delta Q_1(t) = \frac{1}{C_1^*} \exp\left(-\frac{t}{C_2^*}\right) \cdot t \tag{60}$$

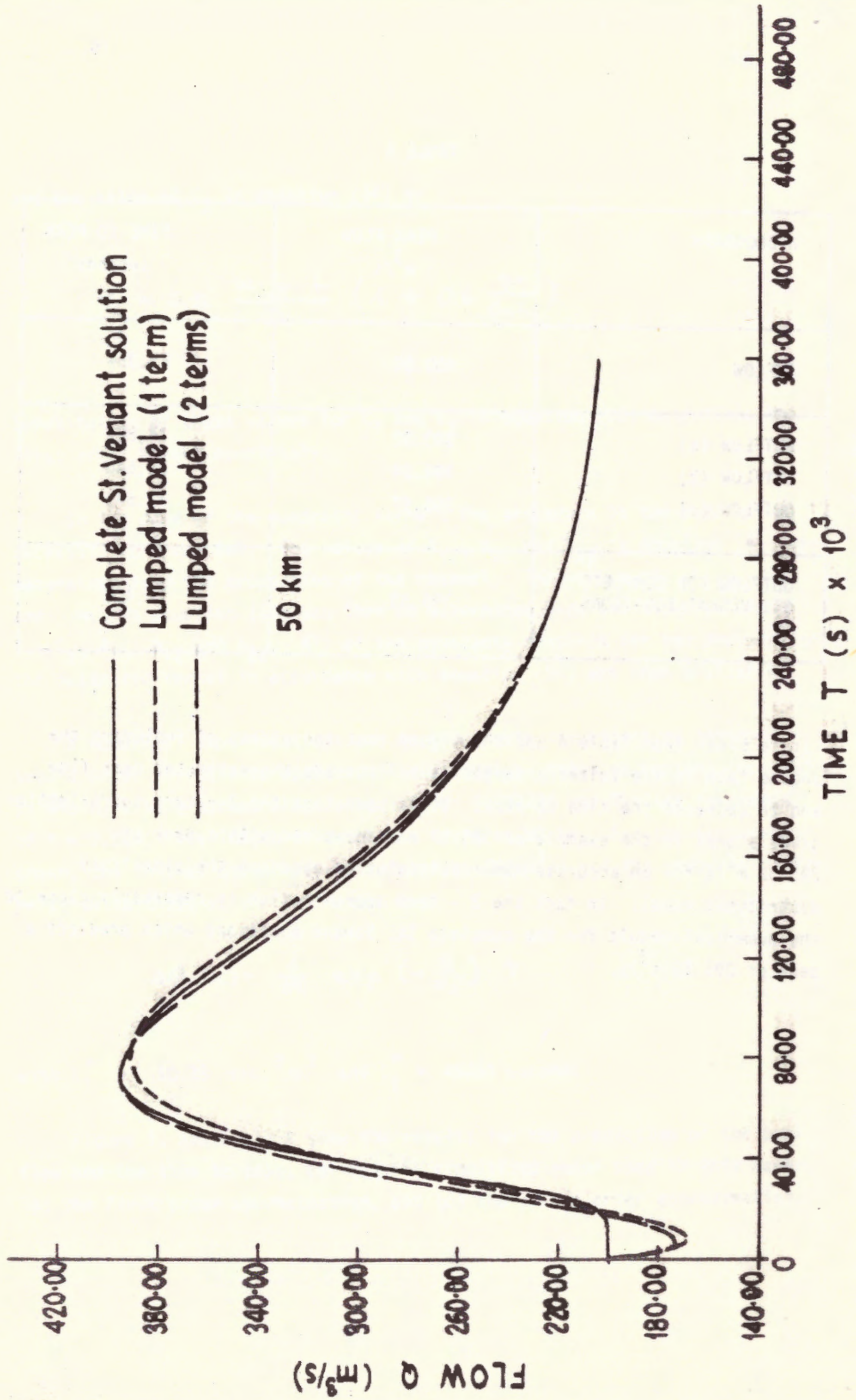
with $C_1^* = 90.78 \ sec.^2/m^3$ and $C_2^* = 49354$ seconds.

Figure 1. and Table A show the results for the prediction of the peak flow and the time to peak, for (a) the simplified model used in this paper, (b) the first order approximation, and (c) the second order approximation.

TABLE A

HYDROGRAPH	PEAK FLOW m^3/s	TIME TO PEAK seconds
INFLOW	400.00	49,354
OUTFLOW (a)	393.32	74,880
OUTFLOW (b)	391.84	80,640
OUTFLOW (c)	395.97	72,840
OUTFLOW FOR COMPLETE ST. VENANT EQUATIONS	395.53	74,400

It is clear from Table A and the figure that the effect of including the second term in the Volterra series is to increase the estimated peak flow and to decrease the time to peak. It is seen that the 2 - term approximation gives a peak in the example of 395.97 m^3/s compared with a peak of 393.32 m^3/s for an accurate numerical solution based on the simplified distributed model. In fact the 2 - term approximation is remarkably close to the numerical result for the complete St. Venant equations which predicts a peak of 395.53 m^3/s.



THE USE OF VOLTERRA SERIES IN THE MODELLING OF FLOW IN AN OPEN CHANNEL

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