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EFFECT OF DOWNSTREAM CONTROL IN DIFFUSION ROUTING

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Abstract

The diffusion analogy model is obtained from the St. Venant equations by means of an approximation of the inertia terms. The relative effects of the upstream and downstream conditions on the water level at any intermediate point in the reach are evaluated for this parabolic model. The results are applicable to any shape of cross-section and to any type of friction law.

1. INTRODUCTION

The hydraulic formulation for unsteady flow in open channels requires two boundary conditions and in the case of tranquil flow (i.e. Froude number less than unity) one of these is at the downstream end of the channel. Most hydrologic methods of flood routing are formulated in terms of an upstream boundary condition only. The present paper examines the effect of neglecting the downstream boundary condition on a simplified but acceptable mathematical model for unsteady flow in an open channel.

The method chosen is to examine analytically the solution of the parabolic form of the linearized St. Venant equations so that the relative effect of the upstream and downstream boundary conditions on the conditions at any intermediate point can be compared.

2. BASIC EQUATION OF FLOOD ROUTING – ST. VENANT EQUATIONS

The one-dimensional equation of continuity for unsteady flow in an open channel without lateral inflow is given by

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0 \tag{1}$$

in which $Q(x, t)$=discharge; $A(x, t)$=cross-sectional area of flow.

If the assumption is made that only accelerations in the direction of motion need to be taken into account then the equation for the conservation of linear momentum in this direction can be written in terms of the same variables as (Dooge et al., 1982)

$$g(1-F^2)\frac{A}{T}\frac{\partial A}{\partial x}+\frac{2Q}{A}\frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial t}=gA(S_0-S_f), \tag{2}$$

where T is the width of the channel at the water surface, i.e.

$$T=\frac{dA}{dy}, \tag{3}$$

S_0 is the bottom slope, y is the depth of flow, F is the Froude number defined by

$$F^2=\frac{Q^2T}{gA^3}. \tag{4}$$

The friction slope S_f depends on the type of friction law assumed, the shape of a cross--section, the flow at the section and the area of the flow. In this discussion the friction slope S_f is taken in the completely general form and may be written as

$$S_f=f(A, Q, \text{shape}, \text{roughness}). \tag{5}$$

The problem of the flood routing involves the solution of the above set of non-linear hyperbolic equations subject to given initial conditions and two appropriate boundary conditions. For the case of tranquil flow, one of these boundary conditions must be prescibed at each end of the reach.

No analytical solution is available and the problem must be solved by some method of numerical approximation, or by some simplification of the non-linear momentum equation. To analytically determine the effect of downstream boundary condition on the solution of equations (1) and (2) some simplification will be assumed.

3. LINEARIZATION OF ST. VENANT EQUATIONS

The complete non-linear St. Venant equations can be simplified by considering the first-order variation from a steady state trajectory. Then we obtain a linear approximation to the solution of the problem. To compute the linearized equations we make use of:

a) expansion of the nonlinear terms in equation (2) in Taylor series around the uniform steady state (Q_0, A_0) and the limitation of this expansion to the first-order increments $Q'(x, t)$, $A'(x, t)$ defined as

$$Q(x, t)-Q_0=Q'(x, t)+e(Q), \tag{6}$$

$$A(x, t)-A_0=A'(x, t)+e(A), \tag{7}$$

where $e(Q)$ and $e(A)$ represent the higher order terms than linear (error of approximation),

b) substitution of equations (6) and (7) to equations (1) and (2) and neglecting the higher order terms. The resulting equations are:

$$\frac{\partial Q'}{\partial x}+\frac{\partial A'}{\partial t}=0\,, \tag{8}$$

$$g\,(1-F_0^2)\frac{A_0}{T_0}\frac{\partial A'}{\partial x}+\frac{2Q_0}{A_0}\frac{\partial Q'}{\partial x}+\frac{\partial Q'}{\partial t}=gA'\left(-\frac{\partial S_f}{\partial Q}Q'-\frac{\partial S_f}{\partial A}A'\right). \tag{9}$$

It is convenient to transform the two first order linear equations given by equations (8) and (9) in the dependent variables $Q'(x, t)$ and $A'(x, t)$ into a single second order partial differential equation in the single dependent variable $A'(x, t)$. This may be done by:

a) differentiating equation (8) with respect to t,

b) differentiating equation (9) with respect to x,

c) making the necessary substitutions in order to eliminate the variable $Q'(x, t)$.

The resulting equation

$$g\,(1-F_0^2)\frac{A_0}{T_0}\frac{\partial^2 A'}{\partial x^2}-\frac{2Q_0}{A_0}\frac{\partial^2 A'}{\partial x\,\partial t}-\frac{\partial^2 A'}{\partial t^2}=gA_0\left(-\frac{\partial S_f}{\partial A}\frac{\partial A'}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial A'}{\partial t}\right) \tag{10}$$

is a second order differential equation for the perturbation $A'(x, t)$ from the steady uniform reference area A_0. The form of boundary condition for this equations can be derived from an accurate continuous record level. Equation (10) is the generalized form of the equation derived by Deymie (1939) for the case of wide rectangular channel with Chezy friction.

Alternatively, one could eliminate $A'(x, t)$ from equations (8) and (9) and obtain a single second order equation in $Q'(x, t)$ which is the same as equation (10). The equation in terms of $A'(x, t)$ is prefered because it allows for the use of more natural boundary conditions.

4. DERIVATION OF PARABOLIC EQUATION

We now consider the question of whether the hyperbolic equation (10) can be further simplified. An analysis of the order of magnitude of terms in equation (10) indicates, for values of the Froude number usually encountered that the second and third terms on the left-hand side of equation (10) are usually much smaller than the first term and that all three terms are smaller than the terms on the right-hand side of the equation (Henderson, 1966; Kuchment, 1972). It must be remembered that the fact that some terms are relatively small in a differential equation does not guarantee that they have no effect on the solution and can be neglected. Instead of completely neglecting the second and third terms in equation (10), it is preferable to express them as functions of the first term on the basis of lower order approximation to the solution of the equation. This low order approximation is given by neglecting all terms on the left-hand side of equation (10) to obtain

$$-\frac{\partial S_f}{\partial A}\frac{\partial A'}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial A'}{\partial t}=0\,. \tag{11}$$

Since we have for the reference condition of uniform steady flow

$$\frac{dS_f}{dA}=0 \tag{12}$$

then we can write

$$\frac{\partial S_f}{\partial A}+\frac{\partial S_f}{\partial Q}\frac{dQ}{dA}=0, \tag{13}$$

where dQ/dA is evaluated at the reference condition.

We can use the latter relationship to write equation (11) as

$$\frac{dQ}{dA}\frac{\partial A'}{\partial x}+\frac{\partial A'}{\partial t}=0 \tag{14}$$

which is the linear kinematic wave equation with a celerity given by

$$c=\frac{dQ}{dA}=\frac{mQ_0}{A_0}, \tag{15}$$

where m is a number which represents the ratio of the kinematic wave celerity to the reference velocity and depends on the friction law and the shape of the channel.

Equation (14) can be used to evaluate the relative magnitudes of the terms of the left-hand side of equation (10)

$$-\frac{2Q_0}{A_0}\frac{\partial^2 A'}{\partial x\,\partial t}=\frac{2Q_0}{A_0}\frac{dQ}{dA}\frac{\partial^2 A'}{\partial x^2}=2m\left(\frac{Q_0}{A_0}\right)^2\frac{\partial^2 A'}{\partial x^2}, \tag{16}$$

$$-\frac{\partial^2 A'}{\partial t^2}=-\left(\frac{dQ}{dA}\right)^2\frac{\partial^2 A'}{\partial x^2}=-m^2\left(\frac{Q_0}{A_0}\right)^2\frac{\partial^2 A'}{\partial x^2}. \tag{17}$$

It can be seen from equations (16) and (17) that the second and third terms on the left-hand side of equation (10) are opposite in sign and of the same order of magnitude. It can also be seen that for low values of Froude number they will be small compared to the first term. If the approximation represented by equation (14) is used to express the second and third terms on the left-hand side of equation (10) in terms of the first term as in equations (16) and (17), we would have

$$\frac{1}{T_0}[1-(m-1)^2F_0^2]\frac{\partial^2 A'}{\partial x^2}=-\frac{\partial S_f}{\partial A}\frac{\partial A'}{\partial x}+\frac{\partial S_f}{\partial Q}\frac{\partial A'}{\partial t}. \tag{18}$$

Equation (18) is parabolic in form in contrast to equation (10) which is hyperbolic. The equation is of the same form as the governing equation for diffusion processes so it is convenient to write equation (18) in the diffusion analogy form

$$D\frac{\partial^2 A'}{\partial x^2}=c\frac{\partial A'}{\partial x}+\frac{\partial A'}{\partial t} \tag{19}$$

in which D is a "hydraulic diffusivity coefficient"

$$D=\frac{1}{T_0}\frac{1-(m-1)^2F_0^2}{\dfrac{\partial S_f}{\partial Q}} \tag{20}$$

and c is an advective velocity

$$c=\frac{dQ}{dA}=-\frac{\dfrac{\partial S_f}{\partial A}}{\dfrac{\partial S_f}{\partial Q}}. \tag{21}$$

It should be noted that in the parabolic case of flood routing model two boundary conditions are required. These boundary condition must be prescribed at the upstream ($x=0$) and the downstream ($x=L$) ends of the reach. They are usually expressed in Dirichlet form, i.e. the values of

$$A_u(t)=A'(0, t) \tag{22}$$

and

$$A_d(t)=A'(L, t) \tag{23}$$

are prescribed. This corresponds to the practical situation where water level is recorded at each end of the channel reach.

So, from the boundary conditions point of view, the parabolic model can be taken as the approximation of the hyperbolic model only for the case of tranquil flow (i.e. the Froude number less than 1).

The degree of the approximation involved in replacing the complete St. Venant equation (10) by the diffusion analogy model (19) - (21) has been discussed by Dooge and Napiórkowski (1983). In that paper the upper limits for the dimensionless wave number are given for certain prescribed levels of error.

5. LAPLACE TRANSFORM OF SIMPLIFIED EQUATION – SOLUTION IN THE TRANSFORM DOMAIN

The parabolic version of the routing problem can be conveniently solved by the use of the Laplace transform technique. Since equation (19) represents perturbation from the initial condition, the initial value of the dependent variable $A'(x, t)$ and its derivatives will all be zero. For this case of zero initial condition, equation (19) when transformed to the Laplace transform domain becomes

$$D\frac{d^2a}{dx^2}=c\frac{da}{dx}+sa, \tag{24}$$

where $a(x, s)$ is the Laplace transform of $A'(x, t)$. Equation (24) is a second-order homogeneous ordinary equation, so the solution can be written in the general form:

$$a(x, s)=B(s)\exp[\lambda_1(s)x]+C(s)\exp[\lambda_2(s)x], \tag{25}$$

where λ_1 and λ_2 are the roots of the characteristic equation for equation (24), and $B(s)$, $C(s)$ are functions of s to be determined so the appropriate boundary conditions are satisfied.

At the upstream boundary ($x=0$) we have

$$a_u(s)=B(s)+C(s) \tag{26}$$

and at the downstream boundary ($x=L$) we have

$$a_d(s)=B(s)\exp(\lambda_1 L)+C(s)\exp(\lambda_2 L), \tag{27}$$

where $a_u(s)$ and $a_d(s)$ are the Laplace transforms of $A_u(t)$ and $A_d(t)$, respectively. Solving equations (26), (27) for the unknown functions and substituting these values in equation (25) we get

$$a(x, s)=h_u(x, s)\, a_u(s)+h_d(x, s)\, a_d(s), \tag{28}$$

where

$$h_u(x, s)=\exp\left(\frac{cx}{2D}\right)\frac{\sinh\left[(L-x)\sqrt{\frac{c^2}{4D^2}+\frac{s}{D}}\right]}{\sinh\left(L\sqrt{\frac{c^2}{4D^2}+\frac{s}{D}}\right)} \tag{29}$$

is the system function (i.e. the Laplace transform of the impulse response) for an upstream input, and

$$h_d(x, s)=\exp\left[-(L-c)\frac{c}{2D}\right]\frac{\sinh\left(x\sqrt{\frac{c^2}{4D^2}+\frac{s}{D}}\right)}{\sinh\left(L\sqrt{\frac{c^2}{4D^2}+\frac{s}{D}}\right)} \tag{30}$$

is the system function for a downstream input.

6. INVERSION TO THE TIME DOMAIN

The impulse responses represented by equations (29) and (30) can be inverted to the time domain by using the standard transform pairs given by Doetsch (1961 – transforms 185 and 186). By adopting these standard transform pairs we can write the inverse of equation (29) as

$$h_u(x, t)=\exp\left(\frac{cx}{2D}-\frac{c^2 t}{4D}\right)\sum_{-\infty}^{+\infty}\frac{2nL+x}{2\sqrt{\pi D}\, t^{3/2}}\exp\left[-\frac{(2nL+x)^2}{4Dt}\right] \tag{31}$$

and the inverse of equation (30) as

$$h_d(x, t)=\exp\left[-\frac{c(L-x)}{2D}\right]\sum_{-\infty}^{+\infty}\frac{2nL+L-x}{2\sqrt{\pi D t}^{\,3/2}}\exp\left[-\frac{(2nL+L-x)^2}{4Dt}\right]. \tag{32}$$

The series (31) and (32) are particularly suitable for practical calculations when t is small because then the factors

$$\exp\left[-\frac{(2nL+x)^2}{4Dt}\right]$$

are all small.

For large t the other forms are more suitable (Doetsch, 1961 – transforms 185b and 186b)

$$h_u(x,t)=\exp\left(\frac{cx}{2D}-\frac{c^2t}{4D}\right)\frac{2\pi D}{L^2}\sum_1^{+\infty} n\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2Dt}{L^2}\right) \tag{33}$$

and

$$h_d(x,t)=\exp\left[-\frac{c(L-x)}{2D}-\frac{c^2t}{4D}\right]\left(\frac{-2\pi D}{L^2}\right)\sum_1^{+\infty}(-1)^n n\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2Dt}{L^2}\right). \tag{34}$$

Thus the solution of flood routing problem in the time domain for the finite reach reads

$$A'(x,t)=h_u(x,t)*A_u(t)+h_d(x,t)*A_d(t) \tag{35}$$

in which $h_u(x, t)$ is given by either of equivalent forms of equation (31) or equation (33) and $h_d(x, t)$ by either equation (32) or equation (34).

If we wish to consider the limiting case of downstream flow for a semi-infinite reach (i.e. the case where the downstream control is so distant from the section of interest that the boundary condition has no influence) we can set $L\to\infty$ in equation (31) in which event the series reduces to the single term $n=0$ and we obtain

$$h_u(x,t)=\frac{x}{2\sqrt{\pi D}\,t^{3/2}}\exp\left[-\frac{(x-ct)^2}{4Dt}\right] \tag{36}$$

which corresponds to solution obtained for semi-infinite channel by Dooge (1967, 1973).

The case of upstream flow in a semi-infinite channel (i.e. where upstream boundary condition has no influence) is obtained by shifting the origin to the downstream end of the channel so that $x=-L$ at the upstream end and $x=0$ at the downstream end. Equation (32) then can be expressed in terms of x' where $x'=x-L$ as

$$h_d'(x,t)=\exp\left(\frac{cx'}{2D}-\frac{c^2t}{4D}\right)\sum_{-\infty}^{+\infty}\frac{2nL-x'}{2\sqrt{\pi D}t^{3/2}}\exp\left[-\frac{(2nL-x')^2}{4Dt}\right]. \tag{37}$$

Setting $L\to\infty$ the series again reduces to a single term and is given by

$$h_d'(x,t)=\frac{-x'}{2\sqrt{\pi D}\,t^{3/2}}\exp\left[-\frac{(x'-ct)^2}{4Dt}\right]. \tag{38}$$

In comparing equation (38) with equation (36) it should be remembered that x takes only positive values in equation (36) and that x' takes only negative values in equation (38). Consequently the degree of damping is greater in equation (38) than in equation (36).

7. EFFECT OF LOCATION OF DOWNSTREAM BOUNDARY

The water level at any intermediate point in the reach is determined by the upstream boundary condition $A_u(t)$ and the downstream boundary condition $A_d(t)$ in accordance with equation (35). It is clear that if the value of $A_d(t)$ is very much larger than $A_u(t)$, then this downstream boundary condition will have a dominant influence on conditions throughout most of the reach. In most cases, however, the two boundary conditions will be of the same order of magnitude. Accordingly it is instructive to examine the relative magnitude of the impulse responses $h_u(x, t)$ and $h_d(x, t)$ for typical conditions. Comparison of equations (31) and (32) enables us to compare readily the influence of an upstream and downstream impulse response on condition at the mid-point of the channel. Substitution of $x=L/2$ in equations (31) and (32) indicates that for all times we have at the mid-point of the channel a ratio of downstream influence to upstream influence given by

$$r\left(\frac{L}{2}, t\right)=\frac{h_d\left(\frac{L}{2}, t\right)}{h_u\left(\frac{L}{2}, t\right)}=\exp\left(-\frac{cL}{2D}\right), \tag{39}$$

where L is the length of the reach, c is the kinematic wave velocity given by equation (21) and D is the channel diffusivity defined by equation (20). Consequently we see that at the mid point of the channel the two impulse responses will have the same shape but the response to the downstream boundary condition will have a smaller effect.

It is interesting to evaluate the ratio given by equation (39) for a typical case. Thus if we have a channel with the following properties: bottom slope $S_0=0.000102$, Chezy coefficient $C_f=70$, length $L=60$ km, width of channel $T=100$ m, and flow in steady state $Q_0=200\ \mathrm{m^3/s}$. We have from equation (20):

$$D=9675\ \mathrm{m^2/s},$$

$$c=1.5\ \mathrm{m/s}.$$

Inserting these values in equation (39) gives

$$r\left(\frac{L}{2}, t\right)=0.955\times 10^{-2}$$

which is seen to be very small indeed.

For the last example the conditions at $x=30$ km are in effect independent of the downstream boundary condition at 60 km so that the level at 30 km given by

$$A'(x, t)=h_u(x, t)*A_u(t) \tag{40}$$

is accurate to 1% and downstream flood routing can be safely used. It is instructive to ask the question how much closer to the point $x=30$ km must the downstream control be to effect seriously conditions at that point. This has been calculated on the basis of equations (31) and (32) for the channel properties already assumed.

The position of the downstream control has two effects on the value at a fixed point in a reach. Firstly, there is the contribution of the convolution of the downstream boundary

Table 1

The ratios of the peak and time to peak of the downstream impulse response to the peak and time to peak of the upstream impulse response for various values of length of the reach

L [km]	$\frac{\max h_d(30, t)}{\max h_u(30, t)}$	$\frac{t_p[h_d(30, t)]}{t_p[h_u(30, t)]}$
60	0.009	1.000
55	0.025	0.755
50	0.068	0.522
45	0.207	0.317
35	0.238	0.046

Table 2

The effect of the downstream control on the peak and time to peak of the upstream impulse response for various values of length of the reach

L [km]	$\frac{\max h_u(30, t)_L}{\max h_u(30, t)_\infty}$	$\frac{t_p[h_u(30, t)]_L}{t_p[h_u(30, t)]_\infty}$
60	1.000	1.000
55	1.000	1.000
50	0.999	0.999
45	0.997	0.996
40	0.965	0.948
35	0.781	0.868

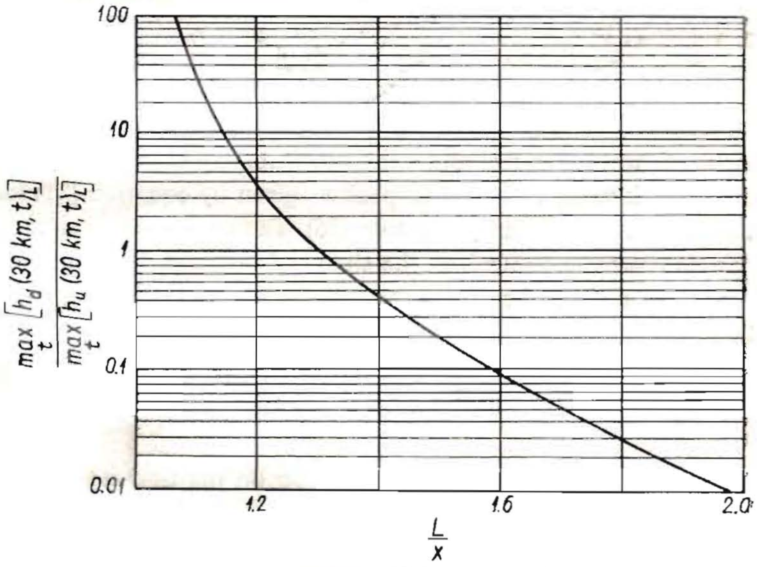

Fig. 1. Ratio of the maximum values of the downstream impulse response (equation (48)) and the upstream impulse response (equation (47)) at $x=30$ km and for various of length of the reach (L)

condition $A_d(t)$ with the downstream impulse response $h_d(x, t)$. Secondly there is the effect of the value of L on the upstream impulse response given by equation (31).

The calculated values of the peak [max $h_d(x, t)$] and the time to peak (t_p) of $h_d(x, t)$ for various values of length of the reach (L) are given in Table 1 and in Fig. 1 as ratios of the values for the impulse response due to upstream inflow given by equation (31). It is clear from the table that the backwater effect as measured by $h_d(x, t)$ is only effective for this particular channel when the control is closed than 20 km to the $x=30$ km point which is being examined.

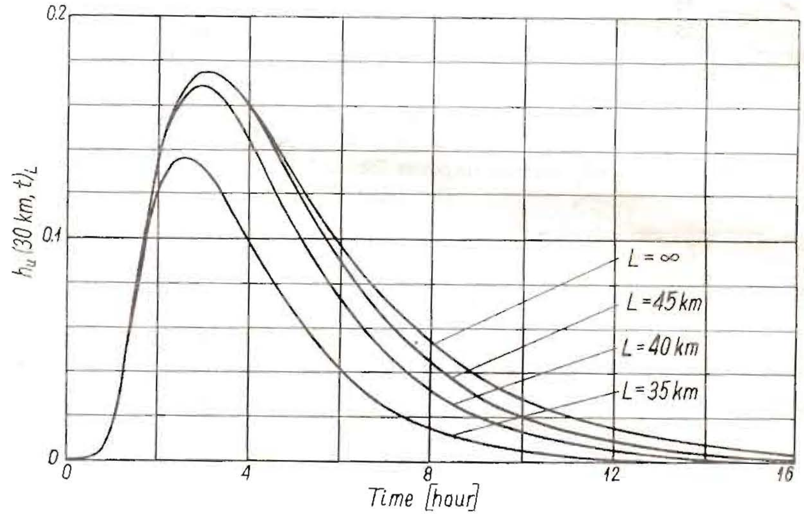

Fig. 2. The effect of downstream boundary condition on the shape of upstream impulse response (equation (47)) at $x=30$ km and for various of length of the reach (L)

The effect of the downstream control on the upstream impulse function can be evaluated by relating the peak value and time to peak as given by equation (31) to the values for a semi-infinite channel as given by equation (36). The results are as shown in Table 2. The effect of the downstream control on the shape of the upstream impulse response is shown in Fig. 2. In this case, it is clear that the simpler form of equation (36) is adequate for peak prediction unless the control is no closed than 15 km.

8. CONCLUSION

The effect of the downstream boundary condition on the flow in a channel reach is examined on a analytical way. To obtaine a simple form of solution the St. Venant equations are simplified to diffusion analogy model by means of approximation of the inertia terms. The relative effects of the upstream and downstream boundary conditions at any intermediate point in the reach are evaluated. The obtained results are applicable to any shape of cross-section and any type of friction law.

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REFERENCES

Deymie Ph., 1939, *Propagation d'une intumescence allongee (problem aval)*, Proc. 5-th Intern. Congress Appl. Mechan., John Wiley and Sons Inc., New York, 537 - 544.

Doetsch G., 1961, *Guide to the Applications of Laplace Transforms*, Van Nostrand, New York—London.

Dooge J. C. I., 1967, *Linear Theory of Open Channel Flow. II – Parabolic Linear Solution* (unpublished), Dept. Civil Engin., Univ. College, Cork.

Dooge J. C. I., 1973, *Linear Theory of Hydrologic Systems*, Techn. Bull. 1468, Agricultural Res. Service, USDA, Washington D. C.

Dooge J. C. I., Strupczewski W. G., Napiórkowski J. J., 1982, *Hydrodynamic derivation of storage parameters of the Muskingum model*, J. Hydrol., 54, 371 - 387.

Dooge J. C. I., Napiórkowski J. J., 1983, *Applicability of diffusion analogy in flood routing*, J. Hydrol. Div. ASCE (in press).

Henderson F. M., 1966, *Open Channel Flow*, Macmillan, New York.

Kuchment L. S., 1972, *Matematicheskoe modelirovanie rechnogo stoka* (*Mathematical Modelling of River Flow*), Gidrometeoizdat., Leningrad.

WPŁYW DOLNEGO WARUNKU BRZEGOWEGO NA TRANSFORMACJĘ FALI POWODZIOWEJ W MODELU ANALOGU DYFUZYJNEGO

Streszczenie

Model analogu dyfuzyjnego otrzymano na podstawie równań St. Venanta przez aproksymację składników bezwładnościowych. Określono względny wpływ górnego i dolnego warunku brzegowego na poziom wody w odcinku koryta otwartego. Otrzymane rezultaty mają zastosowanie dla przekroju poprzecznego o dowolnym kształcie i dowolnego prawa tarcia.