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APPLICABILITY OF DIFFUSION ANALOGY IN FLOOD ROUTING

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Abstract

A number of forms of the diffusion analogy approximation to the linearised St. Venant equation for flood routing in channels are compared. It is suggested that the diffusion analogy based on the kinematic wave approximation of certain terms be used. The upper limits for the dimensionless wave number are given for certain prescribed levels of error in the diffusion analogy predictions for the phase velocity and for the attenuation per unit length.

1. INTRODUCTION

The degree of approximation involved in replacing the complete St. Venant equations by a diffusion analogy method or by a kinematic wave method has been discussed by a number of authors. Some of these authors have approached the subject through harmonic analysis. This can be done either by means of a frequency analysis of the linearised St. Venant equations with harmonic boundary conditions (Grijsen and Vreugdenhil, 1976; Vreugdenhil, 1972, 1977) or by wave number analysis of the linearised equations (Menendez and Norscini, 1982; Ponce and Simons, 1977; Ponce et al., 1978).

The present note comments briefly on three points: (1) the existence of more than one form of the diffusion analogy; (2) the selection of that form of diffusion analogy which approximates most closely to the complete St. Venant equations; (3) the possibility of formulating a simple criterion for the range of applicability of this model of the diffusion analogy. The discussion will, for the sake of brevity, concentrate on the special case of a wide rectangular channel with Chezy friction and on wave number analysis.

2. COMPLETE LINEAR EQUATIONS

The linearised St. Venant equations for one-dimensional unsteady flow in a broad rectangular channel with Chezy friction may be written as (Deymie, 1935; Ponce and Simons, 1977):

$$\frac{\partial y}{\partial t}+v_0\frac{\partial y}{\partial x}+y_0\frac{\partial v}{\partial x}=0, \tag{1}$$

$$\frac{\partial y}{\partial x}+\frac{v_0}{g}\frac{\partial v}{\partial x}+\frac{1}{g}\frac{\partial v}{\partial t}=S_0\left(\frac{y}{y_0}-\frac{2v}{v_0}\right), \tag{2}$$

where y is the perturbation in depth from the reference depth y_0, v is the perturbation in velocity from the reference velocity v_0, g is the acceleration due to gravity, S_0 is the bottom slope of the channel, t is the elapsed time, and x is the distance along the channel.

By eliminating $v(x, t)$ from equations (1) and (2) we can obtain a single second order equations in $y(x, t)$ given by:

$$(gy_0-v_0^2)\frac{\partial^2 y}{\partial x^2}-2v_0\frac{\partial^2 y}{\partial x\,\partial t}=3gS_0\frac{\partial y}{\partial x}+\frac{2gS_0}{v_0}\frac{\partial y}{\partial t}. \tag{3}$$

The above equation can be expressed conveniently in dimensionless form (Woolhiser and Liggett, 1967) by using the bottom slope of the channel (S_0) and the depth and velocity (y_0, v_0) of the steady uniform reference condition about which perturbations are taken. Thus we can write

$$y'=\frac{y}{y_0}, \tag{4}$$

$$x'=\frac{x}{\dfrac{y_0}{S_0}}, \tag{5}$$

$$t'=\frac{t}{\dfrac{y_0}{S_0 v_0}} \tag{6}$$

and transform equation (3) to

$$0.5(1-F_0^2)\frac{\partial^2 y'}{\partial x'^2}-F_0^2\frac{\partial^2 y'}{\partial x'\partial t'}-0.5F_0^2\frac{\partial^2 y'}{\partial t'^2}=1.5\frac{\partial y'}{\partial x'}+\frac{\partial y'}{\partial t'}, \tag{7}$$

where $F_0=v_0/\sqrt{gy_0}$ is the Froude number for the reference flow condition.

3. VARIETY OF DIFFUSION ANALOGY MODELS

In the case of many river channels, the second and third terms in equation (2) are of an order of magnitude smaller than the other three terms in the equation (Cunge et al., 1980; Henderson, 1966; Kuchment, 1972). If these two smaller terms are neglected so that equation (2) becomes

$$\frac{\partial y}{\partial x}=S_0\left(\frac{y}{y_0}-2\frac{v}{v_0}\right) \tag{8}$$

and $v(x, t)$ is now eliminated between equation (1) and equation (8) we obtain

$$\frac{\partial y}{\partial t}+1.5v_0\frac{\partial y}{\partial x}=\frac{v_0\,y_0}{2S_0}\frac{\partial^2 y}{\partial x^2} \tag{9}$$

instead of equation (3). The approximation represented by equation (9) is of the same form as the convective-diffusion equation

$$\frac{\partial y}{\partial t}+c\frac{\partial y}{\partial x}=D\frac{\partial^2 y}{\partial x^2} \tag{10}$$

and consequently known solutions of the latter equation can be applied to flood routing (Hayami, 1951; Schonfeld, 1948). For this approximation based on neglecting terms in equation (2) (Ponce and Simons, 1977; Ponce et al., 1978) the parameters of the convective-diffusion equation are related to the parameters of the channel and the reference flow conditions by

$$c=1.5v_0 \tag{11}$$

which is the kinematic wave speed and

$$D=\frac{v_0\,y_0}{2S_0} \tag{12}$$

which is equivalent hydraulic diffusivity of the channel.

If, alternatively, we examine the dimensionless form of the equation, a somewhat different approximation is suggested. As the Froude number approaches zero, then the second and third term in equation (7) obviously tend to zero and the same must be true of the corresponding terms in equation (3). If these terms are neglected then we also have a convective-diffusion equation. In this case the convective parameter c is again given by equation (11) but the equivalent channel diffusivity is

$$D=(1-F_0^2)\frac{v_0\,y_0}{2S_0} \tag{13}$$

which will differ from that given by the first approximation particularly for higher Froude numbers.

Yet another form of diffusion analogy can be derived if instead of neglecting small terms entirely, they are represented by one of other of the surviving terms on the basis of the kinematic wave approximation (Dooge and Harley, 1967). For the kinematic wave approximation (Lighthill and Whitham, 1955) we can write the solution for the perturbation in depth as:

$$\frac{y}{y_0}=f(x-1.5v_0\,t). \tag{14}$$

Insertion of this solution in equation (1) reveals that to satisfy continuity we must write

$$\frac{v}{v_0}=0.5f(x-1.5v_0 t) \tag{15}$$

where the same function can be used since there is by definition no phase shift for the kinematic wave solution (Menendez and Norscini, 1982). This lower order solution can be used to approximate either the terms neglected in equation (2) or the terms neglected in equation (3).

If we wish to approximate the second term in equation (2) on the basis of the kinematic approximation then we write

$$\frac{v_0}{g}\frac{\partial v}{\partial x}=\frac{v_0^2}{2g}f'(x-1.5v_0 t)=0.5\frac{v_0^2}{gy}\frac{\partial y}{\partial x}. \tag{16}$$

Similarly we write the third term as

$$\frac{1}{g}\frac{\partial v}{\partial t}=-1.5\frac{v_0^2}{2g}f'(x-1.5v_0 t)=-0.75\frac{v_0^2}{gy_0}\frac{\partial y}{\partial x}. \tag{17}$$

Inserting these two approximations into equation (2) we obtain

$$\left(1-\frac{F_0^2}{4}\right)\frac{\partial y}{\partial x}=S_0\left(\frac{y}{y_0}-\frac{2v}{v_0}\right). \tag{18}$$

If $v(x, t)$ is eliminated between equations (1) and (18) we get

$$\frac{\partial y}{\partial t}+1.5v\frac{\partial y}{\partial x}=\left(1-\frac{F_0^2}{4}\right)\frac{v_0 y_0}{2S_0}\frac{\partial^2 y}{\partial x^2} \tag{19}$$

which lead to a value of

$$D=\left(1-\frac{F_0^2}{4}\right)\frac{v_0 y_0}{2S_0} \tag{20}$$

for the equivalent channel diffusivity.

If alternatively we wish to make the approximation in equation (3) then we write the second term as

$$-2v_0\frac{\partial^2 y}{\partial x\,\partial t}=3v_0^2 f''(x-1.5v_0 t)=3v_0^2\frac{\partial^2 y}{\partial x^2} \tag{21}$$

and the third term as

$$-\frac{\partial^2 y}{\partial t^2}=-2.25v_0^2 f''(x-1.5v_0 t)=-2.25v_0^2\frac{\partial^2 y}{\partial x^2}. \tag{22}$$

Substitution of these approximations in equation (3) also gives the convective diffusion equation in the form of equation (19).

4. CHOICE OF FORM OF DIFFUSION ANALOGY

The three forms of the diffusion analogy discussed in the last section all agree in predicting the convective parameter as

$$c = 1.5v_0 \tag{11}$$

which is the kinematic wave velocity. Accordingly all of them will predict the first moment or lag of the linear channel response (LCR) as

$$U'_1(LCR) = \frac{x}{1.5v_0} \tag{23}$$

which is identical to the value for the complete linearised equation (Dooge and Harley 1967). They differ however in their prediction of the equivalent channel diffusivity D excep for the limiting case of $F=0$. Accordingly the question arises of whether any one particula form of diffusion analogy approximation can be shown to be preferable to the others.

On general grounds one could expect that the models based on the approximations of terms through the kinematic wave approximation would be preferable to those based on complete neglect of these terms. These general considerations are reinforced by comparing some properties of equation (19) with other known results in open channel hydraulics. Firstly, the model represented by equation (19) unlike the other two models, indicates that diffusivity will become negative (i.e. disturbances will amplify) for $F_0 > 2$. Secondly, the second moment about the centre of the area of the solution of the complete St. Venant equation represented by equation (3) for a delta function input is given by Dooge and Harley (1967):

$$u_2 = \frac{2}{3}(1 - 0.25F_0^2)\frac{y_0}{S_0 x}\left(\frac{x}{1.5v_0}\right)^2 \tag{24}$$

and for the diffusion analogy represented by equation (10) is given by

$$u_2 = \frac{2D}{cx}\left(\frac{x}{c}\right)^2. \tag{25}$$

Using the value of c from equation (11) which is common to all forms of the diffusion analogy and equating these two values for the second moment we get

$$D = (1 - 0.25F_0^2)\frac{v_0 y_0}{2S_0} \tag{26}$$

which is the value already obtained by using the kinematic wave solution to approximate terms in equations (2) or (3).

It is suggested that any discussion of the range of applicability of the diffusion analogy should be confined to this form of the analogy which reproduces exactly the first and second moments of the complete linear solution. In the final section of this note, the wave analysis method used by Ponce and Simons (1977) will be applied to the question of the range of applicability of this particular model.

5. WAVE NUMBER ANALYSIS OF DIFFUSION ANALOGY

The solution for a harmonic perturbation in depth of either the complete equation or of the diffusion analogy approximation can be sought in the form

$$y=y_m \exp[\mathrm{i}(\sigma x-\beta t)], \tag{27}$$

where σ is the real wave number and β is the complex propagation factor. Alternatively this solution can be written in terms of a dimensionless wave number

$$\sigma'=\frac{y_0}{S_0}\sigma \tag{28}$$

and of a dimensionless propagation factor β' given by

$$\beta'=\frac{y_0}{S_0 v_0}\beta \tag{29}$$

so that the solution can be written as

$$y=y_m \exp[\mathrm{i}(\sigma' x'-\beta' t')]. \tag{30}$$

Either the substitution of equation (27) into equation (3) and the use of equations (28) and (29) or the substitution of equation (30) into equation (7) gives

$$F_0^2(\beta')^2-2(\sigma' F_0^2-\mathrm{i})\beta'-[\sigma'(1-F_0^2)+3\sigma'\mathrm{i}]=0. \tag{31}$$

Equation (31) corresponds to the characteristic equation derived by Ponce and Simons (1977) but in present paper is derived without using the assumption of a zero phase shift between depth and velocity which is only true for the limiting case of $\sigma'=0$ (the kinematic wave approximation).

The dimensionless phase velocity for the complete equation is given by

$$c_c'=\frac{c_c}{v_0}=\frac{\beta_R'}{\sigma'}. \tag{32}$$

The logarithmic decrement (the attenuation over a single wave length) is given by

$$\delta_c'=2\pi\frac{\beta_I'}{\beta_R'}, \tag{33}$$

where β_R' and β_I' are the real and imaginary parts of the complex propagation factor β' which is defined by equation (31).

Substitution of equation (27) into the general equation (10) for diffusion analogy gives

$$\beta=c\sigma-\mathrm{i}D\sigma^2 \tag{34}$$

which from equations (11) and (12) – or equation (13), or equation (20) – can be written as

$$\beta=1.5v_0\sigma+\mathrm{i}(1-rF_0^2)\sigma^2, \tag{35}$$

where the value of r depends on the particular form of the diffusion analogy model.

Accordingly, the dimensionless phase velocity corresponding to equation (32) is

$$c'_d = 1.5 \tag{36}$$

and the dimensionless logarithmic decrement correspond to equation (33) is

$$\delta'_d = 2\pi(1 - rF_0^2)\frac{\sigma'}{3}, \tag{37}$$

where r depends on the model used.

The ratio of the attenuation over a single wave length for a diffusion analogy model to the attenuation for the complete solution will be given by

$$\exp(\delta'_d - \delta'_c) = \exp\left\{-2\pi\left[(1 - rF_0^2)\frac{\sigma'}{3} - \frac{\beta'_I}{\beta'_R}\right]\right\}. \tag{38}$$

Fig. 1 shows the attenuation ratio for the primary wave as a function of the dimensionless wave number σ' for the diffusion analogy model used by Ponce and Simons (1977) for which $r = 0$ for the case of subcritical flow. Fig. 2 shows the same ratio for the diffusion

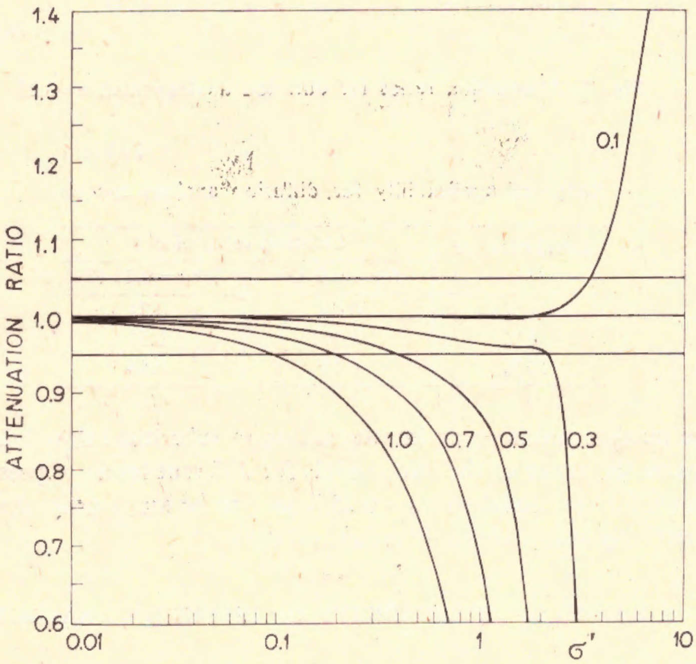

Fig. 1. Attenuation ratios for diffusion analogy with $r = 0$

analogy model proposed by Dooge and Harley (1967) in which $r = 0.25$. The fact that the latter model gives a closer approximation over a wider range of dimensionless wave numbers at any given Froude number reinforces the arguments put forward to support this model in the last section. There will be no difference in attenuation for the limiting case of

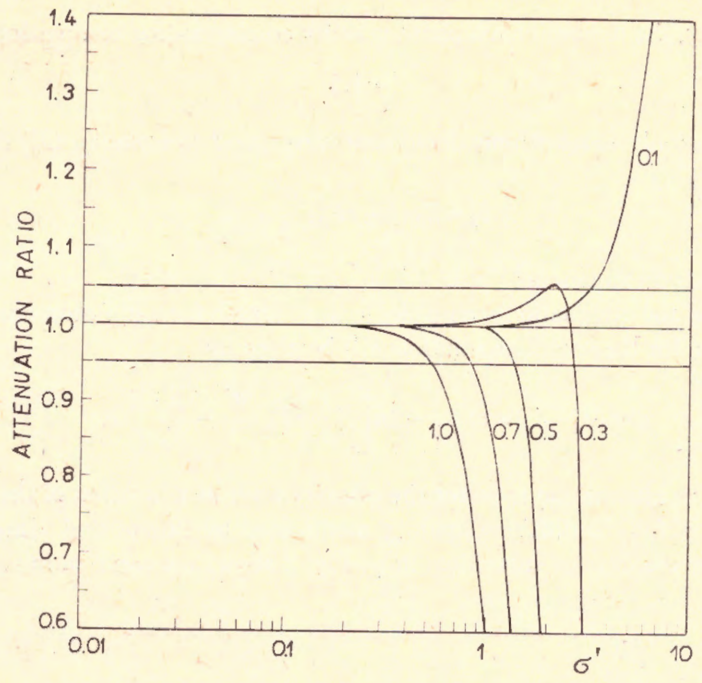

Fig. 2. Attenuation ratios for diffusion analogy with $r=0.25$

Table 1

Range of applicability for diffusion analogy models

Level of error (1)	Limiting value of σ' for $r=0$ (2)	Limiting value of σ' for $r=0.25$ (3)
1%	0.02	0.31
5%	0.097	0.51
10%	0.19	0.64

$F_0=0$ and the maximum difference in attenuation in subcritical flow will be for $F_0=1$. Table 1 compares the range of the two models for different levels of prescribed error for $F_0=1$. The range for the model with $r=0.25$ is seen to be many times that for the model with $r=0$ particularly at low lev els of prescribed error.

6. RANGE OF APPLICABILITY OF DIFFUSION ANALOGY

The phase velocity for the complete equation is obtained by solving equation (31) for the complex propagation factor and applying equation (32). The resulting value of c_c' for the complete equations is presented graphically by Ponce and Simons (1977). The error in phase velocity for the diffusion analogy can be determined from

$$c_R = \frac{c_d'}{c_e'} . \tag{39}$$

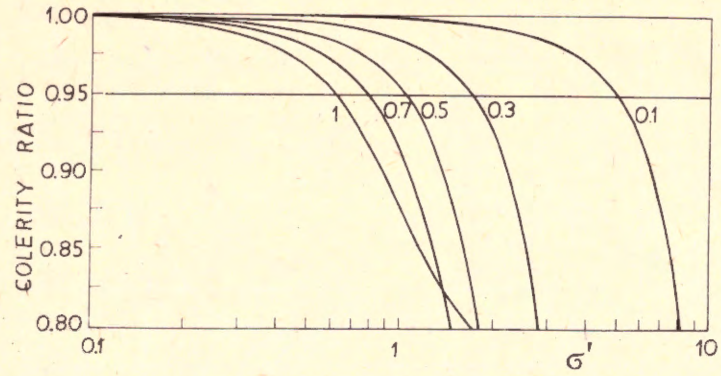

Fig. 3. Error in diffusion celerity for subcritical flow

This ratio is plotted in Fig. 3 as a function of the dimensionless wave number. From this figure it can be seen that all forms of the diffusion analogy give a good approximation of the phase velocity for wave numbers less than the limit defined by

$$\sigma' < 0.62 \tag{40}$$

to ensure that the error is less than 5%. The criteria for other levels are given in the second column of Table 2. The above results are applicable to all versions of the diffusion analogy discussed earlier.

In the case of the attenuation the results will differ for the various models and the discussion is confined to the Dooge and Harley (1967) model which a number of criteria

Table 2

Error levels in phase velocity and attenuation per unit length

Level of error (1)	Limiting value of σ' for c (2)	Limiting value of σ' for δ (3)
1%	0.27	0.41
5%	0.62	0.60
10%	0.89	0.71

indicate to be the most accurate approximation. In the last section the ratio of the diffusion analogy attenuation to the complete equation for a single wave length was approximated and plotted as a function of the dimensionless wave number in Fig. 2. In practice we need an estimate of the error over a fixed length of channel rather than the error over individual wave lengths. The logarithmic decrement for a fixed length is given by

$$\bar{\delta} = \sigma' x' \frac{\beta_I'}{\beta_R'} . \tag{41}$$

The ratio of the two attenuations for a fixed length is given by

$$\exp[(\delta_d - \delta_c)\sigma' x'] = \exp\left\{-2\pi\sigma' x'\left[(1 - rF_0^2)\frac{\sigma'}{3} - \frac{\beta_I'}{\beta_R'}\right]\right\} . \tag{42}$$

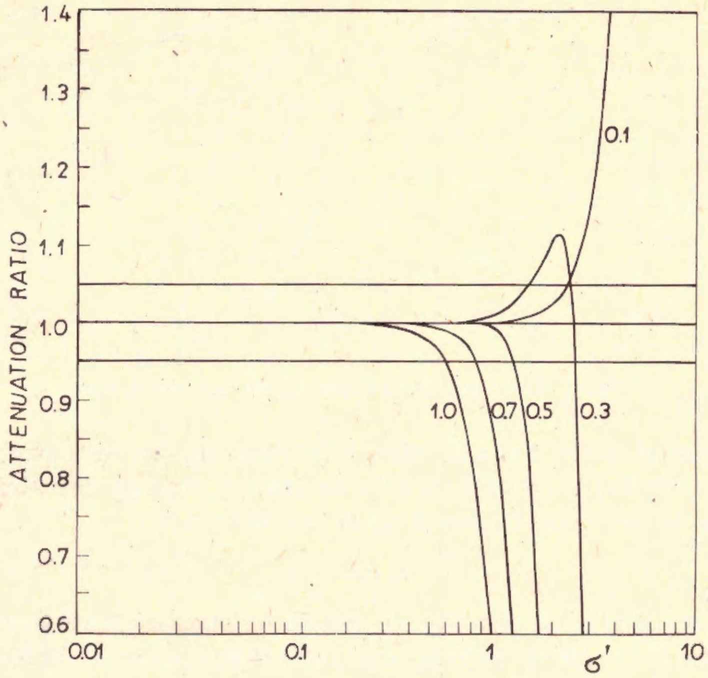

Fig. 4. Error in diffusion analogy attenuation per unit length for subcritical flow

The ratio is shown in Fig. 4 for the case of $x'=1$ which is usually taken as representing a relatively short channel. For multiplies of this length the attenuation ratio can be obtained by raising the ratio for unity length to the appropriate power. Fig. 4 suggests an appropriate criterion for 5% error in a unit length as

$$\sigma' < 0.60. \tag{43}$$

The maximum dimensionless wave length for other levels of error attenuation ratio per unit length are shown in the third column of Table 2.

7. CONCLUSIONS

Three alternate forms of a convection-diffusion equation suitable for flood routing applications has been presented. The first is the classic one, based on neglecting inertia altogether in linearised St. Venant equations, in which the hydraulic diffusivity is independent of Froude number. The second form is based on the partial neglect of inertia. The equivalent diffusivity differs from that given by the classic form particularly for higher Froude numbers. The third form ot diffusion analogy is derived by approximating the inertia terms on the basis of the kinematic wave solution. This form reproduces exactly the first and second moments of the complete linear solution. This indicates, that this form of the diffusion analogy is the most suitable of the three alternative forms.

The range of applicability of third form for flood routing applications is discussed in terms of the wave analysis method for a number of Froude numbers between 0 and 1 and a number of dimensionless wave numbers ($\sigma y_0/S_0$) between 0.01 and 10.

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ZASTOSOWANIE ANALOGU DYFUZYJNEGO W MODELOWANIU TRANSFORMACJI FALI POWODZIOWEJ

Streszczenie

Porównano trzy modele dyfuzji konwekcyjnej, aproksymujące zlinearyzowane równania St. Venanta. Do symulowania transformacji fali powodziowej zaleca się stosować model otrzymany przez oszacowanie składników inercyjnych za pomocą fali kinematycznej. Przeanalizowano błędy tłumienia i prędkości fazowej tego modelu. Podano graniczne wartości bezwymiarowej liczby falowej dla wybranych poziomów błędów.