THE EFFECT OF THE DOWNSTREAM BOUNDARY CONDITIONS IN THE LINEARIZED ST VENANT EQUATIONS

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SUMMARY

The two-point boundary problem for the full hyperbolic St Venant equation, in which both upstream and downstream boundary conditions are taken into account, is discussed. The upstream and downstream transfer functions for the linearized equation are derived analytically for a channel reach of finite length. The effect of the secondary boundary condition is to produce a series solution for each of these transfer functions.

1. Linearized St Venant equations

WHEN only one space dimension is taken into account, the equation of continuity for the unsteady flow in an open channel in the absence of lateral inflow is given by

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0, \tag{1}$$

where $Q(x, t)$ is the discharge, $A(x, t)$ is the cross-sectional area, x is the distance from the upstream boundary and t is the elapsed time.

The equation for the conservation of linear momentum, in open-channel flow, first formulated by St Venant in 1871 (**1**) is usually written in the form (see (**2**, **3**))

$$\frac{\partial y}{\partial x}+\frac{v}{g}\frac{\partial v}{\partial x}+\frac{1}{g}\frac{\partial v}{\partial t}=S_0-S_f, \tag{2}$$

where $y(x, t)$ is the depth of flow, $v(x, t)$ is the average velocity of the cross-section, S_0 is the bottom slope and $S_f(v, y)$ is the friction slope defined by the equilibrium condition for steady uniform flow,

$$\tau_0=\gamma R S_f, \tag{3}$$

in which $\tau_0(x, t)$ is the average shear stress along the perimeter of the cross-section, γ is the weight density of the water, and $R(x, t)$ is the hydraulic radius (the ratio of area to wetted perimeter) of the cross-section.

Since the continuity equation given by (1) is linear in $Q(x, t)$ and $A(x, t)$, it seems appropriate to adopt discharge and area as the dependent variables and to express the nonlinear momentum equation in terms of the same variables. This may be done through the use of the diagnostic equation

$$Q = vA, \tag{4}$$

which by definition connects the discharge $Q(x, t)$, the mean velocity $v(x, t)$ and the area of flow $A(x, t)$. When (4) is used to eliminate velocity from (2) we obtain

$$(1 - F^2)g\bar{y}\frac{\partial A}{\partial x} + \frac{2Q}{A}\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} = gA(S_0 - S_f), \tag{5}$$

where $\bar{y}$ is the hydraulic mean depth, defined by

$$\bar{y}(x, t) = A(x, t)/T(x, t); \tag{6a}$$

$T(x, t)$ represents the width of the channel at the water surface and is defined by

$$T(x, t) = dA/dy, \tag{6b}$$

and $F(x, t)$ is the Froude number defined by

$$F^2(x, t) = Q^2T/gA^3, \tag{7}$$

which is an important parameter of the flow conditions.

The friction slope depends on the type of friction law assumed, the shape and roughness of the cross-section, the flow of the section and the area of flow. It can be written in very general form as

$$S_f = f(A, Q, \text{shape, roughness}). \tag{8a}$$

For any given shape and roughness of the cross-section and any given friction law—whether laminar, smooth turbulent or rough turbulent (Chezy, Manning or logarithmic)—the friction slope can be expressed as a function of the flow Q and the area of flow A. Thus for Chezy friction we have in general that

$$S_f = Q^2/C^2A^2R(A), \tag{8b}$$

where C is the Chezy friction parameter, and for the Manning formula in metric units

$$S_f = n^2[Q^2/A^2R^{\frac{4}{3}}(A)], \tag{8c}$$

where n is the Manning friction parameter.

The most convenient way of linearizing the highly nonlinear momentum equation (5) is to express the dependent variables in the forms

$$Q(x, t) = Q_0 + Q'(x, t) + e_Q(x, t), \tag{9a}$$

$$A(x, t) = A_0 + A'(x, t) + e_A(x, t), \tag{9b}$$

where Q_0 is a reference condition of steady uniform flow, A_0 is the cross-sectional area corresponding to this reference flow, $Q'(x, t)$ and $A'(x, t)$ are the first-order increments and $e_Q(x, t)$ and $e_A(x, t)$ represent the higher-order terms (that is, the error of the linear approximation).

When equations (9) are substituted in (1) and the higher-order terms neglected we obtain, as the continuity equation for the perturbations Q' and A',

$$\frac{\partial Q'}{\partial x} + \frac{\partial A'}{\partial t} = 0, \tag{10}$$

which is identical in form to (1). When the nonlinear terms in (5) are expanded in Taylor series around the uniform steady state (Q_0, A_0) for the increments defined by (9) and the terms of higher order than the linear ones are neglected, we obtain the linearized momentum equation in the form

$$(1 - F_0^2)g\bar{y}_0\frac{\partial A'}{\partial x} + 2v_0\frac{\partial Q'}{\partial x} + \frac{\partial Q'}{\partial t} = gA_0\left(-\frac{\partial S_f}{\partial Q}Q' - \frac{\partial S}{\partial A}A'\right), \tag{11}$$

where the derivatives of the friction slope $S_f(Q, A)$ with respect to discharge (Q) and area (A) on the right-hand side of the equation are evaluated at the reference conditions.

The variation of the friction slope with discharge at the reference condition for all frictional formulae for rough turbulent flow could be taken as

$$\partial S_f/\partial Q = 2S_0/Q_0. \tag{12}$$

We may for convenience define a parameter m as the ratio of the kinematic wave speed to the average velocity of flow

$$m = c_k/(Q_0/A_0), \tag{13}$$

where c_k is the kinematic wave speed as given by Lighthill and Whitham (**4**)

$$c_k = -(\partial S_f/\partial A)/(\partial S_f/\partial Q) = dQ/dA. \tag{14}$$

The parameter m is a function of the shape of channel and of area of flow (A). For a wide rectangular channel with Chezy friction m is always equal to 3/2 and with Manning friction always equal to 5/3. For shapes of channel other than wide rectangles, m will take on different values. Using these values, the right-hand side of (11) can be written as

$$2gA_0S_0\left(m\frac{A'}{A_0} - \frac{Q'}{Q_0}\right). \tag{15}$$

Since (10) and (11) are linear first-order equations in two variables, they are equivalent to a single second-order equation in one variable. The most general form of this second-order equation is obtained by using the unsteady flow potential introduced by Deymie (**5**) and developed by Supino

(6). This potential can be defined as the function $U'(x, t)$ whose partial derivative with respect to distance gives minus the perturbation in the area of flow, that is,

$$\partial U'/\partial x = -A'(x, t), \tag{16}$$

and whose partial derivative with respect to time gives the perturbation from the reference discharge,

$$\partial U'/\partial t = Q'(x, t). \tag{17}$$

Consequently, the perturbation potential $U'(x, t)$ automatically satisfies the continuity equation (10). When (16) and (17) are substituted in (11), we obtain the dynamic equation for the unsteady flow potential $U'(x, t)$ in the form

$$(1 - F_0^2)g\bar{y}_0 \frac{\partial^2 U'}{\partial x^2} - 2v_0 \frac{\partial^2 U'}{\partial x\, \partial t} - \frac{\partial^2 U'}{\partial t^2} = gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial U'}{\partial x} + \frac{\partial S_f}{\partial Q}\frac{\partial U'}{\partial t}\right). \tag{18}$$

Any linear function of the perturbation potential $U'(x, t)$ will also represent a solution of (18). Since differentiation is a linear operation, both $A'(x, t)$ and $Q'(x, t)$ will also be governed by equations of the same form as (18). The choice of dependent variable in any given problem will be governed largely by the form in which the boundary conditions are given.

2. Boundary conditions for linearized St Venant equations

The equation to be solved is hyperbolic in form. Accordingly there are two real characteristics defined by

$$dx/dt = v_0 \pm (g\bar{y}_0)^{\frac{1}{2}}, \tag{19}$$

along which the discontinuities in the derivatives of the solution will propagate. For a Froude number less than one (tranquil flow) the secondary characteristic direction involving the negative root will be in an upstream direction and the flow within the range of influence of the condition at the downstream boundary will be affected by that boundary condition.

For tranquil flow in a given length of channel there will be four solution zones as shown in Fig. 1. In zone A the solution depends only on the double initial condition along $t = 0$. In zone B the solution depends on the upstream boundary condition along $x = 0$ and the condition established along the leading primary characteristic by the solution in zone A. In zone C the solution depends on the downstream boundary condition along $x = L$ and the conditions established along the leading secondary characteristic by the solution in zone A. In zone D, the solution depends on both the upstream and downstream boundary conditions and, through the zone A solution, on the double initial condition. For any point in zone D, the solution will depend on the history of the characteristic variables associated with the

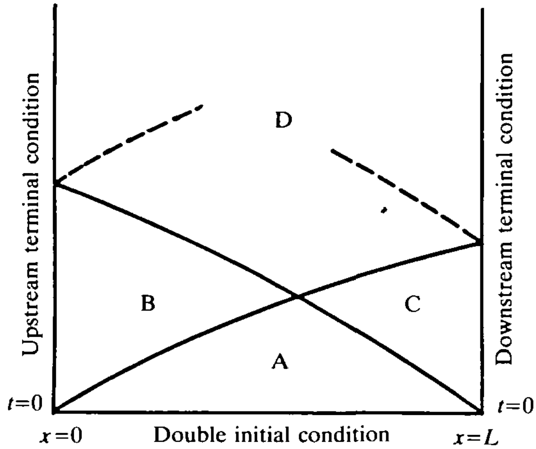

FIG. 1. Zones of tranquil flow

primary and secondary characteristics passing through that particular point (x, t). For later times in zone D these characteristics will traverse the length of the channel many times and the characteristic variables will have been affected by the appropriate boundary condition at each reversal of the characteristic.

The problem of unsteady flow in rivers and canals can be classified on the basis of the nature of the boundary conditions. In problems of flood routing, the aim is to predict the level or the flow at the downstream end of the channel when given the level or the flow at the upstream end. In estuarine hydraulics, the aim is to predict levels or velocities at various points in the channel, given the variation of water level at the downstream end. In either case the problem can only be adequately posed and solved if both boundary conditions are specified. In hydrologic flood routing, only the upstream boundary condition is properly specified and the downstream boundary is either ignored or crudely approximated. By studying the linearized St Venant equations for a finite channel reach with a properly defined boundary condition at each end we can provide a basis for the analysis of the errors involved in the solution due to the inadequate specification of one of the boundary conditions. If both boundary conditions are given as Dirichlet conditions for the same variable, either $Q'(t)$ or $A'(t)$ being given at each end, then the solution will obviously be sought using the prescribed variable as the dependent variable in (18). If one boundary condition is given in terms of $Q'(t)$ and the other in terms of $A'(t)$ then the continuity equation (10) can be used to convert the latter boundary condition from a Dirichlet condition in $A'(t)$ to a Neumann condition in $Q'(t)$ and the problem can be solved in terms of $Q'(x, t)$.

3. Solution for finite channel reach

The solution of (18) for an upstream boundary condition and a semi-infinite wide rectangular channel with Chezy friction is well known (**4**, **5**, **7**). The solution for any other shape of channel and any other friction law does not involve any additional complexity except for the replacement of the value for the parameter m in (13) by the general expression used to define it (Dooge *et al.* (**8**)). The two-point boundary problem in which both an upstream and a downstream boundary condition are taken into account has not, as far as the authors are aware, been reported in the hydrologic literature.

The basic equation to be solved is of the same form as (18) and the dependent variable $f(x, t)$ may be $Q'(x, t)$ or $A'(x, t)$ depending on the nature of the boundary conditions. We shall consider the basic case in which $f(x, t)$ will be prescribed both at the upstream boundary $x = 0$ and at the downstream boundary $x = L$.

The position is to solve the equation

$$(1 - F_0^2)g\bar{y}_0\frac{\partial^2 f}{\partial x^2} - 2v_0\frac{\partial^2 f}{\partial x\,\partial t} - \frac{\partial^2 f}{\partial t^2} = gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial f}{\partial x} + \frac{\partial S_f}{\partial Q}\frac{\partial f}{\partial t}\right), \tag{20}$$

subject to the double initial condition

$$f(x, 0) = 0, \tag{21}$$

$$\partial f(x, 0)/\partial t = 0, \tag{22}$$

and subject to the boundary conditions

$$f(0, t) = f_u(t), \tag{23}$$

$$f(L, t) = f_d(t). \tag{24}$$

The solution can be sought in terms of the Laplace transform

$$\bar{f}(x, s) = \int_0^\infty \exp(-st) f(x, t)\, dt, \tag{25}$$

which, when substituted into (20) and taking account of (21) and (22), gives

$$(1 - F_0^2)g\bar{y}_0\frac{d^2\bar{f}}{dx^2} + \left(-2v_0 s + \frac{\partial S_f}{\partial A}gA_0\right)\frac{d\bar{f}}{dx} - \left(s^2 + gA_0\frac{\partial S_f}{\partial Q}s\right)\bar{f} = 0, \tag{26}$$

which is a second-order homogeneous ordinary differential equation. The general solution can be written in the form

$$\bar{f}(x, s) = A_1(s)\exp[\lambda_1(s)x] + A_2(s)\exp[\lambda_2(s)x], \tag{27}$$

where λ_1 and λ_2 are the roots of the characteristic equation for (26) and are given by

$$\lambda_{1,2}(s) = es + f \pm (as^2 + bs + c)^{\frac{1}{2}}. \tag{28}$$

Here the parameters a, b, c, e and f are functions of the channel parameters and are given as follows:

$$a = \frac{1}{g\bar{y}_0(1-F_0^2)^2}, \tag{29a}$$

$$b = T_0 \frac{Q_0(1-F_0^2)\,\partial S_f/\partial Q - A_0F_0^2\,\partial S_f/\partial A}{Q_0(1-F_0^2)^2} = \frac{2S_0}{v_0\bar{y}_0}\frac{1+(m-1)F_0^2}{(1-F_0^2)^2}, \tag{29b}$$

$$c = \frac{T_0^2(\partial S_f/\partial A)^2}{4(1-F_0^2)^2} = m^2\left(\frac{S_0}{\bar{y}_0}\right)^2\frac{1}{(1-F_0^2)^2}, \tag{29c}$$

$$d = \tfrac{1}{4}b^2 - ac, \tag{29d}$$

$$e = \frac{1}{(g\bar{y}_0)^{\frac{1}{2}}}\frac{F_0}{(1-F_0^2)}, \tag{29e}$$

$$f = -\frac{T_0\,\partial S_f/\partial A}{2(1-F_0^2)} = m\frac{S_0}{\bar{y}_0(1-F_0^2)}. \tag{29f}$$

The above relationships hold for any shape of channel and for any law of rough turbulent friction.

The functions $A_1(s)$ and $A_2(s)$ in (27) can be determined from the boundary conditions. For the upstream boundary condition at $x=0$ equation (27) becomes

$$f_u(s) = A_1(s) + A_2(s), \tag{30}$$

and for the downstream boundary condition at $x = L$ we have

$$f_d(s) = A_1(s)\exp(\lambda_1 L) + A_2(s)\exp(\lambda_2 L), \tag{31}$$

where $f_u(s)$ and $f_d(s)$ are the Laplace transforms of the upstream and downstream boundary conditions. Solving (30) and (31) for the unknown functions $A_1(s)$ and $A_2(s)$ we obtain

$$A_1(s) = \frac{f_u(s)\exp(\lambda_2 L) - f_d(s)}{\exp(\lambda_2 L) - \exp(\lambda_1 L)}, \tag{32a}$$

$$A_2(s) = \frac{f_d(s) - f_u(s)\exp(\lambda_1 L)}{\exp(\lambda_2 L) - \exp(\lambda_1 L)}, \tag{32b}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are given by (28). Substituting these values in (27) the solution for $f(x, s)$ can be written in terms of $h_u(x, s)$ and $h_d(x, s)$. These formulations may be defined as the Laplace transforms of the responses of the channel reach to delta-function inputs at the upstream and downstream ends, respectively. Accordingly we write

$$\bar{f}(x, s) = h_u(x, s)f_u(s) + h_d(x, s)f_d(s) \tag{33}$$

and evaluate the downstream response $h_u(x, s)$ and the upstream response $h_d(x, s)$ by combining (27), (28), (32) and (33). The downstream linear channel response of (33) is given by

$$h_u(x, s) = \exp[(es + f)x] \frac{\sinh\{(as^2 + bs + c)^{\frac{1}{2}}(L - x)\}}{\sinh\{(as^2 + bs + c)^{\frac{1}{2}}L\}} \tag{34}$$

and the Laplace transform of the upstream linear channel response is given by

$$h_d(x, s) = \exp[-(es + f)(L - x)] \frac{\sinh\{(as^2 + bs + c)^{\frac{1}{2}}x\}}{\sinh\{(as^2 + bs + c)^{\frac{1}{2}}L\}}. \tag{35}$$

The solutions given by (34) and (35) fulfill the conditions for the Laplace transform that

$$h_u(s) \to 0 \quad \text{as } s \to \infty, \tag{36a}$$

$$h_d(s) \to 0 \quad \text{as } s \to \infty. \tag{36b}$$

The result for downstream movement in a semi-infinite channel can be obtained from (34) by letting $L \to \infty$ and thus obtaining

$$h_u(x, s) = \exp\{exs + fx - (as^2 + bs + c)^{\frac{1}{2}}x\}. \tag{37}$$

This is the general form of result obtained previously for a wide rectangular channel with Chezy friction in (**4**, **5**, **7**, **8**).

4. Solution in the time domain

It remains to invert (34) and (35) from the Laplace-transform domain to the original time domain. It is only necessary to invert the two linear channel response functions to the time domain since any boundary conditions can be accounted for by convolution with the linear channel responses; thus,

$$f(x, t) = h_u(x, t) * f_u(t) + h_d(x, t) * f_d(t). \tag{38}$$

Equation (34) can be rewritten in the form

$$h_u(x, s) = \exp(esx + fx) \times \\ \times \frac{\exp\{-(as^2 + bs + c)^{\frac{1}{2}}x\} - \exp\{-(as^2 + bs + c)^{\frac{1}{2}}(2L - x)\}}{1 - \exp\{-2L(as^2 + bs + c)^{\frac{1}{2}}\}}. \tag{39}$$

The inversion of (39) to the time domain is not straightforward though the term $\exp(exs)$ is easily interpreted as a time shift and $\exp(fx)$ as an amplification or damping factor independent of time.

We can expand the denominator of (39) into a convergent series

$$[1 - \exp\{-2L(as^2 + bs + c)^{\frac{1}{2}}\}]^{-1} = \sum_{n=0}^{\infty} \exp\{-2nL(as^2 + bs + c)^{\frac{1}{2}}\} \tag{40}$$

and operate on it term by term. When this is done we can write

$$h_u(x, s) = \sum_{n=0}^{\infty} \exp\{exs + fx - (2nL + x)(as^2 + bs + c)^{\frac{1}{2}}\}$$
$$- \sum_{n=1}^{\infty} \exp\{exs + fx - (2nL - x)(as^2 + bs + c)^{\frac{1}{2}}\}. \quad (41)$$

The explicit formulation of the upstream transfer function in the time domain may be obtained by using the standard transform pair given by Doetsch (9). For the expression in the transform domain we use

$$\exp\{-x(as^2 + bs + c)^{\frac{1}{2}}\} - \exp(-bx/2a^{\frac{1}{2}} - a^{\frac{1}{2}}xs), \quad (42a)$$

and for the corresponding function in the time domain,

$$(d/a)^{\frac{1}{2}}x \exp(-bt/2a)(t^2 - ax^2)^{-\frac{1}{2}} I_1[d^{\frac{1}{2}}(t^2 - ax^2)^{\frac{1}{2}}/a] 1(t - a^{\frac{1}{2}}x), \quad (42b)$$

where $I_1[\ ]$ is a modified Bessel function of the first kind and $1(\ )$ is a unit step function. By adopting this standard transform pair it is possible to invert (41) to the time domain, where the solution is found to have two distinct parts. Thus we can write

$$h_u(x, t) = h_u^1(x, t) + h_u^2(x, t). \quad (43)$$

The first part of the solution, which may be termed the head of the wave, is given by

$$h_u^1(x, t) = \sum_{n=0}^{\infty} \exp(-2nL\alpha_1 - \alpha_2 x)\, \delta(t - nt_0 - x/c_1)$$
$$- \sum_{n=0}^{\infty} \exp[-2(n+1)L\alpha_1 + \alpha_3 x]\, \delta[t - (n+1)t_0 - x/c_2], \quad (44)$$

where

$$\alpha_1 = \frac{b}{2a^{\frac{1}{2}}} = \frac{S_0}{\bar{y}_0} \frac{1 + (m-1)F_0}{(1 - F_0^2)F_0}, \quad (45a)$$

$$\alpha_2 = \frac{b}{2a^{\frac{1}{2}}} - f = \frac{S_0}{\bar{y}_0} \frac{1 - (m-1)F_0}{(1 + F_0)F_0}, \quad (45b)$$

$$\alpha_3 = \frac{b}{2a^{\frac{1}{2}}} + f = \frac{S_0}{\bar{y}_0} \frac{1 + (m-1)F_0}{(1 - F_0)F_0}, \quad (45c)$$

$$c_1 = 1/(a^{\frac{1}{2}} - e) = v_0 + (g\bar{y}_0)^{\frac{1}{2}}, \quad (45d)$$

$$c_2 = -1/(a^{\frac{1}{2}} + e) = v_0 - (g\bar{y}_0)^{\frac{1}{2}}, \quad (44e)$$

$$t_0 = L/c_1 - L/c_2. \quad (45f)$$

It can be seen that the head of the wave moves downstream at the dynamic speed c_1 in the form of a delta function of exponentially declining volume proportional to $\exp(-\alpha_2 x)$. At $x = L$ the delta function is reflected with inversion of sign and is then propagated upstream at the speed c_2 and with a heavier damping factor $\exp[-\alpha_3(L-x)]$. The second part of the solution, which may be termed the body of the wave, is given by

$$\begin{aligned} h_u^2(x,t) = {} & \sum_{n=0}^{\infty} \exp(-\beta_1 t + \beta_2 x) h(1/c_1 - 1/c_2)(2nL + x) \times \\ & \times \frac{I_1[2h\{(t - nt_0 - x/c_1)(t + nt_0 - x/c_2)\}^{\frac{1}{2}}]}{\{(t - nt_0 - x/c_1)(t + nt_0 - x/c_2)\}^{\frac{1}{2}}} 1(t - nt_0 - x/c_1) \\ & - \sum_{n=0}^{\infty} \exp(-\beta_1 t + \beta_2 x) h(1/c_1 - 1/c_2)[2(n+1)L - x] \times \\ & \times \frac{I_1[2h[\{t - (n+1)t_0 - x/c_2\}\{t + (n+1)t_0 - x/c_1\}]^{\frac{1}{2}}]}{[\{t - (n+1)t_0 - x/c_2\}\{t + (n+1)t_0 - x/c_1\}]^{\frac{1}{2}}} \times \\ & \times 1(t - (n+1)t_0 - x/c_2). \end{aligned} \tag{46}$$

The remaining parameters are given by

$$\beta_1 = \frac{b}{2a} = \frac{S_0 v_0}{\bar{y}_a} \frac{1 + (m-1)F_0}{F_0}, \tag{47a}$$

$$\beta_2 = f - \frac{be}{2a} = (m-1)\frac{S_0}{y_0}, \tag{47b}$$

$$h = \frac{d^{\frac{1}{2}}}{a} = \frac{S_0 v_0}{y_0} \frac{(1 - F_0^2)[1 - (m-1)^2 F_0^2]}{2F_0^2}. \tag{47c}$$

For the upstream transfer function the head of the wave is given by

$$\begin{aligned} h_d^1(x,t) = {} & \sum_{n=0}^{\infty} [-2nL\alpha_1 - \alpha_3(L-x)]\,\delta[t - nt_0 + (L-x)/c_2] \\ & - \sum_{n=0}^{\infty} [-(2n+1)L\alpha_1 - fL - \alpha_2 x]\,\delta[t - nt_0 + L/c_2 - x/c_1] \end{aligned} \tag{48}$$

and the body of the wave is given by

$$\begin{aligned} h_d^2(x,t) = {} & \sum_{n=0}^{\infty} \exp[-\beta_1 t - \beta_2(L-x)] h(1/c_1 - 1/c_2)[(2n+1)L - x] \times \\ & \times \frac{I_1[2h(\{t - nt_0 + (L-x)/c_2\}\{t + nt_0 + (L-x)/c_1\})^{\frac{1}{2}}]}{[\{t - nt_0 + (L-x)/c_2\}\{t + nt_0 + (L-x)/c_1\}]^{\frac{1}{2}}} \times \\ & \times 1(t - nt_0 + (L-x)/c_2) \\ & - \sum_{n=0}^{\infty} \exp[-\beta_1 t - \beta_2(L-x)] h(1/c_1 - 1/c_2)[(2n+1)L + x] \times \\ & \times \frac{I_1[2h\{(t + nt_0 - x/c_2 + L/c_1)(t - nt_0 + L/c_2 - x/c_1)\}]^{\frac{1}{2}}}{[(t + nt_0 - x/c_2 + L/c_1)(t - nt_0 + L/c_2 - x/c_1)]^{\frac{1}{2}}} \times \\ & \times 1(t - nt + L/c - x/c). \end{aligned} \tag{49}$$

Since the modified Bessel function is itself represented by an infinite series

$$I_1(2x) = x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!(k+1)!}, \tag{50}$$

the solution is in the form of a doubly infinite series which seems too complicated for practical application in river-flow forecasting. However, due to heavy damping only the few first terms of the upstream and downstream transfer functions would normally be required. The solution for the special case of the downstream movement of the flood waves in a semi-infinite channel is well known (**4**, **5**, **7**, **10**). The general solution for a semi-infinite channel has recently been published (**8**). This special case of the semi-infinite channel has the solution

$$h_u(x, t) = \delta(t - x/c_1) \exp(-a_2 x) +$$
$$+ \exp(-\beta_1 t + \beta_2 x) h\left(\frac{x}{c_1} - \frac{x}{c_2}\right) \frac{I_1[2h\{(t - x/c_1)(t - x/c_2)\}^{\frac{1}{2}}]}{[(t - x/c_1)(t - x/c_2)]^{\frac{1}{2}}} 1(t - x/c_1), \tag{51}$$

which corresponds to the use of only the first term in (44) and (46).

5. Summary and conclusions

The effect of the downstream boundary condition on unsteady flow in rivers is explored through the analysis of the linearized St Venant equations. It is found that there are two effects of the inclusion of the downstream boundary condition: (1) the direct upstream transmission of the effect of the downstream boundary condition in accordance with (33) and (35); (2) the generalization of the downstream response to the upstream impulse from a single term to an infinite series as in (34) because of the continual reflection of the upstream generated wave up and down the channel. The form of the two impulse–response functions indicates that the infinite series involved are highly convergent. The verification of this for particular cases is outside the scope of the present paper. Some numerical examples for the special case of low Froude numbers have been published (**8**, **11**). The extension of these numerical investigations is at present underway for the case of any Froude number between zero and unity and to answer such practically important questions as the effect of the assumption of a steady-state rating curve at the downstream end on the accuracy of the solution.

The linearized solution obtained in this way can be used to evaluate the relative effectiveness of the procedures used in numerical methods of solution to handle the downstream boundary condition in the absence of a definite control. These practical procedures include the use of a steady rating curve or the use for computational purposes of a length of channel considerably longer than the channel reach of interest (**3**). The solution is also useful as a basis for evaluating the extra degree of approximation in hydrologic routing methods due to neglect of the downstream boundary condition over and above the approximation involved in the simplification of the dynamic St Venant equation or its replacement by a conceptual model (**7**).

REFERENCES

1. B. DE SAINT-VENANT, *C.r. Acad. Sci. Paris* **73** (1871) 148–154; 237–240.
2. F. M. HENDERSON, *Open Channel Flow* (Macmillan, London 1966).
3. J. A. CUNGE, F. M. HOLLY and A. VERWEY, *Practical Aspects of Computational River Hydraulics* (Pitman, London 1980).
4. M. J. LIGHTHILL and G. P. WHITHAM, *Proc. R. Soc.* A **229** (1955) 281–316.
5. PH. DEYMIE, *Revue Generale de l'Hydraulique* **3** (1935) 138–142.
6. G. SUPINO, *ibid.* **57** (1950) 144.
7. J. C. I. DOOGE and B. M. HARLEY, In *Proc. Int. Hydrol. Symp., Fort Collins, Colorado* **8**/1 (1967) 57–63.
8. ——, J. J. NAPIORKOWSKI and W. G. STRUPCZEWSKI, *Acta Geophysica Polonica* to appear.
9. G. DOETSCH, *Guide to the Applications of Laplace Transforms* (Van Nostrand, New York 1961).
10. J. C. I. DOOGE, Z. W. KUNDZEWICZ and J. J. NAPIORKOWSKI, *Hydrological Sci. J.* **28** (1983) 391–402.
11. —— and J. J. NAPIORKOWSKI, *Acta Geophysica Polonica* **32** (1984) 363–373.